SOME REFINEMENTS OF THE HEINZ INEQUALITIES

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Abstract. This paper aims to discuss Heinz inequalities for unitarily invariant norms. We present some refinements of the Heinz inequalities for matrices due to Kittaneh [Integr. Equ. Oper. Theory, 68:519-527, 2010]. Our results generalize the results shown by Feng [J. Inequal. Appl., 2012:18, 2012], Wang [J. Inequal. Appl., 2013:424, 2013] and Yan et al. [J. Inequal. Appl. 2014:50, 2014].

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1. Introduction

Let $M_{m,n}$ be the space of $m \times n$ complex matrices and $M_n = M_{n,n}$. Let $|||\cdot|||$ denote any unitarily invariant norm on M_n. So, $|||UAV||| = |||A|||$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. The *Ky Fan k-norm* $|||\cdot|||_{(k)}$ is defined as

$$|||A|||_{(k)} = \sum_{j=1}^{k} s_j(A), \quad k = 1, \cdots, n,$$

where $s_1(A) \geq s_2(A) \geq \cdots \geq s_{n-1}(A) \geq s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. The *Schattenp-norm* $|||\cdot|||_p$ is defined as

$$|||A|||_p = \left(\sum_{j=1}^{n} s_j^p(A)\right)^{1/p} = (\operatorname{tr}|A|^p)^{1/p}, \quad 1 \leq p < \infty.$$

It is known that these norms are unitarily invariant [1].

Let $A, B, X \in M_n$ such that A and B are positive semidefinite. Then, for every unitarily invariant norm, the function

$$\varphi(v) = |||A^v X B^{1-v} + A^{1-v} X B^v|||$$

is convex on $[0,1]$, attains its minimum at $v = \frac{1}{2}$ and attains its maximum at $v = 0$ and $v = 1$. Moreover, $\varphi(v) = \varphi(1-v)$ for $0 \le v \le 1$.

Bhatia and Davis proved Heinz inequalities in [2] that if $A, B, X \in M_n$ such that A and B are positive semidefinite, for $0 \le v \le 1$ and for every unitarily invariant norm, then

$$\begin{aligned} 2|||A^{\frac{1}{2}} X B^{\frac{1}{2}}||| &\le |||A^v X B^{1-v} + A^{1-v} X B^v||| \\ &\le |||AX + AB|||. \end{aligned} \tag{1}$$

For more information on Heinz inequality for matrices, the reader is referred to [2]-[7].

By the convexity of function $\varphi(v) = |||A^v X B^{1-v} + A^{1-v} X B^v|||$, Kittaneh [3], Feng [4], Wang [5] and Yan et al [6] got some refinements of (1). In this paper, we also present several refinements of (1). Our results are generalization of results shown in [3]-[6].

2. Main results

In this section, we present several improvement refinements of the Heinz inequalities, to do this, we need the following lemmas.

Lemma 1. (Hermite-Hadamard Integral Inequality) [3] *Let f be a real valued convex function on the interval $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}.$$

Lemma 2. *Let f be a real valued convex function on the interval $[a,b]$. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le \frac{1}{2n}\left[(n-1)f(a) + 2f\left(\frac{a+b}{2}\right) + (n-1)f(b)\right] \\ &\le \frac{f(a)+f(b)}{2}, \end{aligned}$$

where $n \ge 2$ is an integer.

Proof. By Lemma 1, we can easily verify the inequality

$$\frac{1}{2n}\left[(n-1)\,f\,(a)+2f\left(\frac{a+b}{2}\right)+(n-1)\,f(b)\right]\leq\frac{f\,(a)+f\,(b)}{2}.$$

Then, we will prove the following inequality:

$$\frac{1}{b-a}\int_a^b f\,(t)\,dt\leq\frac{1}{2n}\left[(n-1)\,f\,(a)+2f\left(\frac{a+b}{2}\right)+(n-1)\,f(b)\right].$$

Using Lemma 1, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f\,(t)dt &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} f\,(t)dt+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b f\,(t)dt\\ &\leq\frac{1}{b-a}\left[\frac{f\,(a)+f\left(\frac{a+b}{2}\right)}{2}\cdot\frac{b-a}{2}+\frac{f\left(\frac{a+b}{2}\right)+f\,(b)}{2}\cdot\frac{b-a}{2}\right]\\ &=\frac{1}{4}\left[f\,(a)+2f\left(\frac{a+b}{2}\right)+f\,(b)\right]\\ &=\frac{1}{2n}\left[\frac{n}{2}f\,(a)+nf\left(\frac{a+b}{2}\right)+\frac{n}{2}f\,(b)\right]\\ &\leq\frac{1}{2n}\left[\frac{n}{2}f\,(a)+2f\left(\frac{a+b}{2}\right)+\frac{n-2}{2}\left(f\,(a)+f\,(b)\right)+\frac{n}{2}f\,(b)\right]\\ &=\frac{1}{2n}\left[(n-1)\,f\,(a)+2f\left(\frac{a+b}{2}\right)+(n-1)\,f(b)\right].\end{aligned}$$

This completes the proof. ∎

Applying Lemma 2 to the function $\varphi\,(v)=|||A^vXB^{1-v}+A^{1-v}XB^v|||$ on the interval $[u,1-u]$ when $0\leq u<\frac{1}{2}$, and on the interval $[1-u,u]$ when $\frac{1}{2}<u\leq 1$, we achieve a refinement of the first inequality in (1).

Theorem 1. *Let $A,B,X\in M_n$ such that A and B are positive definite, for $0\leq u\leq 1$ and for every unitarily invariant norm. Then*

$$\begin{aligned}2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\leq\frac{1}{|1-2u|}\left|\int_u^{1-u}|||A^vXB^{1-v}+A^{1-v}XB^v|||dv\right|\\ &\leq\frac{1}{n}\left[(n-1)|||A^uXB^{1-u}+A^{1-u}XB^u|||+2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right]\\ &\leq|||A^uXB^{1-u}+A^{1-u}XB^u|||,\end{aligned}\tag{2}$$

where $n\geq 2$ is an integer.

Proof. If $0 \le u < \frac{1}{2}$, then, by Lemma 2, we have

$$\begin{aligned}\varphi\left(\frac{1-u+u}{2}\right) &\le \frac{1}{1-2u}\int_u^{1-u}\varphi(v)\,dv \\ &\le \frac{1}{2n}\left[(n-1)\,\varphi(u)+2\varphi\left(\frac{1-u+u}{2}\right)+(n-1)\,\varphi(1-u)\right] \\ &\le \frac{\varphi(u)+\varphi(1-u)}{2}.\end{aligned}$$

That is,

$$\begin{aligned}\varphi\left(\frac{1}{2}\right) &\le \frac{1}{1-2u}\int_u^{1-u}\varphi(v)\,dv \\ &\le \frac{1}{n}\left[(n-1)\varphi(u)+\varphi\left(\frac{1}{2}\right)\right] \\ &\le \varphi(u),\end{aligned}$$

where $\varphi(v) = |||A^v XB^{1-v} + A^{1-v}XB^v|||$. Thus

$$\begin{aligned}2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\le \frac{1}{1-2u}\int_u^{1-u}|||A^vXB^{1-v}+A^{1-v}XB^v|||dv \\ &\le \frac{1}{n}\left[(n-1)|||A^uXB^{1-u}+A^{1-u}XB^u|||+2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right] \\ &\le |||A^uXB^{1-u}+A^{1-u}XB^u|||.\end{aligned} \tag{3}$$

If $\frac{1}{2} < u \le 1$, then the proof is similar to the case $0 \le u < \frac{1}{2}$, so we obtain

$$\begin{aligned}2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\le \tfrac{1}{2u-1}\int_{1-u}^{u}|||A^vXB^{1-v}+A^{1-v}XB^v|||dv \\ &\le \frac{1}{n}\left[(n-1)|||A^uXB^{1-u}+A^{1-u}XB^u|||+2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right] \\ &\le |||A^uXB^{1-u}+A^{1-u}XB^u|||.\end{aligned} \tag{4}$$

Hence,

$$\begin{aligned}&\lim_{u\to\frac{1}{2}}\frac{1}{|1-2u|}\left|\int_u^{1-u}|||A^vXB^{1-v}+A^{1-v}XB^v|||dv\right| \\ &= \lim_{u\to\frac{1}{2}}\frac{1}{n}\left[(n-1)|||A^uXB^{1-u}+A^{1-u}XB^u|||+2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right] \\ &= 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||.\end{aligned}$$

The inequalities in (2) follow by combining the inequalities (3) and (4). This completes the proof. ∎

Applying Lemma 2 to the function $\varphi(v) = |||A^v XB^{1-v} + A^{1-v}XB^v|||$ on the interval $\left[u, \frac{1}{2}\right]$ when $0 \le u < \frac{1}{2}$, and on the interval $\left[\frac{1}{2}, u\right]$ when $\frac{1}{2} < u \le 1$, we obtain the following result.

Theorem 2. *Let $A, B, X \in M_n$ such that A and B are positive definite. For $0 \le u \le 1$ and for every unitarily invariant norm. Then*

$$\begin{aligned} &|||A^{\frac{1+2u}{4}} XB^{\frac{3-2u}{4}} + A^{\frac{3-2u}{4}} XB^{\frac{1+2u}{4}}||| \\ &\quad \le \frac{2}{|1-2u|} \left| \int_u^{\frac{1}{2}} |||A^v XB^{1-v} + A^{1-v}XB^v||| dv \right| \\ &\quad \le \frac{1}{2n} \big[(n-1)|||A^u XB^{1-u} + A^{1-u}XB^u||| \\ &\quad +2|||A^{\frac{1+2u}{4}} XB^{\frac{3-2u}{4}} + A^{\frac{3-2u}{4}} XB^{\frac{1+2u}{4}}||| + 2(n-1)|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\big] \\ &\quad \le \frac{1}{2}\left(|||A^u XB^{1-u} + A^{1-u}XB^u||| + 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right), \end{aligned} \tag{5}$$

where $n \ge 2$ is an integer.

Proof. The proof is similar to Theorem 1, so we omit it. ■

Inequalities (5) and the first inequality in (1) yield the following refinement of the first inequality in (1).

Corollary 1. *Let $A, B, X \in M_n$ such that A and B are positive definite. For $0 \le u \le 1$ and for every unitarily invariant norm. Then*

$$\begin{aligned} 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\le |||A^{\frac{1+2u}{4}} XB^{\frac{3-2u}{4}} + A^{\frac{3-2u}{4}} XB^{\frac{1+2u}{4}}||| \\ &\le \frac{2}{|1-2u|} \left| \int_u^{\frac{1}{2}} |||A^v XB^{1-v} + A^{1-v}XB^v||| dv \right| \\ &\le \frac{1}{2n}[(n-1)|||A^u XB^{1-u} + A^{1-u}XB^u||| \\ &+2|||A^{\frac{1+2u}{4}} XB^{\frac{3-2u}{4}} + A^{\frac{3-2u}{4}} XB^{\frac{1+2u}{4}}||| + 2(n-1)|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||] \\ &\le \frac{1}{2}\left(|||A^u XB^{1-u} + A^{1-u}XB^u||| + 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||\right) \\ &\le |||A^u XB^{1-u} + A^{1-u}XB^u|||, \end{aligned} \tag{6}$$

where $n \ge 2$ is an integer.

It should be noticed here that in the inequalities (5) and (6)

$$\lim_{u \to \frac{1}{2}} \frac{1}{|1-2u|} \left| \int_u^{\frac{1}{2}} |||A^v XB^{1-v} + A^{1-v}XB^v||| dv \right| = |||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||.$$

In the sequel, we get another refinement of the second inequality in (1).

Applying Lemma 2 to the function $\varphi(v) = |||A^v X B^{1-v} + A^{1-v} X B^v|||$ on the interval $[0, u]$ when $0 < u \leq \frac{1}{2}$, and on the interval $[u, 1]$ when $\frac{1}{2} \leq u < 1$, we obtain the following theorem.

Theorem 3. *Let $A, B, X \in M_n$ such that A and B are positive definite. Then*

1. *for $0 \leq u \leq \frac{1}{2}$ and for every unitarily invariant norm,*

$$\begin{aligned} & |||A^{\frac{u}{2}} X B^{1-\frac{u}{2}} + A^{1-\frac{u}{2}} X B^{\frac{u}{2}}||| \\ & \leq \frac{1}{u} \int_0^u |||A^v X B^{1-v} + A^{1-v} X B^v||| dv \\ & \leq \frac{1}{2n} [(n-1)|||AX + XB||| + 2|||A^{\frac{u}{2}} X B^{1-\frac{u}{2}} + A^{1-\frac{u}{2}} X B^{\frac{u}{2}}||| \\ & +(n-1)|||A^u X B^{1-u} + A^{1-u} X B^u|||] \\ & \leq \frac{1}{2} \left(|||AX + XB||| + |||A^u X B^{1-u} + A^{1-u} X B^u||| \right), \end{aligned} \tag{7}$$

where $n \geq 2$ is an integer,

2. *for $\frac{1}{2} \leq u \leq 1$ and for every unitarily invariant norm,*

$$\begin{aligned} & |||A^{\frac{1+u}{2}} X B^{\frac{1-u}{2}} + A^{\frac{1-u}{2}} X B^{\frac{1+u}{2}}||| \\ & \leq \frac{1}{1-u} \int_u^1 |||A^v X B^{1-v} + A^{1-v} X B^v||| dv \\ & \leq \frac{1}{2n} [(n-1)|||AX + XB||| + 2|||A^{\frac{1+u}{2}} X B^{\frac{1-u}{2}} + A^{\frac{1-u}{2}} X B^{\frac{1+u}{2}}||| \\ & +(n-1)|||A^u X B^{1-u} + A^{1-u} X B^u|||] \\ & \leq \frac{1}{2} \left(|||AX + XB||| + |||A^u X B^{1-u} + A^{1-u} X B^u||| \right), \end{aligned} \tag{8}$$

where $n \geq 2$ is an integer.

Proof. The proof is similar to Theorem 1, so we omit it. ∎

In view of the fact that the function $\varphi(v) = |||A^v X B^{1-v} + A^{1-v} X B^v|||$ is decreasing on the interval $\left[0, \frac{1}{2}\right]$ and increasing on the interval $\left[\frac{1}{2}, 1\right]$, by Theorem 3, we have the following result, which is a refinement of the second inequality in (1).

Corollary 2. *Let $A, B, X \in M_n$ such that A and B are positive definite. Then*

1. *for* $0 \le u \le \frac{1}{2}$ *and for every unitarily invariant norm*

$$
\begin{aligned}
&|||A^u X B^{1-u} + A^{1-u} X B^u||| \\
&\quad \le |||A^{\frac{u}{2}} X B^{1-\frac{u}{2}} + A^{1-\frac{u}{2}} X B^{\frac{u}{2}}||| \\
&\quad \le \frac{1}{u} \int_0^u |||A^v X B^{1-v} + A^{1-v} X B^v||| dv \\
&\quad \le \frac{1}{2n}[(n-1)|||AX + XB||| + 2|||A^{\frac{u}{2}} X B^{1-\frac{u}{2}} + A^{1-\frac{u}{2}} X B^{\frac{u}{2}}||| \\
&\quad +(n-1)|||A^u X B^{1-u} + A^{1-u} X B^u|||] \\
&\quad \le \frac{1}{2}\left(|||AX + XB||| + |||A^u X B^{1-u} + A^{1-u} X B^u|||\right) \\
&\quad \le |||AX + XB|||,
\end{aligned} \tag{9}
$$

where $n \ge 2$ is an integer.

2. *for* $\frac{1}{2} \le u \le 1$ *and for every unitarily invariant norm*

$$
\begin{aligned}
&|||A^u X B^{1-u} + A^{1-u} X B^u||| \\
&\quad \le |||A^{\frac{1+u}{2}} X B^{\frac{1-u}{2}} + A^{\frac{1-u}{2}} X B^{\frac{1+u}{2}}||| \\
&\quad \le \frac{1}{1-u} \int_u^1 |||A^v X B^{1-v} + A^{1-v} X B^v||| dv \\
&\quad \le \frac{1}{2n}[(n-1)|||AX + XB||| + 2|||A^{\frac{1+u}{2}} X B^{\frac{1-u}{2}} + A^{\frac{1-u}{2}} X B^{\frac{1+u}{2}}||| \\
&\quad +(n-1)|||A^u X B^{1-u} + A^{1-u} X B^u|||] \\
&\quad \le \frac{1}{2}\left(|||AX + XB||| + |||A^u X B^{1-u} + A^{1-u} X B^u|||\right) \\
&\quad \le |||AX + XB|||,
\end{aligned} \tag{10}
$$

where $n \ge 2$ is an integer.

It should be noticed that in the inequalities (7) to (10), we have

$$
\begin{aligned}
&\lim_{u \to 0} \frac{1}{u} \int_0^u |||A^v X B^{1-v} + A^{1-v} X B^v||| dv \\
&\qquad = \lim_{u \to 1} \frac{1}{1-u} \int_u^1 |||A^v X B^{1-v} + A^{1-v} X B^v||| dv \\
&\qquad = |||AX + XB|||.
\end{aligned}
$$

Remark 1. The three special values $n = 2$, $n = 16$ and $n = 4$ give the refinements of Heinz inequalities obtained in [4], [5] and [6], respectively.

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