(i, j)-ω-b-OPEN SETS AND (i, j)-ω-b-CONTINUITY IN BITOPOLOGICAL SPACES

Carlos Carpintero

Department of Mathematics
Universidad de Oriente
Núcleo de Sucre, Cumaná
Venezuela
e-mail: carpintero.carlos@gmail.com

Sabir Hussain

Department of Mathematics
Islamia University Bahawalpur
Pakistan
and
Department of Mathematics
College of Science
Qassim University
P.O.BOX 6644, Buraydah 51482
Saudi Arabia
e-mail: sabiriub@yahoo.com

Ennis Rosas

Department of Mathematics
Universidad de Oriente
Núcleo de Sucre, Cumaná
Venezuela
e-mail: ennisrafael@gmail.com

Abstract. As a generalization of (i, j)-b-open sets in bitopological spaces, we introduce and explore the notions of (i, j)-ω-b-open sets. We also develop its relationship with already defined generalizations of b-open sets. Moreover we define and discuss the properties of (i, j)-ω-b-continuous functions.

Keywords: bitopological spaces, (i, j)-ω-semiopen sets, (i, j)-ω-semiclosed sets.

AMS(MOS) Subject Classification Codes [2010]: 54A05,54C05,54C08.

1. Introduction

In [5], Kelly initiated the study of bitopological spaces. Thereafter a lot of work have been done to generalize the topological concepts to bitopological setting.

Andrejevic [2] introduced the concept of b- open sets and extended this notions to a bitopological spaces. Recently in [4], Hdeib introduced the notions ω-closed set as generalization of closed sets. A point $x \in X$ is called a condensation point of A, if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is said to be ω-closed [4], if it contains all its condensation points. The complement of a ω-closed set is said to be ω-open. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The set of all ω-open sets in (X, τ) is denoted by τ_ω, τ_ω is a topology on X finer than τ. In this paper, as a generalization of (i, j)-b-open sets in bitopological spaces, we introduce and explore the notions of (i, j)-ω-b open sets. We also develop its relationship with already defined generalizations of b-open sets. Moreover we define and discuss the properties of (i, j)-ω-b-continuous functions. For a subset A of X, the closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-b- open, if $A \subseteq \tau_i\text{-}cl(\tau_j\text{-}Int(A)) \cup \tau_i\text{-}Int(\tau_j\text{-}Cl(A))$, where $i \neq j$, $i, j = 1, 2$. The complement of a (i, j)-b-open set is said to be a (i, j)-b-closed. The (i, j)-b-closure of A, denoted by (i, j)-b-$cl(A)$ is defined to be the intersection of all (i, j)-b-closed sets containing A. The (i, j)-b-interior of A, denoted by (i, j)-b-$Int(A)$ is defined to be the union of all (i, j)-b-open sets contained in A. The family of all (i, j)-b-open (respectively (i, j)-b-closed) subsets of a space (X, τ_1, τ_2) is denoted by (i, j)-$BO(X)$, (respectively (i, j)-$BC(X)$). A function $f : (X, \tau_1, \tau_2) \mapsto (Y, \sigma_1, \sigma_2)$ is said to be (i, j)-b- continuous, if the inverse image of every σ_i-open set in (Y, σ_1, σ_2) is (i, j)-b-open in (X, τ_1, τ_2), where $i \neq j$, $i, j = 1, 2$. Observe that a σ_i-open set U in (Y, σ_1, σ_2) means $U \in \sigma_i$.

2. (i, j)-ω-b-open sets

A set X equipped with two topologies is called a bitopological space. Throughout this paper, spaces (X, τ_1, τ_2) (or simply X) always means a bitopological spaces on which no separation axioms are assumed unless explicitly stated.

Definition 1 A subset A of a bitopological space X is (i, j)-ω-b-open, if for each $x \in A$ there exists a (i, j)-b- open subset U_x containing x such that $U_x - A$ is a countable set. The complement of a (i, j)-ω-b-open is said to be (i, j)-ω-b-closed set.

The family of all (i, j)-ω-b-open (respectively (i, j)-ω-b-closed) subsets of a space (X, τ_1, τ_2) is denoted by (i, j)-ω-$BO(X)$, (respectively (i, j)-ω-$BC(X)$). Also the family of all (i, j)-ω-b-open sets of (X, τ_1, τ_2) containing x is denoted by (i, j)-ω-$BO(X, x)$.

Example 2.1 Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{b, c\}, X\}$. Then $\{a\}$ is a (i, j)-ω-b-open but not (i, j)-b-open.

Example 2.2 Let X be the real line, $\tau_1 = \tau_2 =$ the usual topology. Q is (1,2)-ω-b-open set but not either 12-bω-open neither 12-ωb-open, see [6], for the definition of ij-bω-open and ij-ωb-open.

Example 2.3 Let $X = A \cup B \cup C \cup D$, where A, B, C, D are disjoint uncountable sets, $\tau_1 = \tau_2 = \{\emptyset, A, B, A \cup B, A \cup B \cup C, X\}$. Then $A \cup C$ is a (i,j)-ω-b-open but not (i,j)-ωb-open set.

It is well known that every semiopen (respectively preopen) set is a b-open set, in consequence, every (i,j)-ω-semiopen (respectively (i,j)-ω-propen) set is an (i,j)-ω-b-open set and therefore the results obtained in this article generalize the results obtained in [3] (respectively [8]).

Remark 2.4 It is easy to see in Example 2.3, the set $A \cup C$ is (i,j)-ω-b-open but is not (i,j)-ω-preopen set.

Theorem 2.5 *Let A be a subset of a bitopological space X. A is an (i,j)-ω-b-open if and only if for every $x \in A$, there exists a (i,j)-b-open set U_x containing x and a countable subset C such that $U_x - C \subseteq A$.*

Proof. Let A be an (i,j)-ω-b-open set and $x \in A$, then by Definition 1, there exists an (i,j)-b-open subset U_x containing x such that $U_x - A$ is countable. Let $C = U_x - A = U_x \cap (X - A)$. Then $U_x - C \subseteq A$. Conversely, let $x \in A$. Then by hypothesis, there exists a (i,j)-b-open subset U_x containing x and a countable subset C such that $U_x - C \subseteq A$. Thus $U_x - A \subseteq C$ and $U_x - A$ is countable and the result follows. ∎

Theorem 2.6 *Let C be a subset of a bitopological space X. If C is an (i,j)-ω-b-closed set, then $C \subseteq K \cup B$, for some (i,j)-b-closed subset K and a countable subset B.*

Proof. If C is a (i,j)-ω-b-closed set, then its complement $X - C$ is a (i,j)-ω-b-open set and therefore by Theorem 2.5, for every $x \in X - C$, there exists a (i,j)-b-open set U containing x and a countable set B such that $U - B \subseteq X - C$. Thus $C \subseteq X - (U - B) = X - (U \cap (X - B)) = (X - U) \cup B$, let $K = X - U$. Follows that $C \subseteq K \cup B$ and K is an (i,j)-b-closed set. ∎

Theorem 2.7 *The union of any family of (i,j)-ω-b-open sets is an (i,j)-ω-b-open set.*

Proof. Let $\{A_\alpha : \alpha \in I\}$ be a collection of (i,j)-ω-b-open subsets of X, then for every $x \in \bigcup_{\alpha \in I} A_\alpha$, $x \in A_\alpha$, for some $\alpha \in I$. Hence, using Definition 1, there exists a (i,j)-b-open subset U containing x, such that $U - A_\alpha$ is countable. Now as $U - (\bigcup_{\alpha \in I} A_\alpha) \subseteq U - A_\alpha$, it follows that $U - (\bigcup_{\alpha \in I} A_\alpha)$ is countable. In consequence, $\bigcup_{\alpha \in I} A_\alpha$ is an (i,j)-ω-b-open set. ∎

Definition 2 Let A be a subset of a bitopological space X, the union of all (i,j)-ω-b-open sets contained in A is called the (i,j)-ω-b-interior of A and is denoted by $(i,j) - \omega$-b-Int(A). The intersection of all (i,j)-ω-b-closed sets of X containing A is called the (i,j)-ω-b-closure of A and is denoted by (i,j)-$\omega$$-b-Cl(A)$.

Remark 2.8 By Theorem 2.7, The (i,j)-ω-b-$Cl(A)$ is a (i,j)-ω-b-closed set and the (i,j)-ω-$b - Int(A)$ is a (i,j)-ω-b-open set.

Theorem 2.9 *Let X be a bitopological space and $A, B \subseteq X$. Then the following properties hold:*

(1) (i,j)-ω-b-$Int((i,j)$-ω-b-$Int(A)) = (i,j)$-ω-b-$Int(A)$.

(2) *If* $A \subseteq B$, *then* (i,j)-ω-b-$Int(A) \subseteq (i,j)$-ω-b-$Int(B)$.

(3) (i,j)-ω-b-$Int(A \cap B) \subseteq (i,j)$-$\omega$-$b$-$Int(A) \cap (i,j)$-$\omega$-$b$-$Int(B)$.

(4) (i,j)-ω-b-$Int(A) \cup (i,j)$-ω-b-$Int(B) \subseteq (i,j)$-ω-b-$Int(A \cup B)$.

(5) (i,j)-ω-b-$Int(A)$ *is the largest* (i,j)-ω-b-*open subset of* X *contained in* A.

(6) A *is* (i,j)-ω-b-*open if and only if* $A = (i,j)$-ω-b-$Int(A)$.

(7) (i,j)-ω-b-$Cl((i,j)$-ω-b-$Cl(A)) = (i,j)$-ω-b-$Cl(A)$.

(8) *If* $A \subseteq B$, *then* (i,j)-ω-b-$Cl(A) \subseteq (i,j)$-ω-b-$Cl(B)$.

(9) (i,j)-ω-b-$Cl(A) \cup (i,j)$-ω-b-$Cl(B) \subseteq (i,j)$-ω-b-$Cl(A \cup B)$.

(10) (i,j)-ω-b-$Cl(A \cap B) \subseteq (i,j)$-$\omega$-$b$-$Cl(A) \cap (i,j)$-$\omega$-$b$-$Cl(B)$.

Proof. (1), (2), (6), (7) and (8) follow directly from the definition 1 of (i,j)-ω-b-open and (i,j)-ω-b-closed sets. (3), (4) and (5) follow from (2). (9) and (10) follow by applying (8). ■

Example 2.10 Let X be the real line, $\tau_1 = \{\emptyset, \Re, Q^c\}$ and $\tau_2 = \{\emptyset, \Re, Q, Q^c\}$. Take $A = (0,1)$, $B = (1,2)$, then (i,j)-ω-b-$Cl(A \cap B) \subset (i,j)$-$\omega$-$b$-$Cl(A) \cap (i,j)$-$\omega$-$b$-$Cl(B)$.

Example 2.11 Let X be the real line, $\tau_1 = \{\emptyset, \Re, Q\}$ and $\tau_2 = \{\emptyset, \Re, Q\}$. The collection of (i,j)-$BO(X)$ is $\{\emptyset, \Re, Q\}$. take $A = Q$, $B = \{\pi\}$. Then (i,j)-ω-b-$Cl(A) = Q$, (i,j)-ω-b-$Cl(B) = \{\pi\}$ and (i,j)-ω-b-$Cl(A) \cup (i,j)$-ω-b-$Cl(B) \subset (i,j)$-ω-b-$Cl(A \cup B)$.

Remark 2.12 Observe that the collection (i,j)-ω-$BO(X)$ forms a minimal structure.

The following theorem give a characterization of the (i,j)-ω-b-closure of a set.

Theorem 2.13 *Let A be a subset of a bitopological space X and $x \in X$. Then $x \in (i,j)$-ω-b-$Cl(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in (i,j)$-ω-$BO(X,x)$.*

Proof. Suppose that $x \in (i,j)$-ω-b-$Cl(A)$ and we show that $U \cap A \neq \emptyset$, for all $U \in (i,j)$-ω-$BO(X,x)$. Suppose on the contrary that there exists $U \in (i,j)$-ω-$BO(X,x)$ such that $U \cap A = \emptyset$, then $A \subseteq X - U$ and $X - U$ is a (i,j)-ω-b-closed set. This follows that (i,j)-ω-b-$Cl(A) \subseteq (i,j)$-ω-b-$Cl(X - U) = X - U$. Since $x \in (i,j)$-ω-b-$Cl(A)$, we have $x \in X - U$ and hence $x \notin U$. Which contradicts the fact that $x \in U$. Therefore, $U \cap A \neq \emptyset$.

Conversely, suppose on the contrary that $U \cap A \neq \emptyset$ for every $U \in (i,j)$-ω-$BO(X,x)$. We shall prove that $x \in (i,j)$-ω-b-$Cl(A)$. Suppose that $x \notin (i,j)$-ω-b-$Cl(A)$, let $U = X - (i,j)$-ω-b-$Cl(A)$, then $U \in (i,j)$-ω-$BO(X,x)$ and $U \cap A = (X - ((i,j)$-ω-b-$Cl(A))) \cap A \subseteq (X - A) \cap A = \emptyset$. This is a contradiction to the fact that $U \cap A \neq \emptyset$. Hence $x \in (i,j)$-ω-b-$Cl(A)$. ∎

The following theorem give the duality between the (i,j)-ω-b-closure and the (i,j)-ω-b-interior of a set.

Theorem 2.14 *Let A be a subset of a bitopological space X. The following property holds:*

(1) (i,j)-ω-b-$Cl(X\backslash A) = X\backslash(i,j)$-$\omega$-$b$-$Int(A)$.

Proof. (1). Let $x \in X\backslash(i,j)$-ω-b-$Cl(A)$. Then by Teorem 2.13, there exists $V \in (i,j)$-ω-$BO(X,x)$ such that $V \cap A = \emptyset$ and hence we obtain $x \in (i,j)$-ω-b-$Int(X\backslash A)$. This shows that $X\backslash(i,j)$-ω-b-$Cl(A) \subset (i,j)$-ω-b-$Int(X\backslash A)$. Now consider $x \in (i,j)$-ω-b-$Int(X\backslash A)$. Since (i,j)-ω-b-$Int(X\backslash A) \cap A = \emptyset$, we obtain $x \notin (i,j)$-ω-b-$Cl(A)$. Therefore, we have, (i,j)-ω-b-$Cl(X\backslash A) = X\backslash(i,j)$-$\omega$-$b$-$Int(A)$. ∎

Definition 3 Let A be a subset of a bitopological space X. A is said an (i,j)-ω-b-neighborhood of a point $x \in X$ if there exists an (i,j)-ω-b-open set W such that $x \in W \subset A$.

Theorem 2.15 *Let A be a subset of a bitopological space X. A is an (i,j)-ω-b-open set if and only if it is a (i,j)-ω-b-neighborhood of each of its points.*

Proof. Let A be an (i,j)-ω-b-open set of X. Then by definition 3, A is an (i,j)-ω-b-neighborhood of each of its points. Conversely, suppose that A is an (i,j)-ω-b-neighborhood of each of its points. Then for each $x \in A$, there exists $S_x \in (i,j)$-ω-$BO(X,x)$ such that $S_x \subset A$. Then $A = \bigcup\{S_x : x \in A\}$. Since each S_x is an (i,j)-ω-b-open, using Theorem 2.7, A is an (i,j)-ω-b-open in X. ∎

Theorem 2.16 *Let X be a bitopological space. If each nonempty (i,j)-ω-b-open set of X is uncountable, then (i,j)-b-$Cl(A) = (i,j)$-ω-b-$Cl(A)$, for each subset $A \in \tau_1 \cap \tau_2$.*

Proof. Always, (i,j)-ω-$b-Cl(A) \subseteq (i,j)$-$b-Cl(A)$. Conversely, let $x \in (i,j)$-$b-Cl(A)$ and B an (i,j)-ω-b-open set containing x. Using Theorem 2.5, there exists an (i,j)-b-open set V containing x and a countable set C such that $V - C \subseteq B$. Follows that $(V - C) \cap A \subseteq B \cap A$ and so $(V \cap A) - C \subseteq B \cap A$. Now $x \in V$, $x \in (i,j)$-$b-Cl(A)$ such that $V \cap A \neq \emptyset$ where $V \cap A$ is a (i,j)-ω-b-open set, since V is a (i,j)-b-open set and $A \in \tau_1 \cap \tau_2$. Using the hypothesis, each nonempty (i,j)-ω-b-open set of X is uncountable and so is $(V \cap A)\backslash C$. Thus $B \cap A$ is uncountable. Therefore, $B \cap A \neq \emptyset$ implies that $x \in (i,j)$-ω-$b - Cl(A)$. ∎

The following theorem give under some conditions, the collection (i,j)-ω-$BO(X))$ is a topology.

Theorem 2.17 *Let X be a bitopological space. If every (i,j)-b-open subset of X is τ_i-open in X. Then $(X,(i,j)$-ω-$BO(X))$ is a topological space.*

Proof. 1. $\emptyset, X$ belong to (i,j)-ω-$BO(X)$.

2. Let U, V be element of (i,j)-ω-$BO(X)$ and suppose that $x \in U \cap V$. Then by Definition 1, there exist (i,j)-b-open sets G, H in X containing x such that $G\backslash U$ and $H\backslash V$ are countable. Since $(G\cap H)\backslash(U\cap V) = (G\cap H)\cap((X\backslash U)\cup(X\backslash V)) \subseteq (G\cap(X\backslash U))\cup(H\cap(X\backslash V))$ implies that $(G\cap H)\backslash(U\cap V)$ is a countable set and by hypothesis, the intersection of two (i,j)-b-open set is (i,j)-b-open. Hence $U\cap V \in (i,j)$-ω-$BO(X))$.

3. The union follows directly from Theorem 2.7. ■

The following example shows that the converse of the Theorem 2.17 not necessarily is true.

Example 2.18 In the Example 2.1, the collection of $(1,2)$-ω-$BO(X) = P(X)$, in consequence, is a topology on X, but the set $\{a\}$ is $(1,2)$-b-open and $\{a\} \notin \tau_1$

3. (i,j)-ω-b-continuous functions

Definition 4 Let (X,τ_1,τ_2) and (Y,σ_1,σ_2) be bitopological spaces. A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is:

(1) (i,j)-ω-b-continuous, if the inverse image of every σ_i-open set of Y is (i,j)-ω-b-open in (X,τ_1,τ_2), where $i\neq j$, i,j=1, 2.

(2) (i,j)-b-continuous, if the inverse image of every σ_i-open set of Y is (i,j)-b-open in (X,τ_1,τ_2), where $i\neq j$, i,j=1, 2.

Theorem 3.1 *Every (i,j)-b-continuous function is (i,j)-ω-b-continuous.*

Proof. The proof follows from the fact that every (i,j)-b-open set is (i,j)-ω-b-open. ■

However, the converse may be not true.

Example 3.2 Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$, $\tau_2 = \{\emptyset,\{a\},X\}$, $\sigma_1 = \{\emptyset,\{a,b\},X\}$, $\sigma_2 = \{\emptyset,\{a,c\},X\}$. Then the identity function $f:(X,\tau_1,\tau_2)\to(X,\sigma_1,\sigma_2)$ is (i,j)-ω-b-continuous but not (i,j)-b-continuous.

Remark 3.3 Since every (i,j)-ω-preopen set is (i,j)-ω-b-open, then every (i,j)-ω-precontinuous function [8] is (i,j)-ω-b-continuous but not conversely.

Theorem 3.4 *Let (X,τ_1,τ_2) and (Y,σ_1,σ_2) be a bitopological spaces and $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ a function, the following statements are equivalent:*

(1) *f is (i,j)-ω-b-continuous;*

(2) *For each point $x \in X$ and each σ_i-open set F in Y such that $f(x) \in F$, there is a (i,j)-ω-b-open set A in X such that $x \in A$, and $f(A) \subset F$;*

(3) *The inverse image of each σ_i-closed set in Y is a (i,j)-ω-b-closed in X;*

(4) $f((i,j)\text{-}\omega\text{-}b - Cl(A)) \subseteq \sigma_i\text{-}cl(f(A))$, *for every $A \subseteq X$;*

(5) $(i,j)\text{-}\omega\text{-}b\text{-}Cl(f^{-1}(B)) \subseteq f^{-1}(\sigma_i\text{-}cl(B))$, *for every $B \subseteq Y$;*

(6) $f^{-1}(\sigma_i\text{-}Int(C)) \subseteq (i,j)\text{-}\omega\text{-}b\text{-}Int(f^{-1}(C))$, *for every $C \subseteq Y$.*

Proof. (1)⇒(2): Let $x \in X$ and F be a σ_i-open set of Y containing $f(x)$. By (1), $f^{-1}(F)$ is (i,j)-ω-b-open in X. Let $A = f^{-1}(F)$. Then $x \in A$ and $f(A) \subset F$.

(2)⇒(1): Let F be σ_i-open in Y and let $x \in f^{-1}(F)$. Then $f(x) \in F$. By (2), there is a (i,j)-ω-b-open set U_x in X such that $x \in U_x$ and $f(U_x) \subseteq F$ implies $x \in U_x \subseteq f^{-1}(F)$. Hence $f^{-1}(F)$ is a (i,j)-ω-b-open in X.

(1)⇔(3): This follows from the fact that for any subset B of Y, $f^{-1}(Y \backslash B) = X \backslash f^{-1}(B)$.

(3)⇒(4): Let A be a subset of X. Since $A \subseteq f^{-1}(f(A))$, we have $A \subseteq f^{-1}(\sigma_i\text{-Cl}(f(A)))$. By hypothesis $f^{-1}(\sigma_i\text{-Cl}(f(A)))$ is a (i,j)-ω-b-closed set in X and hence $(i,j)\text{-}\omega\text{-}b\text{-}Cl(A)) \subseteq f^{-1}(\sigma_i\text{-Cl}(f(A)))$. Follows $f((i,j)\text{-}\omega\text{-}b\text{-}Cl(A))) \subseteq f(f^{-1}(\sigma_i\text{-Cl}(f(A))) \subseteq \sigma_i\text{-Cl}(f(A))$.

(4)⇒(3): Let F be any σ_i-closed subset of Y. Then $f((i,j)\text{-}\omega\text{-}b\text{-}Cl(f^{-1}(F)) \subseteq \sigma_i\text{-}cl(f(f^{-1}(F))) \subseteq \sigma_i\text{-}cl(F) = F$. Therefore, the $(i,j)\text{-}\omega\text{-}b\text{-}Cl(f^{-1}(F)) \subseteq f^{-1}(F)$. Consequently, $f^{-1}(F)$ is a (i,j)-ω-b-closed set in X.

(4)⇒(5): Let $B \subseteq Y$. Now, $f((i,j)\text{-}\omega\text{-}b\text{-}Cl(f^{-1}(B))) \subseteq \sigma_i\text{-Cl}(f(f^{-1}(B))) \subseteq \sigma_i\text{-Cl}(B)$. Consequently, $(i,j)\text{-}\omega\text{-}b - Cl(f^{-1}(B)) \subseteq f^{-1}(\sigma_i\text{-Cl}(B))$.

(5)⇒(4): Let $B = f(A)$ where $A \subseteq X$. Then, $(i,j)\text{-}\omega\text{-}b\text{-}Cl(A) \subseteq (i,j)\text{-}\omega\text{-}b - Cl(f^{-1}(B)) \subseteq f^{-1}(\sigma_i\text{-Cl}(B)) = f^{-1}(\sigma_i\text{-Cl}(f(A)))$, and hence $f((i,j)\text{-}\omega\text{-}b\text{-}Cl(A)) \subseteq \sigma_i\text{-Cl}(f(A))$.

(1)⇒(6): Let $B \subseteq Y$. Clearly, $f^{-1}(\sigma_i\text{-Int}(B))$ is a (i,j)-ω-b-open and we have $f^{-1}(\sigma_i\text{-Int}(B)) \subseteq (i,j)\text{-}\omega\text{-}b\text{-}Int(f^{-1}\sigma_i\text{-Int}(B)) \subseteq (i,j)\text{-}\omega\text{-}b\text{-}Int(f^{-1}B)$.

(6)⇒(1): Let B be a σ_i-open set in Y. Then $\sigma_i\text{-Int}(B) = B$ and $f^{-1}(B) \subseteq f^{-1}(\sigma_i\text{-Int}(B)) \subseteq (i,j)\text{-}\omega\text{-}b\text{-}Int(f^{-1}(B))$. Hence, we have $f^{-1}(B) = (i,j)\text{-}\omega\text{-}b\text{-}Int(f^{-1}(B))$. This implies that $f^{-1}(B)$ is a (i,j)-ω-b-open in X. ■

Definition 5 Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ a function. The graph $G(f)$ of $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be (i,j)-ω-b-closed in $X \times Y$, if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exists $U \in (i,j)\text{-}\omega\text{-}BO(X, x)$, $i, j = 1, 2$ with $i \neq j$ and a σ_i-open set V of Y containing y such that $(U \times V) \cap G(f) = \emptyset$.

Lemma 3.5 *The graph $G(f)$ of $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i,j)-ω-b-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exists $U \in (i,j)\text{-}\omega\text{-}BO(X, x)$, $i, j = 1, 2$ and $i \neq j$ and a σ_i-open set V of Y containing y such that $f(U) \cap V = \emptyset$.*

Proof. The proof is an immediate consequence of Definition 5. ∎

Theorem 3.6 *If $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is an (i,j)-ω-b-continuous function and (Y,σ_i) is a T_2-space, $i=1,2$, then $G(f)$ is (i,j)-ω-b-closed.*

Proof. Let $(x,y)\in(X\times Y)\setminus G(f)$. Then $y\neq f(x)$. Since (Y,σ_i) is T_1, there exist a σ_i-open set V and W of Y such that $f(x)\in V$ and $y\notin W$ and $V\cap W=\emptyset$. Since f is (i,j)-ω-b-continuous, there exists $U\in(i,j)$-ω-$BO(X,x)$ such that $f(U)\subset V$. Therefore, $f(U)\cap W=\emptyset$. Therefore, by Lemma 3.5, $G(f)$ is (i,j)-ω-b-closed. ∎

Definition 6 A bitopological space X is said to be a (i,j)-ω-b-T_2 space, if for each pair of distinct points $x,y\in X$, there exist $U,V\in(i,j)$-ω-$BO(X)$ containing x and y, respectively, such that $U\cap V=\emptyset$.

Theorem 3.7 *If $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is a (i,j)-ω-b-continuous injective function and (Y,σ_i) is a T_2 space, then (X,τ_1,τ_2) is a ω-b-T_2 space.*

Proof. The proof follows from the Definition 4 and 6. ∎

Theorem 3.8 *If $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is an injective (i,j)-ω-b-continuous function with (i,j)-ω-b-closed graph, then X is an (i,j)-ω-b-T_2 space.*

Proof. Let x_1 and x_2 be any pair of distinct points of X. Then $f(x_1)\neq f(x_2)$, so $(x_1,f(x_2))\in(X\times Y)\backslash G(f)$. Since the graph $G(f)$ is (i,j)-ω-b-closed, then by Lemma 3.5, there exist an (i,j)-ω-b-open set U containing x_1 and $V\in\sigma_i$ containing $f(x_2)$ such that $f(U)\cap V=\emptyset$. Since f is (i,j)-ω-b-continuous, $f^{-1}(V)$ is an (i,j)-ω-b-open set containing x_2 such that $U\cap f^{-1}(V)=\emptyset$. Hence X is (i,j)-ω-b-T_2. ∎

Definition 7 Let A be a subset of a bitopological space X. A collection $\{U_\alpha:\alpha\in I\}$ of (i,j)-b-open subsets of X is called an (i,j)-b-open cover of A, if $A\subseteq\bigcup_{\alpha\in I}U_\alpha$.

Definition 8 A bitopological space X is said to be (i,j)-b-Lindeloff, if every (i,j)-b-open cover of X has a countable subcover. A subset A of bitopological space X is said to be (i,j)-b-Lindeloff relative to X, if every cover of A by (i,j)-b-open sets of X has a countable subcover.

Theorem 3.9 *Let X be a bitopological space. If every (i,j)-b-open subset is (i,j)-b-Lindeloff relative to X. Then every subset is (i,j)-b-Lindeloff relative to X*

Theorem 3.10 *For a bitopological space X. The following properties are equivalent:*

(1) *X is (i,j)-b-Lindeloff.*

(2) *Every countable cover of X by (i,j)-b-open sets has a countable subcover.*

Proof. (2)⇒(1): Since every (i,j)-b-open set is (i,j)-ω-b-open, the proof follows. (1)⇒(2): Let $\{U_\alpha : \alpha \in I\}$ be any cover of X by (i,j)-ω-b-open sets of X. For each $x \in X$, there exists an $\alpha_x \in I$ such that $x \in U_{\alpha_x}$. Since U_{α_x} is an (i,j)-ω-b-open, then using Definition 1, there exists a (i,j)-b-open set V_{α_x} such that $x \in V_{\alpha_x}$ and $V_{\alpha_x} - U_{\alpha_x}$ is countable. The family $\{V_\alpha : \alpha \in I\}$ is a (i,j)-b-open cover of X and X is (i,j)-b-Lindeloff. By Definition 8, the collection $\{V_\alpha : \alpha \in I\}$ has a countable subcover $\{U_{\alpha_{x_i}}\}_{i\in N}$ such that $X = \bigcup_{i\in N} V_{\alpha_{x_i}}$. Since $X = \bigcup_{i\in N}[(V_{\alpha_{x_i}} - U_{\alpha_{x_i}}) \cup U_{\alpha_{x_i}}] = \bigcup_{i\in N}[(V_{\alpha_{x_i}} - U_{\alpha_{x_i}}) \bigcup_{i\in N} U_{\alpha_{x_i}}]$ and $V_{\alpha_{x_i}} - U_{\alpha_{x_i}}$ is a countable set, for each α_{x_i}, there exists a countable subset $I_{\alpha(x_i)}$ of I such that $V_{\alpha_{x_i}} - U_{\alpha_{x_i}} \subseteq \bigcup_{I_{\alpha(x_i)}} U_\alpha$ and therefore $X = \bigcup_{i\in N}(\bigcup_{\alpha\in I_{\alpha(x_i)}} U_\alpha) \cup (\bigcup_{i\in N} U_{\alpha(x_i)})$. ∎

Definition 9 A bitopological space X is called pairwise Lindeloff if each pairwise open cover of X has a countable subcover.

Theorem 3.11 *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a surjective an (i,j)-ω-b-continuous function. If X is (i,j)-b-Lindeloff, then Y is pairwise Lindeloff.*

Proof. Let $\{U_\alpha : \alpha \in I\}$ be any pairwise open cover of Y by σ_i-open sets. Then $\{f^{-1}(U_\alpha) : \alpha \in I\}$ is a (i,j)-ω-b-open cover of X. Since X is (i,j)-b-Lindeloff, there exists a countable subset I_0 of I such that $X = \bigcup_{\alpha\in I_0} U_\alpha$. Therefore, Y is a pairwise Lindeloff. ∎

Definition 10 Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be a bitopological spaces. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called:

1 (i,j)-ω-b-open if $f(U)$ is an (i,j)-ω-b-open set in Y for every τ_i-open set U of X.

2 (i,j)-ω-b-closed if $f(U)$ is an (i,j)-ω-b-closed set in Y for every τ_i-closed set U of X.

The following theorem give a characterization of (i,j)-ω-b-open functions.

Theorem 3.12 *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be a bitopological spaces and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ a function, the following properties are equivalent:*

(1) *f is an (i,j)-ω-b-open.*

(2) *$f(\tau_i - Int(U)) \subseteq (i,j)\text{-}\omega\text{-}b - Int(f(U))$, for each subset U of X.*

(3) *$\tau_i - Int(f^{-1}(V)) \subseteq f^{-1}((i,j)\text{-}\omega\text{-}b - Int(V))$, for each subset V of Y.*

Proof. (1)⇒(2): Let U be any subset of X. Then $\tau_i - Int(U)$ is a τ_i-open set of X. Then $f(\tau_i - Int(U))$ is a (i,j)-ω-b-open set of Y. Since $f(\tau_i - Int(U)) \subseteq f(U)$, $f(\tau_i - Int(U)) = (i,j)\text{-}\omega\text{-}b - Int(f(\tau_i - Int(U))) \subseteq (i,j)\text{-}\omega\text{-}b - Int(f(U))$.

(2)⇒(3): Let V be any subset of Y. Then $f(\tau_i - Int(f^{-1}(V))) \subseteq (i,j)\text{-}\omega\text{-}b\text{-}Int(f(f^{-1}(V)))$. Hence $\tau_i\text{-}Int(f^{-1}(V)) \subseteq f^{-1}((i,j)\text{-}\omega\text{-}b - Int(V))$.

(3)⇒(1): Let U be any τ_i-open set of X. Then $\tau_i - Int(U) = U$. Now, $V = \tau_i - Int(V) \subseteq \tau_i - Int(f^{-1}(f(V)) \subseteq f^{-1}((i,j)\text{-}\omega\text{-}b - Int(f(V)))$. Which implies that $f(V) \subseteq f(f^{-1}((i,j)\text{-}\omega\text{-}b - Int(f(V)))) \subseteq (i,j)\text{-}\omega\text{-}b - Int(f(V))$. Hence $f(V)$ is a (i,j)-ω-b-open set of Y. Thus f is (i,j)-ω-b-open. ∎

Theorem 3.13 *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function, then f is a (i, j)-ω-b-closed function if and only if the (i, j)-ω-$b - Cl(f(V)) \subseteq f(\tau_i - Cl(V)))$ for each subset V of X.*

Proof. Let f be an (i, j)-ω-b-closed function and V be any subset of X. Then $f(V) \subseteq f(\tau_i - Cl(V))$ and $f(\tau_i - Cl(V))$ is an (i, j)-ω-b-closed set of Y. Hence (i, j)-ω-b-$Cl(f(V)) \subseteq (i, j)$-$\omega$-$b$-$Cl(f(\tau_i - Cl(V))) = f(\tau_i - Cl(V)))$.

Conversely, let V be a τ_i-closed set of X. Then $f(V) \subseteq (i, j)$-ω-$b - Cl(f(V)) \subseteq f(\tau_i - Cl(V))) = f(V)$. Hence $f(V)$ is an (i, j)-ω-b-closed set of Y. Therefore, f is an (i, j)-ω-b-closed function. ∎

Definition 11 A bitopological space X is said to be (i, j)-ω-b-connected, if X cannot be expressed as the union of two nonempty disjoint (i, j)-ω-b-open sets.

Example 3.14 The bitopological space defined in Example 2.2 is not (i, j)-ω-b-connected but the bitopological space defined in Example 2.3 is (i, j)-ω-b-connected

Definition 12 A bitopological space X is said to be pairwise connected [7], if it cannot be expressed as the union of two nonempty disjoint sets U and V such that U is τ_i-open and V is τ_j-open, where $i, j = 1, 2$ and $i \neq j$.

Example 3.15 The bitopological space defined in Example 2.3 is pairwise connected

Theorem 3.16 *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a (i, j)-ω-b-continuous function. If X is an (i, j)-ω-b-connected space then $f(X)$ is pairwise connected.*

Proof. The proof is clear. ∎

References

[1] Abd EL-Monsef, M., EL-Atik, A. and EL-Sharkasy, M., *Some Topologies induced by b-open sets,* Kyungpook Math., 45 (2005), 539-547.

[2] D. Andrijevic, D., *On b-open sets*, Mat. Vesnik, 48 (1996), 59-64.

[3] Carpintero, C., Hussain, S., Rosas, E., *(i, j)-ω-semi open sets and (i, j)-ω-semicontinuity in Bitopological spaces*, submitted (2014).

[4] Hdeib, H.Z., *ω-closed mappings*, Revista Colombiana Mat., 16 (1982), 65-78.

[5] Kelly, J.C., *Bitopological spaces, Proc. London Math. Soc* 13(1963), 71-89.

[6] Alswidi, Luay A., Alhosani, Assad M.A., *Weak forms of ω-open sets in bitopological spaces and Connectedness*, European Journal of Scientific Research, 52 (2) (2011), 204-212.

[7] Pervin, W.J., *Connectedness in Bitopological spaces*, Ind. Math., 29 (1967), 369-372.

[8] Rajesh, N., Mustafa, J.M., *On (i, j)-ω-preopen sets,* Proyecciones Journal of Mathematics, 31 (3) (2012), 235-246.

Accepted: 04.01.2015