SOME REFINEMENTS OF THE HEINZ INEQUALITIES

Jianming Xue

Oxbridge College
Kunming University of Science and Technology
Kunming, Yunnan 650106
P.R. China
e-mail: xuejianming104@163.com


Keywords: Heinz inequality; convex function; unitarily invariant norm.


1. Introduction

Let $M_{m,n}$ be the space of $m \times n$ complex matrices and $M_n = M_{n,n}$. Let $||| \cdot |||$ denote any unitarily invariant norm on $M_n$. So, $|||UAV||| = |||A|||$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. The Ky Fan $k$-norm $||| \cdot |||_{(k)}$ is defined as

$$|||A|||_{(k)} = \sum_{j=1}^{k} s_j(A), \quad k = 1, \cdots, n,$$

where $s_1(A) \geq s_2(A) \geq \cdots \geq s_{n-1}(A) \geq s_n(A)$ are the singular values of $A$, that is, the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{1/2}$, arranged in decreasing order and repeated according to multiplicity. The Schatten-$p$-norm $||| \cdot |||_p$ is defined as

$$|||A|||_p = \left( \sum_{j=1}^{n} s_j^p(A) \right)^{1/p} = (\text{tr} |A|^p)^{1/p}, \quad 1 \leq p < \infty.$$

It is known that these norms are unitarily invariant [1].
Let $A, B, X \in M_n$ such that $A$ and $B$ are positive semidefinite. Then, for every unitarily invariant norm, the function

$$\varphi(v) = \|A^v XB^{1-v} + A^{1-v}XB^v\|$$

is convex on $[0, 1]$, attains its minimum at $v = \frac{1}{2}$ and attains its maximum at $v = 0$ and $v = 1$. Moreover, $\varphi(v) = \varphi(1-v)$ for $0 \leq v \leq 1$.

Bhatia and Davis proved Heinz inequalities in [2] that if $A, B, X \in M_n$ such that $A$ and $B$ are positive semidefinite, for $0 \leq v \leq 1$ and for every unitarily invariant norm, then

$$2\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \leq \|A^v XB^{1-v} + A^{1-v}XB^v\| \leq \|AX + AB\|.$$

For more information on Heinz inequality for matrices, the reader is referred to [2]-[7].

By the convexity of function $\varphi(v) = \|A^v XB^{1-v} + A^{1-v}XB^v\|$, Kittaneh [3], Feng [4], Wang [5] and Yan et al [6] got some refinements of (1). In this paper, we also present several refinements of (1). Our results are generalization of results shown in [3]-[6].

2. Main results

In this section, we present several improvement refinements of the Heinz inequalities, to do this, we need the following lemmas.

**Lemma 1.** (Hermite-Hadamard Integral Inequality) [3] Let $f$ be a real valued convex function on the interval $[a, b]$. Then

$$f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(t) \, dt \leq f(a) + f(b).$$

**Lemma 2.** Let $f$ be a real valued convex function on the interval $[a, b]$. Then

$$f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(t) \, dt \leq \frac{1}{2n} \left[ (n-1) f(a) + 2f \left( \frac{a+b}{2} \right) + (n-1) f(b) \right] \leq \frac{f(a) + f(b)}{2},$$

where $n \geq 2$ is an integer.
Proof. By Lemma 1, we can easily verify the inequality
\[ \frac{1}{2n} \left[ (n-1) f(a) + 2f \left( \frac{a+b}{2} \right) + (n-1) f(b) \right] \leq f(a) + f(b). \]

Then, we will prove the following inequality:
\[ \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{1}{2n} \left[ (n-1) f(a) + 2f \left( \frac{a+b}{2} \right) + (n-1) f(b) \right]. \]

Using Lemma 1, we have
\[ \frac{1}{b-a} \int_a^b f(t) dt = \frac{1}{b-a} \int_a^{\alpha b} f(t) dt + \frac{1}{b-a} \int_{\alpha b}^b f(t) dt \]
\[ \leq \frac{1}{b-a} \left[ f(a) + f \left( \frac{a+b}{2} \right) \cdot \frac{b-a}{2} + f \left( \frac{a+b}{2} \right) \cdot \frac{b-a}{2} \right] \]
\[ = \frac{1}{4} \left[ f(a) + 2f \left( \frac{a+b}{2} \right) + f(b) \right] \]
\[ = \frac{1}{2n} \left[ \frac{n}{2} f(a) + nf \left( \frac{a+b}{2} \right) + \frac{n}{2} f(b) \right] \]
\[ \leq \frac{1}{2n} \left[ \frac{n}{2} f(a) + 2f \left( \frac{a+b}{2} \right) + \frac{n-2}{2} (f(a) + f(b)) + \frac{n}{2} f(b) \right] \]
\[ = \frac{1}{2n} \left[ (n-1) f(a) + 2f \left( \frac{a+b}{2} \right) + (n-1) f(b) \right]. \]

This completes the proof. \[ \Box \]

Applying Lemma 2 to the function \( \varphi(v) = |||A^v XB^{1-v} + A^{1-v} XB^v||| \) on the interval \([u, 1-u]\) when \(0 \leq u < \frac{1}{2}\), and on the interval \([1-u, u]\) when \(\frac{1}{2} < u \leq 1\), we achieve a refinement of the first inequality in (1).

**Theorem 1.** Let \(A, B, X \in M_n\) such that \(A\) and \(B\) are positive definite, for \(0 \leq u \leq 1\) and for every unitarily invariant norm. Then
\[ 2|||A^{1/2}XB^{1/2}||| \leq \frac{1}{1-2u} \left| \int_u^{1-u} |||A^v XB^{1-v} + A^{1-v} XB^v||| dv \right| \]
\[ \leq \frac{1}{n} \left[ (n-1)|||A^u XB^{1-u} + A^{1-u} XB^u||| + 2|||A^{1/2}XB^{1/2}||| \right] \]
\[ \leq |||A^u XB^{1-u} + A^{1-u} XB^u|||, \]
where \(n \geq 2\) is an integer.
Proof. If $0 \leq u < \frac{1}{2}$, then, by Lemma 2, we have

\[
\varphi\left(\frac{1 - u + u}{2}\right) \leq \frac{1}{1 - 2u} \int_u^{1-u} \varphi(v) \, dv \\
\leq \frac{1}{2n} \left[ (n-1) \varphi(u) + 2\varphi\left(\frac{1 - u + u}{2}\right) + (n-1) \varphi(1 - u) \right] \\
\leq \varphi(u) + \varphi(1 - u).
\]

That is,

\[
\varphi\left(\frac{1}{2}\right) \leq \frac{1}{1 - 2u} \int_u^{1-u} \varphi(v) \, dv \\
\leq \frac{1}{n} \left[ (n-1) \varphi(u) + \varphi\left(\frac{1}{2}\right) \right] \\
\leq \varphi(u),
\]

where $\varphi(v) = |||A^vXB^{1-v} + A^{1-v}XB^v|||$. Thus

\[
2|||A^{\frac{1}{2}}XB^{\frac{3}{2}}||| \leq \frac{1}{1 - 2u} \int_u^{1-u} |||A^vXB^{1-v} + A^{1-v}XB^v||| \, dv \\
\leq \frac{1}{n} \left[ (n-1)|||A^uXB^{1-u} + A^{1-u}XB^u||| + 2|||A^{\frac{1}{2}}XB^{\frac{3}{2}}||| \right] \\
\leq |||A^uXB^{1-u} + A^{1-u}XB^u|||.
\]

If $\frac{1}{2} < u \leq 1$, then the proof is similar to the case $0 \leq u < \frac{1}{2}$, so we obtain

\[
2|||A^{\frac{1}{2}}XB^{\frac{3}{2}}||| \leq \frac{1}{2u - 1} \int_{1-u}^{u} |||A^vXB^{1-v} + A^{1-v}XB^v||| \, dv \\
\leq \frac{1}{n} \left[ (n-1)|||A^uXB^{1-u} + A^{1-u}XB^u||| + 2|||A^{\frac{1}{2}}XB^{\frac{3}{2}}||| \right] \\
\leq |||A^uXB^{1-u} + A^{1-u}XB^u|||.
\]

Hence,

\[
\lim_{u \to \frac{1}{2}} \frac{1}{1 - 2u} \int_u^{1-u} |||A^vXB^{1-v} + A^{1-v}XB^v||| \, dv = \lim_{u \to \frac{1}{2}} \frac{1}{n} \left[ (n-1)|||A^uXB^{1-u} + A^{1-u}XB^u||| + 2|||A^{\frac{1}{2}}XB^{\frac{3}{2}}||| \right] \\
= 2|||A^{\frac{1}{2}}XB^{\frac{3}{2}}|||.
\]

The inequalities in (2) follow by combining the inequalities (3) and (4). This completes the proof.
Applying Lemma 2 to the function $\varphi(v) = \|A^vXB^{1-v} + A^{1-v}XB^v\|$ on the interval $[u, \frac{1}{2}]$ when $0 \leq u < \frac{1}{2}$, and on the interval $[\frac{1}{2}, u]$ when $\frac{1}{2} < u \leq 1$, we obtain the following result.

**Theorem 2.** Let $A, B, X \in M_n$ such that $A$ and $B$ are positive definite. For $0 \leq u \leq 1$ and for every unitarily invariant norm. Then

$$
\|A^{\frac{1+2u}{4}}XB^{\frac{3-2u}{4}} + A^{\frac{3-2u}{4}}XB^{\frac{1+2u}{4}}\| \\
\leq \frac{2}{1-2u} \left| \int_u^{\frac{1}{2}} \|A^vXB^{1-v} + A^{1-v}XB^v\| dv \right| \\
\leq \frac{1}{2n} \left[ (n-1)\|A^uXB^{1-u} + A^{1-u}XB^u\| + 2\|A^{\frac{1+2u}{4}}XB^{\frac{3-2u}{4}} + A^{\frac{3-2u}{4}}XB^{\frac{1+2u}{4}}\| + 2(n-1)\|A^\frac{3}{2}XB^\frac{1}{2}\| \right] \\
\leq \frac{1}{2} \left( \|A^uXB^{1-u} + A^{1-u}XB^u\| + 2\|A^\frac{3}{2}XB^\frac{1}{2}\| \right),
$$

where $n \geq 2$ is an integer.

**Proof.** The proof is similar to Theorem 1, so we omit it.  

Inequalities (5) and the first inequality in (1) yield the following refinement of the first inequality in (1).

**Corollary 1.** Let $A, B, X \in M_n$ such that $A$ and $B$ are positive definite. For $0 \leq u \leq 1$ and for every unitarily invariant norm. Then

$$
2\|A^\frac{1}{2}XB^\frac{1}{2}\| \leq \|A^{\frac{1+2u}{4}}XB^{\frac{3-2u}{4}} + A^{\frac{3-2u}{4}}XB^{\frac{1+2u}{4}}\| \\
\leq \frac{2}{1-2u} \left| \int_u^{\frac{1}{2}} \|A^vXB^{1-v} + A^{1-v}XB^v\| dv \right| \\
\leq \frac{1}{2n} \left[ (n-1)\|A^uXB^{1-u} + A^{1-u}XB^u\| + 2\|A^{\frac{1+2u}{4}}XB^{\frac{3-2u}{4}} + A^{\frac{3-2u}{4}}XB^{\frac{1+2u}{4}}\| + 2(n-1)\|A^\frac{3}{2}XB^\frac{1}{2}\| \right] \\
\leq \frac{1}{2} \left( \|A^uXB^{1-u} + A^{1-u}XB^u\| + 2\|A^\frac{3}{2}XB^\frac{1}{2}\| \right) \\
\leq \|A^uXB^{1-u} + A^{1-u}XB^u\|,
$$

where $n \geq 2$ is an integer.

It should be noticed here that in the inequalities (5) and (6)

$$
\lim_{u \to \frac{1}{2}} \frac{1}{1-2u} \left| \int_u^{\frac{1}{2}} \|A^vXB^{1-v} + A^{1-v}XB^v\| dv \right| = \|A^\frac{3}{2}XB^\frac{1}{2}\|.
$$
In the sequel, we get another refinement of the second inequality in (1).

Applying Lemma 2 to the function \( \varphi (v) = \| |A^v XB^{1-v} + A^{1-v}XB^v| | \) on the interval \([0, u]\) when \(0 < u \leq \frac{1}{2}\), and on the interval \([u, 1]\) when \(\frac{1}{2} \leq u < 1\), we obtain the following theorem.

**Theorem 3.** Let \(A, B, X \in M_n\) such that \(A\) and \(B\) are positive definite. Then

1. for \(0 \leq u \leq \frac{1}{2}\) and for every unitarily invariant norm,

\[
\| |A^{\frac{u}{2}} XB^{\frac{1-u}{2}} + A^{\frac{1-u}{2}}XB^{\frac{u}{2}}| | \leq \frac{1}{u} \int_0^u \| |A^v XB^{1-v} + A^{1-v}XB^v| | \, dv
\]

\[
\leq \frac{1}{2n} [(n-1)\| |AX + XB| | + 2\| |A^{\frac{u}{2}} XB^{\frac{1-u}{2}} + A^{\frac{1-u}{2}}XB^{\frac{u}{2}}| | + (n-1)\| |AX + XB| | + \| |AX + XB| | + \| |AX + XB| |] \),
\]

where \(n \geq 2\) is an integer,

2. for \(\frac{1}{2} \leq u \leq 1\) and for every unitarily invariant norm,

\[
\| |A^{\frac{1+u}{2}} XB^{\frac{1-u}{2}} + A^{\frac{1-u}{2}}XB^{\frac{1+u}{2}}| | \leq \frac{1}{1-u} \int_u^1 \| |A^v XB^{1-v} + A^{1-v}XB^v| | \, dv
\]

\[
\leq \frac{1}{2n} [(n-1)\| |AX + XB| | + 2\| |A^{\frac{1+u}{2}} XB^{\frac{1-u}{2}} + A^{\frac{1-u}{2}}XB^{\frac{1+u}{2}}| | + (n-1)\| |AX + XB| | + \| |AX + XB| | + \| |AX +XB| |] \),
\]

where \(n \geq 2\) is an integer.

**Proof.** The proof is similar to Theorem 1, so we omit it.

In view of the fact that the function \( \varphi (v) = \| |A^v XB^{1-v} + A^{1-v}XB^v| | \) is decreasing on the interval \([0, \frac{1}{2}]\) and increasing on the interval \([\frac{1}{2}, 1]\), by Theorem 3, we have the following result, which is a refinement of the second inequality in (1).
Corollary 2. Let $A, B, X \in M_n$ such that $A$ and $B$ are positive definite. Then

1. for $0 \leq u \leq \frac{1}{2}$ and for every unitarily invariant norm

$$
\begin{align*}
|||A^uXB^{1-u} + A^{1-u}XB^u||| \\
&\leq |||A^{\frac{u}{2}}XB^{1-\frac{u}{2}} + A^{1-\frac{u}{2}}XB^{\frac{u}{2}}||| \\
&\leq \frac{1}{u} \int_0^u |||A^vXB^{1-v} + A^{1-v}XB^v|||\,dv \\
&\leq \frac{1}{2n}[(n-1)|||AX + XB||| + 2|||A^{\frac{u}{2}}XB^{1-\frac{u}{2}} + A^{1-\frac{u}{2}}XB^{\frac{u}{2}}||| \\
&\quad + (n-1)|||A^uXB^{1-u} + A^{1-u}XB^u|||] \\
&\leq \frac{1}{2} (|||AX + XB||| + |||A^uXB^{1-u} + A^{1-u}XB^u|||) \\
&\leq |||AX + XB|||,
\end{align*}
$$

where $n \geq 2$ is an integer.

2. for $\frac{1}{2} \leq u \leq 1$ and for every unitarily invariant norm

$$
\begin{align*}
|||A^uXB^{1-u} + A^{1-u}XB^u||| \\
&\leq |||A^{\frac{1-u}{2}}XB^{\frac{1-u}{2}} + A^{\frac{1-u}{2}}XB^{\frac{1-u}{2}}||| \\
&\leq \frac{1}{1-u} \int_u^1 |||A^vXB^{1-v} + A^{1-v}XB^v|||\,dv \\
&\leq \frac{1}{2n}[(n-1)|||AX + XB||| + 2|||A^{\frac{1-u}{2}}XB^{\frac{1-u}{2}} + A^{\frac{1-u}{2}}XB^{\frac{1-u}{2}}||| \\
&\quad + (n-1)|||A^uXB^{1-u} + A^{1-u}XB^u|||] \\
&\leq \frac{1}{2} (|||AX + XB||| + |||A^uXB^{1-u} + A^{1-u}XB^u|||) \\
&\leq |||AX + XB|||,
\end{align*}
$$

where $n \geq 2$ is an integer.

It should be noticed that in the inequalities (7) to (10), we have

$$
\lim_{u \to 0} \frac{1}{u} \int_0^u |||A^vXB^{1-v} + A^{1-v}XB^v|||\,dv
= \lim_{u \to 1} \frac{1}{1-u} \int_u^1 |||A^vXB^{1-v} + A^{1-v}XB^v|||\,dv
= |||AX + XB|||.
$$
Remark 1. The three special values $n = 2$, $n = 16$ and $n = 4$ give the refinements of Heinz inequalities obtained in [4], [5] and [6], respectively.

Acknowledgments. This research was supported by Scientific Research Fund of Yunnan Provincial Education Department (No. 2013C157).

References


Accepted: 22.12.2014