SUBORDINATION RESULTS FOR A CERTAIN SUBCLASS OF NON-BAZILEVIC ANALYTIC FUNCTIONS DEFINED BY LINEAR OPERATOR

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Abstract. In this work, by making use of the principle of subordination, we introduce a certain subclass of non-Bazilevic analytic functions defined by linear operator. Such results as subordination and superordination, sandwich theorem and inequality properties are given.

1. Introduction

Let A_s denote the class of the functions f of the form

$$f(z) = z + \sum_{n=s+1}^{\infty} a_n z^n, \quad (s \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

If $f(z)$ and $F(z)$ are analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is subordinate to $F(z)$ in $\mathbb{U}$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $|w(z)| \leq 1$ and $f(z) = F(w(z))$, denoted by $f \prec F$ or $f(z) \prec F(z)$. Furthermore, if the function $F(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence (see [10]):

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \text{ and } f(\mathbb{U}) \subset F(\mathbb{U}).$$

Let $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the first order differential subordination:

$$\varphi\left(p(z), zp'(z); z\right) \prec h(z), \tag{2}$$

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then $p(z)$ is a solution of the differential subordination (2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (2) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (2) is called the best dominant. If $p(z)$ and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and if $p(z)$ satisfies first order differential superordination:

$$h(z) \prec \varphi\left(p(z), zp'(z); z\right), \tag{3}$$

then $p(z)$ is a solution of the differential superordination (3). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (3) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (3) is called the best subordinant. For further properties of subordination and superordination, see [10] and [11].

For functions $f, g \in A_s$, where f is given by (1) and g is defined by $g(z) = z + \sum\limits_{n=s+1}^{\infty} b_n z^n$, then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) = f(z) * g(z) = z + \sum_{n=s+1}^{\infty} a_n b_n z^n.$$

For the functions $f, g \in A_s$, we define the linear operator $D^k_{\alpha,\beta,\lambda} : A_k \to A_k$ (for $k = 0, 1, 2, \ldots$), $0 < \alpha \le 1,\ 0 < \beta \le 1, \lambda \ge 0$, and $z \in \mathbb{U}$ by:

$$\begin{aligned} D^0_{\alpha,\beta,\lambda}(f * g)(z) &= (f * g)(z), \\ D^1_{\alpha,\beta,\lambda}(f * g)(z) &= D_{\alpha,\beta,\lambda}(f * g)(z) \\ &= [1 - \lambda(\alpha + \beta - 1)](f * g)(z) + z\lambda(\alpha + \beta - 1)[(f * g)(z)]' \\ &= z + \sum_{n=s+1}^{\infty} [\lambda(\alpha + \beta - 1)(n - 1) + 1]\, a_n b_n z^n, \end{aligned}$$

and (in general)

$$\begin{aligned} D^k_{\alpha,\beta,\lambda}(f * g)(z) &= D_{\alpha,\beta,\lambda}\left(D^{k-1}_{\alpha,\beta,\lambda}(f * g)(z)\right) \\ &= z + \sum_{n=s+1}^{\infty} [\lambda(\alpha + \beta - 1)(n - 1) + 1]^k\, a_n b_n z^n, \quad (\lambda \ge 0). \end{aligned} \tag{4}$$

Using (4), it is easy to verify that

$$\begin{aligned} &\lambda(\alpha + \beta - 1)z[D^k_{\alpha,\beta,\lambda}(f * g)(z)]' \\ &\quad = D^{k+1}_{\alpha,\beta,\lambda}(f * g)(z) + [1 - \lambda(\alpha + \beta - 1)]D^k_{\alpha,\beta,\lambda}(f * g)(z). \end{aligned} \tag{5}$$

Remark 1. For $b_n = C(\delta, n)$, the operator $D^k_{\alpha,\beta,\lambda}(f * g)(z)$ extends to $D^k_{\alpha,\beta,\delta,\lambda} f(z)$, where the operator $D^k_{\alpha,\beta,\delta,\lambda} f(z)$ was introduced and studied by Alamoush and Darus, which generalizes many other operators (see [1]), where

$$C(\delta, n) = \begin{pmatrix} n + \delta - 1 \\ \delta \end{pmatrix}.$$

Definition 1. A function $f \in A_s$ is said to be in the class $N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B)$ if it satisfies the following subordination condition:

$$(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz}, \tag{6}$$

where ($g \in A_s$, $\rho \in \mathbb{C}$, $0 < \mu < 1$, $-1 \le B < A \le 1$, $A \neq B$, $A \in \mathbb{R}$, and $D^k_{\alpha,\beta,\lambda}f(z)$ as defined on (4)). Here all the powers are the principal values.

Furthermore, the function $f \in N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;\varpi)$ if and only if $f,g \in A_s$ and

$$Re\left\{(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}\right\} > \varpi,$$

where ($0 \le \varpi < 1$; $z \in \mathbb{U}$).

We note that:

If $k=0$, and $b_n = 1$, then the class $N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B)$ reduces to the class $N(\rho,\mu;A,B)$ which is defined by Wang el. at [5]. If $k=0$, $\rho=-1$, $n=1$, $A=1$, $B=-1$ and $b_n=1$, then the class $N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B)$ reduces to the class of non-Bazilevic functions which introduced by Obradovic [13]. If $k=0$, $\rho=-1$, $n=1$, $A=1-2\varpi$, $B=-1$ and $b_n=1$, then the class $N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B)$ reduces to the class of non-Bazilevic functions of order $\varpi(0 \le \varpi < 1)$ which was given by Tuneski and Darus [12]. Other works related to non-Bazilevic can be found in ([2]-[7]).

In the present paper, we discuss and prove the subordination and superordination properties, sandwich theorem and inequality properties for the class $N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B)$.

2. Preliminary results

In order to establish our main results, we need the following definition and lemmas.

Definition 2. [9]. Denote by Q the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} \setminus E(f)$, where

$$E(f) = \left\{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} f(z) = \infty\right\},$$

and such that $f'(\zeta) \neq 0$ for $\zeta \in \overline{\mathbb{U}} \setminus E(f)$

Lemma 1. [10] *Let the function $h(z)$ be analytic and convex (univalent) in $\mathbb{U}$ with $h(0)=1$. Suppose also that the function $g(z)$ given by*

$$g(z) = 1 + c_k z^k + c_{k+1}z^{k+1} + ... \tag{7}$$

is analytic in $\mathbb{U}$. *If*

$$g(z)+\frac{zg'(z)}{\gamma}\prec h(z),\quad (Re(\gamma)>0;\gamma\neq 0;z\in\mathbb{U}),\tag{8}$$

then

$$g(z)\prec q(z)=\frac{\gamma}{k}z^{-\frac{\gamma}{k}}\int h(t)t^{\frac{\gamma}{k}-1}dt\prec h(z),$$

and $q(z)$ *is the best dominant of* (8).

Lemma 2. [8] *Let* $q(z)$ *be a convex univalent function in* $\mathbb{U}$ *and let* $\sigma\in\mathbb{C}$, $\eta\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ *with*

$$Re\left(1+\frac{zq''(z)}{q(z)}\right)>\max\left\{0,-Re\left(\frac{\sigma}{\eta}\right)\right\}.$$

If the function $g(z)$ *is analytic in* $\mathbb{U}$ *and*

$$\sigma g(z)+\eta zg'(z)\prec\sigma q(z)+\eta zq'(z),$$

then $g(z)\prec q(z)$ *and* $q(z)$ *is the best dominant.*

Lemma 3. [9] *Let* $q(z)$ *be a convex univalent function in* $\mathbb{U}$ *and let* $k\in\mathbb{C}$. *Further assume that* $Re(k)>0$. *If*

$$g(z)\in H[q(0),1]\cap Q,$$

and

$$g(z)+kzg'(z)$$

is univalent in $\mathbb{U}$, *then*

$$q(z)+kzq'(z)\prec g(z)+kzg'(z),$$

implies $q(z)\prec g(z)$ *and* $q(z)$ *and* q *is the best subordinant.*

Lemma 4. [14] *Let* F *be analytic and convex in* $\mathbb{U}$. *If*

$$f,g\in A\quad\text{and}\quad f,g\prec F$$

then

$$\lambda f+(1-\lambda)g\prec F,\ (0\leq\lambda\leq 1).$$

Lemma 5. [15] *Let*

$$f(z)=1+\sum_{n=1}^{\infty}a_nz^n$$

be analytic in U *and*

$$g(z)=1+\sum_{n=1}^{\infty}b_nz^n$$

be analytic and convex in U. *If* $f(z)\prec g(z)$, *then*

$$|a_n|<|b_1|\quad(n\in\mathbb{N}).$$

3. Main results

We begin by presenting our first subordination property given by Theorem 1.

Theorem 1. *For $g \in A_s$, $\rho \in \mathbb{C}$, $0 < \mu < 1$, $-1 \leq B < A \leq 1$, $A \neq B$, $A \in \mathbb{R}$, and $D^k_{\alpha,\beta,\lambda}(f*g)$ as defined by (4). Let $f(z) \in N^k_{\alpha,\beta,\lambda}(g, \rho, \mu; A, B)$ with $Re(\rho) > 0$. Then*

$$\begin{aligned}(9)\qquad &\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \prec q(z)\\ &= \frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1+Azu}{1+Bzu}du \prec \frac{1+Az}{1+Bz}\end{aligned}$$

and $q(z)$ is the best dominant.

Proof. Define the function $g(z)$ by

$$g(z) = \left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \quad (z \in \mathbb{U}). \tag{10}$$

Then $g(z)$ is of the form (7) and analytic in $\mathbb{U}$ with $g(0) = 1$. Taking logarithmic differentiation of (10) with respect to z and using (5), we deduce that

$$\begin{aligned}(11)\qquad &(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}\\ &= g(z) + \frac{\lambda(\alpha+\beta-1)\rho}{\mu}zg'(z).\end{aligned}$$

Since $f(z) \in N^k_{\alpha,\beta,\lambda}(g, \rho, \mu; A, B)$, we have

$$g(z) + \frac{\lambda(\alpha+\beta-1)\rho}{\mu}zg'(z) \prec \frac{1+Az}{1+Bz}.$$

Applying Lemma 1 to (11) with $\gamma = \dfrac{\mu}{\lambda(\alpha+\beta-1)\rho}$, we get

$$\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \prec q(z) = \frac{\mu}{\lambda(\alpha+\beta-1)s\rho}z^{-\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}}\int_z^1 t^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1+At}{1+Bt}dt$$

$$= \frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1+Azu}{1+Bzu}du \prec \frac{1+Az}{1+Bz}, \tag{12}$$

and $q(z)$ is the best dominant. The proof of Theorem 1 is thus complete. ∎

Theorem 2. *Let $q(z)$ be univalent in $\mathbb{U}$, $\rho \in \mathbb{C}^*$. Suppose also that $q(z)$ satisfies the following inequality:*

$$Re\left(1+\frac{zq''(z)}{q(z)}\right) > \max\left\{0, -Re\left(\frac{\mu}{\lambda(\alpha+\beta-1)\rho}\right)\right\}. \tag{13}$$

If $f \in A_s$ satisfies the following subordination condition:

$$\begin{aligned}(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} &-\rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}\\ &\prec q(z)+\frac{\lambda(\alpha+\beta-1)\rho}{\mu}zq'(z),\end{aligned} \tag{14}$$

then

$$\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Let the function $g(z)$ be defined by (10). We know that (11) holds true. Combining (11) and (14), we find that

$$g(z)+\frac{\lambda(\alpha+\beta-1)\rho}{\mu}zg'(z) \prec q(z)+\frac{\lambda(\alpha+\beta-1)\rho}{\mu}zq'(z). \tag{15}$$

By using Lemma 2 and (15), we easily get the assertion of Theorem 2. ∎

Taking $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 2, we get the following result.

Corollary 1. *Let $\rho \in \mathbb{C}$ and $-1 \le B < A \le 1$. Suppose also that*

$$Re\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -Re\left(\frac{\mu}{\lambda(\alpha+\beta-1)\rho}\right)\right\}.$$

If $f \in A_s$ satisfies the following subordination:

$$(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}$$

$$\prec \frac{1+Az}{1+Bz}+\frac{\lambda(\alpha+\beta-1)\rho}{\mu}\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz},$$

and the function $\frac{1+Az}{1+Bz}$ is the best dominant.

Now, by making use of Lemma 3, we now derive the following superordination result.

Theorem 3. *Let $q(z)$ be convex univalent in $\mathbb{U}$, $\rho \in \mathbb{C}$ with $Re(\rho) > 0$. Also let*

$$\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \in H[q(0),1] \cap Q$$

and

$$(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}$$

be univalent in $\mathbb{U}$. If $f \in A_s$ satisfies the following superordination:

$$q(z) + \frac{\lambda(\alpha+\beta-1)\rho}{\mu} zq'(z)$$

$$\prec (1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\delta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu},$$

then

$$q(z) \prec \left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}$$

and the function $q(z)$ is the best subordinant.

Proof. Let the function $g(z)$ be defined by (10). Then

$$q(z) + \frac{\lambda(\alpha+\beta-1)\rho}{\mu} zq'(z) \prec (1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}$$

$$-\rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} = g(z) + \frac{\lambda(\alpha+\beta-1)\rho}{\mu} zg'(z).$$

An application of Lemma 3 yields the assertion of Theorem 3.

Taking $q(z) = \dfrac{1+Az}{1+Bz}$ in Theorem 3, we get the following result.

Corollary 2. *Let $\rho \in \mathbb{C}$ and $-1 \le B < A \le 1$ with $Re(\rho) > 0$. Suppose also that*

$$\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \in H[q(0),1] \cap Q,$$

and

$$(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}$$

be univalent in $\mathbb{U}$. If $f \in A_s$ satisfies the following superordination:

$$\frac{1+Az}{1+Bz} + \frac{\lambda(\alpha+\beta-1)\rho}{\mu}\frac{(A-B)z}{(1+Bz)^2}$$

$$\prec (1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu},$$

then

$$\frac{1+Az}{1+Bz} \prec \left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}$$

and the function $\dfrac{1+Az}{1+Bz}$ *is the best subordinant.*

Combining Theorems 2 and 3, we easily get the following "Sandwich-type result".

Theorem 4. *Let q_1 be convex univalent and let q_2 be univalent in $\mathbb{U}$, $\rho \in \mathbb{C}$ with $Re(\rho) > 0$. Let q_2 satisfies* (13). *If*

$$\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \in H[q_1(0),1] \cap Q$$

and

$$(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}$$

be univalent in $\mathbb{U}$, also

$$q_1(z) + \frac{\lambda(\alpha+\beta-1)\rho}{\mu}zq_1'(z) \prec (1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}$$

$$-\rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} = q_2(z) + \frac{\lambda(\alpha+\beta-1)\rho}{\mu}zq_2'(z),$$

then

$$q_1(z) \prec \left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \prec q_2(z).$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and dominant.

Next, we consider the following:

Theorem 5. *If $\rho > 0$ and $f \in N^k_{\alpha,\beta,\lambda}(g,\mu,\varpi)$ $(0 \le \varpi < 1)$, then $f \in N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;\varpi)$ for $|z| < R$, where*

$$R = \left(\sqrt{\left(\frac{\lambda(\alpha+\beta-1)s\rho}{\mu}\right)^2 + 1} - \frac{\lambda(\alpha+\beta-1)s\rho}{\mu}\right)^{\frac{1}{s}}. \tag{16}$$

The bound R is the best possible.

Proof. We begin by writing

$$\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} = \varpi + (1-\varpi)g(z) \quad (z \in \mathbb{U},\ 0 \le \varpi < 1). \tag{17}$$

Then, clearly, the function $g(z)$ is of the form (7), is analytic and has a positive real part in $\mathbb{U}$. By taking the derivatives of both sides of (17), we get

$$\frac{1}{1-\varpi}\left\{(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \varpi\right\}$$

$$= g(z) + \frac{\lambda(\alpha+\beta-1)\rho}{\mu}zg'(z). \tag{18}$$

By making use of the following well-known estimate (see [16], Theorem 1):

$$\frac{|zg'(z)|}{Re\,\{g(z)\}} \le \frac{2sr^s}{1-2r^{2s}} \quad (|z| = r < 1)$$

in (18), we obtain

$$Re\left(\frac{1}{1-\varpi}\left\{(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \varpi\right\}\right)$$

$$\ge Re\,\{g(z)\}\left(1 - \frac{2\lambda(\alpha+\beta-1)\rho s r^s}{\mu(1-r^{2s})}\right). \tag{19}$$

It is seen that the right-hand side of (19) is positive, provided that $r < R$, where R is given by (16). In order to show that the bound R is the best possible, we consider the function $f(z) \in A_s$ defined by

$$\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} = \varpi + (1-\varpi)\frac{1+z^s}{1-z^s} \ (z \in U,\ 0 \le \varpi < 1).$$

Noting that

$$\frac{1}{1-\varpi}\left\{(1+\rho)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \varpi\right\}$$

$$= \frac{1+z^s}{1-z^s} + \frac{2\lambda(\alpha+\beta-1)\rho s z^s}{\mu(1-z^s)^2} = 0$$

for $|z| = R$, we conclude that the bound is the best possible. Theorem 5 is thus proved.

Now, we give the inclusion properties:

Theorem 6. *Let $\rho_2 \geq \rho_1 \geq 0$ and $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$. Then*

$$N^k_{\alpha,\beta,\lambda}(g, \rho_2, \mu; A_2, B_2) \subset N^k_{\alpha,\beta,\lambda}(g, \rho_1, \mu; A_1, B_1). \tag{20}$$

Proof. Let $f \in N^k_{\alpha,\beta,\lambda}(g, \rho_2, \mu; A_2, B_2)$. Then we have

$$(1+\rho_2)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho_2 \frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \prec \frac{1+A_2z}{1+B_2z}.$$

Since $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, we easily find that

$$\begin{aligned}(1+\rho_2)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} &- \rho_2 \frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \\ &\prec \frac{1+A_2z}{1+B_2z} \prec \frac{1+A_1z}{1+B_1z},\end{aligned} \tag{21}$$

that is $f \in N^k_{\alpha,\beta,\lambda}(g, \rho_2, \mu; A_1, B_1)$. Thus the assertion of Theorem 6 holds for $\rho_2 = \rho_1 \geq 0$. If $\rho_2 > \rho_1 \geq 0$, by Theorem 1 and (21), we know that $f \in N^k_{\alpha,\beta,\lambda}(g, 0, \mu; A_1, B_1)$, that is,

$$\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \prec \frac{1+A_1z}{1+B_1z}. \tag{22}$$

At the same time, we have

$$\begin{aligned}&(1+\rho_1)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho_1 \frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \\ &= \left(1-\frac{\rho_1}{\rho_2}\right)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \\ &+\frac{\rho_1}{\rho_2}\left[(1+\rho_2)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho_2 \frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\delta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}\right].\end{aligned} \tag{23}$$

Moreover, since $0 \leq \frac{\rho_1}{\rho_2} < 1$ and the function $\frac{1+A_1z}{1+B_1z}$ $(-1 \leq B_1 < A_1 \leq 1)$ is analytic and convex in $\mathbb{U}$. Combining (21)-(23) and Lemma 4, we find that

$$(1+\rho_1)\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho_1 \frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} \prec \frac{1+A_1z}{1+B_1z},$$

that is $f \in N^k_{\alpha,\beta,\lambda}(g, \rho_1, \mu; A_1, B_1)$, which implies that the assertion (20) of Theorem 6 holds.

Theorem 7. *Let $f \in N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B)$ with $\rho > 0$ and $-1 \le B < A \le 1$. Then*

$$\begin{aligned} &\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1-Au}{1-Bu}du \\ (24)\qquad &< \Re\left\{\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}\right\} < \frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1+Au}{1+Bu}du. \end{aligned}$$

The extremal function of (24) *is defined by*

$$\begin{aligned} F(z) &= D^k_{\alpha,\beta,\lambda}(f*g)(z) \\ (25)\qquad &= z\left(\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1+Auz^s}{1+Buz^s}du\right)^{\frac{-1}{\mu}}. \end{aligned}$$

Proof. Let $f \in N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B)$ with $\rho > 0$. From Theorem 1, we know that (9) holds, which implies that

$$\begin{aligned} &\Re\left\{\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}\right\} \\ &< \sup_{z\in U}\Re\left\{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1+Auz}{1+Buz}du\right\} \\ (26)\qquad &\le \frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\sup_{z\in U}\left(\frac{1+Auz}{1+Buz}\right)du \\ &< \frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1+Au}{1+Bu}du \end{aligned}$$

and

$$\begin{aligned} &\Re\left\{\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}\right\} \\ &> \inf_{z\in U}\Re\left\{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1+Auz}{1+Buz}du\right\} \\ (27)\qquad &\ge \frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\inf_{z\in U}\left(\frac{1+Auz}{1+Buz}\right)du \\ &> \frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1-Au}{1-Bu}du. \end{aligned}$$

Combining (26) and (27), we get (24). By noting that the function $F(z)$ defined by (25) belongs to the class $N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B)$, we obtain that equality (24) is sharp. The proof of Theorem 7 is evidently complete. ■

Similarly, by applying the method of proof of Theorem 7, we easily get the following result.

Corollary 3. *Let* $f \in N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B)$ *with* $\rho > 0$ *and* $-1 \leq B < A \leq 1$. *Then*

$$\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1+Au}{1+Bu}du$$

$$< \Re\left\{\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}\right\} \tag{28}$$

$$< \frac{\mu}{\lambda(\alpha+\beta-1)s\rho}\int_0^1 u^{\frac{\mu}{\lambda(\alpha+\beta-1)s\rho}-1}\frac{1-Au}{1-Bu}du.$$

The extremal function of (28) *is defined by* (25).

Theorem 8. *Let*

$$f(z) = z + \sum_{n=s+1}^{\infty} a_n z^n \in N^k_{\alpha,\beta,\lambda}(g,\rho,\mu;A,B), \quad (s \in \mathbb{N} = \{1,2,3,...\}). \tag{29}$$

Then

$$|a_{s+1}| \leq [\lambda(\alpha+\beta-1)+1]^{-k}\,\frac{(A-B)}{|\mu+\lambda(\alpha+\beta-1)\rho||b_{s+1}|}. \tag{30}$$

The inequality (30) *is sharp, with the extremal function defined by* (25).

Proof. Combining (6) and (29), we obtain

$$\begin{aligned}(1+\rho)&\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu} - \rho\frac{D^{k+1}_{\alpha,\beta,\lambda}(f*g)(z)}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\left(\frac{z}{D^k_{\alpha,\beta,\lambda}(f*g)(z)}\right)^{\mu}\\ &= 1+(\mu+\lambda(\alpha+\beta-1)\rho)\left[\lambda(\alpha+\beta-1)+1\right]^k a_{s+1}b_{s+1}z+... \prec \frac{1+Az}{1+Bz}.\end{aligned} \tag{31}$$

An application of Lemma 5 to (31) yields

$$\left|(\mu+\lambda(\alpha+\beta-1)\rho)\left[\lambda(\alpha+\beta-1)+1\right]^k a_{s+1}b_{s+1}\right| < A-B. \tag{32}$$

Thus, from (32), we easily arrive at (30) asserted by Theorem 8. ■

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