ON FULLY STABLE ACTS

Mehdi Sadiq Abbas
Hiba Rabee Baanoon

Department of Mathematics
University of Mustansiriya
Baghdad
Iraq
e-mails: m.abass@uomustansiriyah.edu.iq
h.rabeeh@ymail.com

Abstract. The purpose of this paper is to introduce and investigate the fully stable acts as a concept generalizing fully stable modules but is stronger than that of duo acts. In this study, we consider some properties and characterizations of the class of fully stable acts, and the relations between this class and other well studied classes of acts, like quasi-injective acts and acts satisfying Baer's criterion.

Keywords: fully stable, Baer's criterion, quasi-injective, right S-act.

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1. Preliminaries

Let S be a monoid. A right S-act M_S is a nonempty set M together with a map (written multiplicatively) from $M \times S$ into M satisfying $m1 = m$ and $m(st) = (ms)t$, for all $m \in M$ and $s, t \in S$.

A nonempty subset N of an S-act M_S is S-subact if $ns \in N$ for all $s \in S$ and $n \in N$. We say that M_S is a cyclic S-act if $M_S = uS$ for some $u \in M_S$.

An element $z \in M_S$ is called a *fixed element* of M_S if $zs = z$ for all $s \in S$. The set of all fixed elements of M_S will be denoted by $\mathcal{F}(M)$.

If M_S has a unique fixed element z, then z is called *zero element* of M_S. We will denote the zero element of M_S by $\mathcal{O}$. Every S-act M_S can be extended to an S-act with fixed element z by taking the disjoint union: $M_S \dot{\cup} \{z\}_S$.

A nonempty subset $K \subseteq S$ is called *left ideal* of a monoid S if $SK \subseteq K$; a *right ideal* of S if $KS \subseteq K$; an ideal of S if $KS \subseteq K$ and $SK \subseteq K$.

Recall that, for two S-acts A_S, B_S a mapping $\theta : A_S \to B_S$ is called a *homomorphism of S-acts* or just an *S-homomorphism* if $\theta(as) = \theta(a)s$ for all $a \in A_S$, $s \in S$. The set of all S-homomorphisms from A_S into B_S will be denoted by $\mathrm{Hom}(A_S, B_S)$ or sometimes by $\mathrm{Hom}_S(A, B)$.

Note that if $\theta : A_S \to B_S$ is an S-homomorphism then $\operatorname{Im}\theta = \theta(A_S)$ is a subact of B_S, and the S-homomorphism $f : M_S \to M_S$ is called an endomorphism of M_S.

The set $\mathrm{Hom}_S(M, M)$ which forms a monoid under composition of mappings is denoted by $\mathrm{End}_S(M)$ and is called the endomorphism monoid of M_S.

An equivalence relation ρ on an S-act M_S is called an *S-act congruence* or a *congruence* on M_S, if $(m, n) \in \rho$ implies $(ms, ns) \in \rho$ for $m, n \in M_S$, $s \in S$. If S is a monoid then any right (semigroup) congruence ρ on S is an act congruence on S_S. Also, for an S-act M_S, $H \subset S$, $K \subset M \times M$, $T \subset M$, $J \subset S \times S$.

$$\begin{aligned}
\mathcal{L}_M(H) &= \{(m, n) \in M \times M \mid mx = nx \text{ for all } x \in H\};\\
\mathcal{R}_S(K) &= \{s \in S \mid as = bs \text{ for all } (a, b) \in K\};\\
\mathcal{R}_S(T) &= \{(a, b) \in S \times S \mid ma = mb \text{ for all } m \in T\};\\
\mathcal{L}_M(J) &= \{m \in M \mid ma = mb \text{ for all } (a, b) \in J\}.
\end{aligned}$$

The above is a kind of *annihilator* in S-act. Where $\mathcal{L}_M(H)$ (resp. $\mathcal{L}_M(J)$) are called the left annihilator of H (resp. J) and $\mathcal{R}_S(K)$ (resp. $\mathcal{R}_S(T)$) are called the right annihilator of K (resp. T).

Clearly,

$$\mathcal{R}_S(M) = \{(a, b) \in S \times S \mid ma = mb, \text{ for all } m \in M\}$$

is a right semigroup congruence on S and $\mathcal{R}_S(K)$ is a right ideal of S. If S is commutative then the set

$$\mathcal{L}_M(S) = \{(m, n) \in M \times M \mid mx = nx \text{ for all } x \in S\}$$

is a congruence on M_S and, if $\mathcal{L}_M(J) \neq \emptyset$, then it is a subact of M_S.

Recall that for a family $\{A_i\}$, $i \in I$, of right S-acts their Cartesian product $\prod_{i\in I} A_i$ with the S-action (multiplication) defined by $(a_i)s = (a_i s)$ is the *product* of a family of $\{A_i\}$, $i \in I$ of a right S-act.

The *coproduct* of a family of $\{A_i\}$, $i \in I$ of a right S-act is their disjoint union

$$\coprod_{i\in I} A_i = \left(\bigcup_{i\in I} A_i \times \{i\}\right)$$

with the action of S defined by $(a, i)s = (as, i)$ for $a \in A_i$ and $s \in S$.

For the family $\{A_i : i \in I\}$ of S-acts with a unique fixed element (zero element $\mathcal{O}$), the *direct sum* $\bigoplus_{i\in I} A_i$ is defined as the subact of the product $\prod_{i\in I} A_i$ consisting of all $(a_i)_{i\in I}$ such that $a_i = \mathcal{O}$ for all $i \in I$ except a finite number. We use $\bigoplus_{i\in I} A_i$ only when the S-acts A_i have unique fixed elements.

An S-act M_S is called *injective* if for each S-monomorphism $g : A_S \to B_S$ (where A_S, B_S are any two S-acts) and each S-homomorphism $f : A_S \to M_S$, there exists an S-homomorphism $h : B_S \to M_S$ such that $hg = f$.

A subact B_S is *essential* in an S-act M_S if for any S-act A_S and any S-homomorphism $f : M_S \to A_S$ whose restriction to B is one-to-one, the map f is itself one-to-one. In such a case, we say that M_S is an essential extension of B_S. The minimal injective extension of M_S is called the injective hull of M_S and is denoted by $E(M)$. Note that $E(M)$ is the injective hull of M_S if and only if M_S is essential in $E(M)$ and $E(M)$ is injective [3].

The *Jacobson radical* of an S-act M_S (denoted $\mathcal{J}(M_S)$) is defined by:

$$\mathcal{J}(M_S) = \{m \in M_S \mid \lambda_m \text{ is one-to-one only on one element right ideals of } S\},$$

where the mappings $\lambda_m : S_S \to M_S$ are given by $s \mapsto ms$ for each $s \in S$.

For an S-act M_S with zero element $\mathcal{O}$, the Jacobson radical $\mathcal{J}(M_S)$ is a subact of M_S [4].

2. Fully stable acts

In 1990, M.S. Abbas introduced a class of modules is called a fully stable as follows, a submodule N of an R-module M_R is called stable if $f(N) \subseteq N$ for each R-homomorphism $f : N \to M$, M is called fully stable module in case each submodule of M is stable [1].

In this section, we introduced the fully stable concept as a class of acts, and give several characterizations of fully stable acts. Also we consider the relations between this class and acts satisfying Baer's criterion.

Definition 2.1. Let M_S be an S-act. A subact N_S of M_S is called stable, if $f(N) \subseteq N$ for each S-homomorphism $f : N \to M$. The act M is called fully stable in case each subact of M is stable. A monoid S is fully stable if it is a fully stable S-act.

We have directly from the definition that every fully stable act is duo act, where by a duo S-act M_S we mean an S-act in which every subact N_S is fully invariant (i.e. $f(N) \subseteq N$ for any S-homomorphism $f : M \to M$ [2]).

However, the converse need not to be true in general; for example, it is easy to see that the act $\mathbb{Z}_{(\mathbb{Z},.)}$ of all integers is duo but not fully stable. For, if we define $\alpha : 2\mathbb{Z} \to \mathbb{Z}$ by $2n \mapsto 3n$, then, clearly, α is a $\mathbb{Z}$-homomorphism for which $\alpha(2\mathbb{Z}) \nsubseteq 2\mathbb{Z}$ since $\alpha(2.1) = 3.1 = 3 \notin 2\mathbb{Z}$.

Remarks 2.2.

1. Every subact of a fully stable act is fully stable.

2. The direct sum (hence, product) of fully stable acts need not be fully stable. For instance, let M_S be a fully stable S-act with a unique fixed element (zero element $\mathcal{O}$). The map $f : M \oplus \{\mathcal{O}\} \to M \oplus M$ defined by $f((m, \mathcal{O})) = (\mathcal{O}, m)$ is an S-homomorphism. Hence from the definition of direct sum, there is an element $\mathcal{O} \neq a \in M$ with $f((a, \mathcal{O})) = (\mathcal{O}, a) \notin M \oplus \{\mathcal{O}\}$. Thus $f(M \oplus \{\mathcal{O}\}) \nsubseteq M \oplus \{\mathcal{O}\}$.

3. The coproduct of any family of fully stable acts need not be fully stable. For example, let M_S be a fully stable S-act, $N \times \{1\}$ be a subact of $M \amalg M = M \times \{1\} \dot{\cup} M \times \{2\}$. Define $\theta : N \times \{1\} \to M \amalg M$ by $\theta((n, 1)) = (n, 2)$. Clearly, θ is an S-homomorphism but $\theta(N \times \{1\}) \nsubseteq N \times \{1\}$, since for any $n \in N$ $(n, 2) \notin N \times \{1\}$.

In the following corollary, it is seen that to determine whether an act is fully stable it suffices to consider stability of a very restricted class of subacts.

Corollary 2.3. *An S-act M_S is fully stable if and only if every cyclic subact is stable.*

In the following proposition we give another characterization of fully stable acts which will be used later, when a monoid S is commutative.

Proposition 2.4. *An S-act M_S is fully stable if and only if for each x, y in M, $y \notin xS$ implies $\mathcal{R}_S(x) \nsubseteq \mathcal{R}_S(y)$.*

Proof. Suppose that M is fully stable and that there exist two elements x,$y \in M$ with $y \notin xS$ and $\mathcal{R}_S(x) \subseteq \mathcal{R}_S(y)$, define $f : xS \to M$ by $f(xr) = yr$ for $r \in S$. If $xr_1 = xr_2$ where $r_1, r_2 \in S$, then $(r_1, r_2) \in \mathcal{R}_S(x) \subseteq \mathcal{R}_S(y)$. This implies that $yr_1 = yr_2$, hence $f(xr_1) = f(xr_2)$, and f is well-defined. Clearly, f is an S-homomorphism. Since M is fully stable, we have $f(xS) \subseteq xS$ and $y = f(x) \in xS$ which is a contradiction.

Conversely, assume that there exists a cyclic subact xS of M and an S-homomorphism $\theta : xS \to M$ such that $\theta(xS) \nsubseteq xS$. Then, there exists an element $y \in xS$ such that $\theta(y) \notin xS$. Let $(s, t) \in \mathcal{R}_S(x)$, hence $xs = xt$. So

$$\theta(y)s = \theta(ys) = \theta(xrs) = \theta(xsr) = \theta(xtr) = \theta(xrt) = \theta(yt) = \theta(y)t.$$

Therefore, $(s, t) \in \mathcal{R}_S(y)$ and $\mathcal{R}_S(x) \subseteq \mathcal{R}_S(y)$, which is a contradiction. ∎

It is well-known that the Jacobson radical $\mathcal{J}(M)$ of an S-act is a fully invariant subact [3].

The following proposition gives a kind of subact which is always stable in any act.

Proposition 2.5. *The Jacobson radical of any act is a stable subact.*

Proof. Let M_S be an S-act and $f : \mathcal{J}(M) \to M$ an S-homomorphism. If A is a right ideal of S with more than one element i.e. $|A| \geq 2$, then there exist $a_1 \neq a_2 \in A$ such that $ma_1 = ma_2$. Hence

$$f(m)a_1 = f(ma_1) = f(ma_2) = f(m)a_2, \text{ for } m \in \mathcal{J}(M).$$

So $\lambda_{f(m)}$ is not one-to-one on A. Thus $f(m) \in \mathcal{J}(M)$. ∎

Definition 2.6. Let N_S be a subact of some act M_S. We say that N_S satisfies Baer criterion, if for every S-homomorphism $f : N_S \to M_S$, there exists an element $s \in S$ such that $f(n) = ns$ for each $n \in N_S$. An S-act M_S is said to satisfy Baer criterion if every subact of M_S satisfies Baer criterion.

Proposition 2.7. *If M_S is a fully stable S-act, then M_S satisfies Baer criterion for cyclic subacts* (*where S is a commutative monoid*).

Proof. Let xS be a cyclic subact of M_S and $f : xS \to M$ an S-homomorphism. Since xS is stable, we have $f(xS) \subseteq xS$ and hence $f(x) \in xS$ i.e. there is $t \in S$ such that $f(x) = xt$. Let $w \in xS$, hence $w = xr$ for some $r \in S$ and hence $f(w) \in xS$. So

$$f(w) = f(xr) = f(x)r = (xt)r = x(tr) = x(rt) = (xr)t = wt.$$

Hence there is $t \in S$ such that $f(w) = wt$ for every $w \in xS$. Thus Baer criterion holds for cyclic subacts. ∎

In the following proposition and its corollary, we obtain another characterization of fully stable acts. We assume the monoid S is commutative.

Proposition 2.8. *For an S-act M_S, Baer criterion holds for cyclic subacts if and only if $\mathcal{L}_M(\mathcal{R}_S(x)) = xS$ for all $x \in M$.*

Proof. Assume that the Baer criterion holds for cyclic subacts of M_S. Let $y \in \mathcal{L}_M(\mathcal{R}_S(x))$ and define $\theta : xS \to M$ by $\theta(xr) = yr$ for each $r \in S$. If $xr_1 = xr_2$ where $r_1, r_2 \in S$, then $(r_1, r_2) \in \mathcal{R}_S(x)$, hence $yr_1 = yr_2$ (since $y \in \mathcal{L}_M\mathcal{R}_S(x)$). Thus θ is well-defined. It is clear that θ is an S-homomorphism. By the assumption, there exists an element $t \in S$ such that $\theta(w) = wt$ for each $w \in xS$.

In particular,

$$y = \theta(x) = xt \in xS.$$

This implies that $\mathcal{L}_M(\mathcal{R}_S(x)) \subseteq xS$; since the inclusion $xS \subseteq \mathcal{L}_M(\mathcal{R}_S(x))$ is always true. Hence

$$\mathcal{L}_M(\mathcal{R}_S(x)) = xS.$$

Conversely, assume that $\mathcal{L}_M(\mathcal{R}_S(x)) = xS$ for each $x \in M$. Then, for each S-homomorphism $f : xS \to M$ and $(s,t) \in \mathcal{R}_S(x)$, we have

$$xs = xt \text{ and } f(x)s = f(xs) = f(xt) = f(x)t.$$

Thus $f(x) \in \mathcal{L}_M(\mathcal{R}_S(x)) = xS$. Therefore, $f(x) = xt$ for some $t \in S$. Now, for each $w \in xS$ there exists $r \in S$ such that $w = xr$, hence

$$f(w) = f(xr) = f(x)r = (xt)r = x(tr) = x(rt) = (xr)t = wt.$$

So there exists $t \in S$ such that $f(w) = wt$ for each $w \in xS$. ∎

As we have mentioned earlier, any fully stable S-act satisfies Baer criterion for cyclic subacts, thus we have the following corollary.

Corollary 2.9. *An S-act M_S is fully stable if and only if $\mathcal{L}_M(\mathcal{R}_S(x)) = xS$ for each $x \in M$.*

The results of this section can be summarized together with those of section one, in the following theorem.

Theorem 2.10. *The following statements are equivalent for an S-act M_S.*

1. M_S *is a fully stable act.*
2. *Every cyclic subact of* M_S *is stable.*
3. *For each* x, y *in* M_S, $y \notin xS$ *implies* $\mathcal{R}_S(x) \nsubseteq \mathcal{R}_S(y)$.
4. M_S *satisfies Baer criterion for cyclic subacts.*
5. *For each* x *in* M_S, $\mathcal{L}_M(\mathcal{R}_S(x)) = xS$.

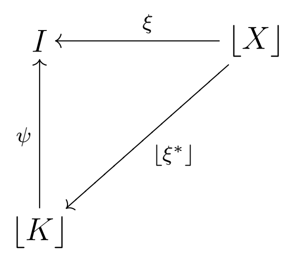
Another characterization of fully stable acts is given here.

Remark 2.11. An S-act M_S is fully stable if and only if for each S-act A_S and for any two homomorphisms $f, g : A \to M$, with g injective (one-to-one mapping), we have $\operatorname{Im} f \subseteq \operatorname{Im} g$.

Proof. ($\Rightarrow$) Let A_S be an S-act and $f, g : A \to M$ S-homomorphisms. By injectivity of g, there exists an S-homomorphism $h : g(A) \to A$ such that $h \circ g = id_A$. Since $g(A)$ is a subact of M, we have $g(A)$ is stable. Hence $f \circ h(g(A)) \subseteq g(A)$. So $f(h \circ g(A)) \subseteq g(A)$ and $f(A) \subseteq g(A)$. ($\Leftarrow$) Let N_S be a subact of M_S and $f : N \to M$ an S-homomorphism. Since the inclusion $i : N \to M$ is an injective homomorphism, we get $f(N) \subseteq i(N) = N$. Thus, M_S is fully stable. ∎

3. Fully stable and quasi-injective acts

Recall that an S-act A_S is called quasi-injective [3] if for each subact B_S of A_S and any S-homomorphism $f : B_S \to A_S$ there exists an S-homomorphism $g : A_S \to A_S$ extending f. We will discuss the relation between quasi-injective and fully stable acts under the assumption that the monoid S is commutative. First, we recall some concepts needed. Given some concrete category C, an object $K \in C$ is called a *cofree* object in C if there exists $I \in Set$ and a mapping $\psi : \lfloor K \rfloor \to I$ such that the following universal property is valid: for every $X \in C$ and every mapping $\xi : \lfloor X \rfloor \to I$ there exists exactly one $\xi^* \in \mathrm{Mor}_C(X, K)$ such that the following diagram in Set is commutative:



We write $Cof(I)$ for K and say that K is I-*cofree.* The set I is called a *cobasis* for K.

For the cofree concept in S-Act, we have the following proposition. But, first, recall that $I^S = \mathrm{Hom}({}_S S_{\{1\}}, {}_{\{1\}} I_{\{1\}})$ is a right S-act and fs for $f \in I^S$, $s \in S$ is defined by $(fs)(t) = f(st)$ for every $t \in S$, $I \neq \emptyset$, see [3, Remark 1.7.20].

Proposition 3.1. [2, p.151] *Let $I \neq \emptyset$. The S-act I^S with $\psi(f) = f(1)$ for all $f \in I^S$ is an $|I|$-cofree object in Act-S.*

The next proposition shows that cofree of a fully stable act is itself a fully stable act.

Proposition 3.2. *If the S-act M_S is a fully stable act, then $(M^S)_S$ is fully stable (where S is a commutative monoid, i.e., the left S-act S is right).*

Proof. Let $f, g \in M^S$ such that $\mathcal{R}_S(g) \subseteq \mathcal{R}_S(f)$, where

$$\mathcal{R}_S(g) = \{(s,t) \in S \times S \mid gs = gt\} \text{ and } \mathcal{R}_S(f) = \{(s,t) \in S \times S \mid fs = ft\}.$$

Since M is a cobasis of M^S, there exists an S-homomorphism $\psi : M^S \to M$ such that $\psi(f) = f(1)$, for each $f \in M^S$. Hence $f(1)$, $g(1) \in M$ and

$$\mathcal{R}_S(g(1)) \subseteq \mathcal{R}_S(f(1)).$$

Since, if $(s_1, s_2) \in \mathcal{R}_S(g(1))$, then $g(1)s_1 = g(1)s_2$, hence $g(s_1) = g(s_2)$ and hence $g(s_1)(1) = g(s_2)(1)$. Now, for each $t \in S$, we have that

$$g(s_1)(t) = g(s_1)(1)t = g(s_2)(1)t = g(s_2)(t)$$

by commutativity of S, hence $(s_1, s_2) \in \mathcal{R}_S(g) \subseteq \mathcal{R}_S(f)$, so that $(s_1, s_2) \in \mathcal{R}_S(f(1))$. Thus,

$$\mathcal{R}_S(g(1)) \subseteq \mathcal{R}_S(f(1)).$$

By full stability of M, we have

$$f(1)S \subseteq g(1)S.$$

Therefore, $f \in gS$ and, by Proposition 2.4, we have that M^S is fully stable S-act. ∎

Now, we ask the following question. Is there a relation between fully stable acts and quasi-injective acts? The following theorem answers this question.

Theorem 3.3. *Every fully stable act is quasi-injective.*

Proof. Let M_S be a fully stable act. Hence, for any subact N_S of M_S and S-homomorphism $\alpha : N \to M$, we have that $\alpha(N) \subseteq N$, that is $\alpha : N \to N$.

By the injectivity of $E(M)$, the map α extends to an S-homomorphism $\beta : E(M) \to E(M)$. But $(M^S)_S$ is a cofree fully stable S-act, hence $(M^S)_S$ is injective fully stable act see Theorem 3.1.5 in [3], but $E(M)$ is a minimal injective extension of M_S, hence $E(M)$ is a subact of $(M^S)_S$ and since every subact of fully stable is fully stable [Remark 2.2.1], hence $E(M)$ is a fully stable act and then $\hat{\beta} : M \to M$ is an extension of α where $\hat{\beta} = \beta|_M$. Therefore, M a quasi-injective. ∎

Corollary 3.4. *The injective hull of fully stable act is fully stable.*

The converse of Theorem 3.3 is not true in general as in the following example.

Example 3.5. Let $S = \{0, 1\}$. Consider the S-act $A = \{\mathcal{O}, a, b, c\}$ with multiplication $0 = b0 = c0 = \mathcal{O}$. The act A_S is injective, so it is must be quasi-injective. But it is not fully stable, because $aS = \{\mathcal{O}, a\} \neq \mathcal{L}_A(\mathcal{R}_S(a)) = A$.

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