ON 2-ABSORBING PRIMARY AND WEAKLY 2-ABSORBING ELEMENTS IN MULTIPLICATIVE LATTICES

Fethi Çallialp
Beykent University
Faculty of Science and Art
Ayazağa-Maslak, Istanbul
Turkey
e-mail: fethicallialp@beykent.edu.tr

Ece Yetkin
Unsal Tekir
Marmara University
Department of Mathematics
Ziverbey, Goztepe, 34722, Istanbul
Turkey
e-mails: yetkinece@gmail.com and
utekir@marmara.edu.tr

Abstract. In this paper, we introduce the concept of 2-absorbing primary and weakly 2-absorbing primary elements which are generalizations of primary and weakly primary elements in multiplicative lattices. Let $L$ be a multiplicative lattice. A proper element $q$ of $L$ is said to be a (weakly) 2-absorbing primary element of $L$ if whenever $a, b, c \in L$ with $(0 \neq abc \leq q)$ $abc \leq q$ implies either $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$. Some properties of 2-absorbing primary and weakly 2-absorbing primary elements are presented and relations among prime, primary, 2-absorbing, weakly 2-absorbing, 2-absorbing primary and weakly 2-absorbing primary elements are investigated. Furthermore, we determine 2-absorbing primary elements in some special lattices and give a new characterization for principal element domains in terms of 2-absorbing primary elements.

Keywords: prime element, primary element, 2-absorbing element, 2-absorbing primary element, multiplicative lattice.

1991 Mathematics Subject Classification: Primary 16F10; Secondary 16F05, 13A15.

1. Introduction

The concept of 2-absorbing ideal in a commutative ring with identity, which is a generalization of prime ideal, was introduced by Badawi in [7] and studied in [8], [12], and [1]. Various generalizations of prime ideals are also studied in
As a generalization of primary ideals the concept of 2-absorbing primary ideals and weakly 2-absorbing primary ideals are introduced in [9] and [10]. Our aim is to extend the concept of 2-absorbing primary ideals of commutative rings to 2-absorbing primary elements of non modular multiplicative lattices and give a characterization for principal element domains in terms of 2-absorbing primary elements.

A multiplicative lattice is a complete lattice $L$ with the least element $0$ and compact greatest element $1$, on which there is defined a commutative, associative, completely join distributive product for which $1$ is a multiplicative identity. An element $a$ of $L$ is said to be compact if whenever $a \leq \bigvee_{\alpha \in I_a} a_\alpha$ implies $a \leq \bigvee_{\alpha \in I_0} a_\alpha$ for some finite subset $I_0$ of $I$. By a C-lattice we mean a (not necessarily modular) multiplicative lattice which is generated under joins by a multiplicatively closed subset $C$ of compact elements. C-lattices can be localized. For any prime element $p$ of $L$, $L_p$ denotes the localization at $F = \{ x \in C \mid x \notin p \}$. For details on C-lattices and their localization theory, the reader is referred to [15] and [19]. We note that in a C-lattice, a finite product of compact elements is again compact. Throughout this paper, $L$ denotes a C-lattice and the set of all compact elements of $L$ is shown by $L_*$. An element $e \in L$ is said to be principal [13], if it satisfies the meet principal property (i) $a \land be = ((a : e) \land b)e$ and join principal property (ii) $(ae \lor b) : e = (b : e) \lor a$. A finite product of meet (join) principal elements of $L$ is again meet (join) principal from [13, Lemma 3.3 and Lemma 3.4].

If every element of $L$ is principal, then $L$ is called a principal element lattice. For more information about principal element lattices, the reader is referred to [3], [16] and [17]. $L$ is called a totally ordered lattice, if any two elements of $L$ are comparable. $L$ is said to be a Prüfer lattice if every compact element is principal.

An element $a \in L$ is said to be proper if $a < 1$. A proper element $p$ of $L$ (weakly, [4]) prime if $(0 \neq ab \leq p)$ $ab \leq p$ implies either $a \leq p$ or $b \leq p$. If $0$ is prime, then $L$ is said to be a domain. An element $m < 1$ in $L$ is said to be maximal if $m < x \leq 1$ implies $x = 1$. It can be easily shown that maximal elements are prime. A maximal element $m$ of $L$ is said to be simple, if there is no element $a \in L$ such that $m^2 < a < m$. $L$ is said to be quasi-local if it contains a unique maximal element. If $L = \{0, 1\}$, then $L$ is called a field. An element $a \in L$ is said to be a strong compact element if both $a$ and $a^\omega = \bigwedge_{n=1}^\infty a^n$ are compact elements of $L$. Strong compact elements have been studied in [16]. For $a \in L$, we define radical of $a$ as $\sqrt{a} = \wedge\{p \in L : p$ is prime and $a \leq p\}$. Note that in a C-lattice $L$, $\sqrt{a} = \wedge\{p \in L : p$ is prime and $a \leq p\} = \vee\{x \in L_* \mid x^n \leq a$ for some $n \in \mathbb{Z}^+\}$. (See also Theorem 3.6 of [21]). A proper element $q$ is said to be (weakly) primary if for every $a, b \in L$, $(0 \neq ab \leq q)$ $ab \leq q$ implies either $a \leq q$ or $b^n \leq q$ for some $n \in \mathbb{Z}^+$, [6]. If $q$ is primary and if $\sqrt{q} = p$ is a prime element, then $q$ is called a $p$-primary element. A principally generated C-lattice domain $L$ is said to be a Dedekind domain, if every element of $L$ is a finite product of prime elements of $L$.

Recall from [18] that a proper element $q$ of $L$ is called a (weakly) 2-absorbing element of $L$ if whenever $a, b, c \in L$ with $(0 \neq abc \leq q)$ $abc \leq q$, then $ab \leq q$ or
ac ≤ q or bc ≤ q. In this paper, we introduce the concepts of 2-absorbing primary and weakly 2-absorbing primary element which are generalizations of primary and weakly primary elements. A proper element \( q \) of \( L \) is said to be a (weakly) 2-absorbing primary element of \( L \) if whenever \( a, b, c \in L \) with \( 0 \neq abc \leq q \) \( abc \leq q \), then \( ab \leq q \) or \( ac \leq \sqrt{q} \) or \( bc \leq \sqrt{q} \).

Among many results in this paper, it is shown (Theorem 2.4) that the radical of a 2-absorbing primary element of \( L \) is a 2-absorbing element of \( L \). It is shown (Theorem 2.6) that if \( q_1 \) is a \( p_1 \)-primary element of \( L \) for some prime element \( p_1 \) of \( L \) and \( q_2 \) is a \( p_2 \)-primary element of \( L \) for some prime element \( p_2 \) of \( L \), then \( q_1q_2 \) and \( q_1 \wedge q_2 \) are 2-absorbing primary elements of \( L \). It is shown (Theorem 2.7) that if radical of \( q \) is primary, then \( q \) is a 2-absorbing primary element. 2-absorbing primary and weakly 2-absorbing primary elements of cartesian product of multiplicative lattices are presented (Theorem 2.20-2.24). A new characterization for principal element domains in terms of 2-absorbing primary elements is established (Theorem 3.30).

2. 2-absorbing primary and Weakly 2-absorbing primary elements

**Definition 2.1**

(1) A proper element \( q \) of \( L \) is called a 2-absorbing primary element of \( L \) if whenever \( a, b, c \in L \) and \( abc \leq q \), then \( ab \leq q \) or \( bc \leq \sqrt{q} \) or \( ac \leq \sqrt{q} \).

(2) A proper element \( q \) of \( L \) is called a weakly 2-absorbing primary element of \( L \) if whenever \( a, b, c \in L \) and \( 0 \neq abc \leq q \), then \( ab \leq q \) or \( ac \leq \sqrt{q} \) or \( bc \leq \sqrt{q} \).

The following theorem is obvious from the definitions, so the proof is omitted.

**Theorem 2.2** Let \( q \) be a proper element of \( L \). Then

(1) If \( q \) is a (weakly) prime element, then \( q \) is a (weakly) 2-absorbing primary element.

(2) If \( q \) is a (weakly) primary element, then \( q \) is a (weakly) 2-absorbing primary element.

(3) If \( q \) is a (weakly) 2-absorbing element, then \( q \) is a (weakly) 2-absorbing primary element.

(4) If \( q \) is a 2-absorbing primary element, then \( q \) is a weakly 2-absorbing primary element.

It is known from [Theorem 1, [15]] that if \( L \) is a Prüfer lattice and \( p \) is a prime element of \( L \), then \( p^n \) is \( p \)-primary element. Thus \( p^n \) is a 2-absorbing primary element of \( L \) for all \( n > 0 \).
Theorem 2.3

(1) An element \( q \in L \) is a 2-absorbing primary element if and only if for any \( a, b, c \in L^* \), \( abc \leq q \) implies either \( ab \leq q \) or \( bc \leq \sqrt{q} \) or \( ac \leq \sqrt{q} \).

(2) An element \( q \in L \) is a weakly 2-absorbing primary element if and only if for any \( a, b, c \in L^* \), \( 0 \neq abc \leq q \) implies either \( ab \leq q \) or \( bc \leq \sqrt{q} \) or \( ac \leq \sqrt{q} \).

Proof. (1) Assume that for any \( a, b, c \in L^* \), \( abc \leq q \) implies either \( ab \leq q \) or \( bc \leq \sqrt{q} \) or \( ac \leq \sqrt{q} \). Let \( a, b, c \in L \), \( abc \leq q \), \( bc \not\leq \sqrt{q} \) and \( ac \not\leq \sqrt{q} \). Then there exist compact elements \( a' \leq a \), \( b' \leq b \) and \( c' \leq c \) such that \( a'b'c' \leq q \). Since \( ac \not\leq \sqrt{q} \) and \( bc \not\leq \sqrt{q} \), there exist compact elements \( a_1 \leq a \), \( c_1 \leq c \), \( c_2 \leq c \) and \( b_1 \leq b \) such that \( a_1c_1 \not\leq \sqrt{q} \) and \( b_1c_2 \not\leq \sqrt{q} \). Put \( c_3 = c_1 \lor c_2 \lor c' \), \( a_2 = a_1 \lor a' \), \( b_2 = b_1 \lor b' \). We show that \( ab \leq q \). Choose compact elements \( a_\alpha \leq a \) and \( b_\alpha \leq b \). Then \((a_2 \lor a_\alpha)c_3(b_2 \lor b_\alpha) \leq q \), \((a_2 \lor a_\alpha)c_3 \not\leq \sqrt{q} \), \((b_2 \lor b_\alpha) \not\leq \sqrt{q} \) and hence by the hypothesis, \((a_2 \lor a_\alpha)(b_2 \lor b_\alpha) \leq q \). So \( a_\alpha b_\alpha \leq q \). Consequently, \( ab \leq q \). Therefore \( q \) is a 2-absorbing element of \( L \). The converse part is obvious.

(2) It can be easily shown similar to (1). □

Theorem 2.4 If \( q \) is a 2-absorbing primary element of \( L \), then \( \sqrt{q} \) is a 2-absorbing element of \( L \).

Proof. Let \( a, b, c \in L \) such that \( abc \leq \sqrt{q} \), \( ac \not\leq \sqrt{q} \) and \( bc \not\leq \sqrt{q} \). Since \( abc \leq \sqrt{q} \), there exists a positive integer \( n \) such that \( (abc)^n = a^n b^n c^n \leq q \). We obtain \( a^n c^n \not\leq \sqrt{q} \) and \( b^n c^n \not\leq \sqrt{q} \). Since \( q \) is 2-absorbing primary, we conclude that \( a^n b^n = (ab)^n \leq q \), and hence \( ab \leq \sqrt{q} \). Thus \( \sqrt{q} \) is a 2-absorbing element of \( L \). □

Theorem 2.5 Let \( q \) be a proper element of \( L \). Then \( \sqrt{q} \) is a (weakly) 2-absorbing element of \( L \) if and only if \( \sqrt{q} \) is a (weakly) 2-absorbing primary element of \( L \).

Proof. Since \( \sqrt{q} = \sqrt{q} \), the proof is clear. □

Theorem 2.6 If \( q \) is a 2-absorbing primary element of \( L \), then one of the following statements must hold.

(1) \( \sqrt{q} = p \) is a prime element,

(2) \( \sqrt{q} = p_1 \land p_2 \), where \( p_1 \) and \( p_2 \) are the only distinct prime elements of \( L \) that are minimal over \( q \).

Proof. Suppose that \( q \) is a 2-absorbing primary element of \( L \). Then \( \sqrt{q} \) is a 2-absorbing element by Theorem 2.4. Since \( \sqrt{\sqrt{q}} = \sqrt{q} \), the claim follows from Theorem 3 in [18]. □

Let \( q \) be a proper element of \( L \). It is known that if \( \sqrt{q} \) is a maximal element of \( L \), then \( q \) is a primary element of \( L \). The following theorem states that it is sufficient that if \( \sqrt{q} \) is a primary element of \( L \), then \( q \) is a 2-absorbing primary element of \( L \). Note that \( \sqrt{q} \) is a (weakly) prime element of \( L \) if and only if \( \sqrt{q} \) is a (weakly) primary element of \( L \) as \( \sqrt{q} = \sqrt{\sqrt{q}} \).
Theorem 2.7 Let \( q \) be a proper element of \( L \).

1. If \( \sqrt{q} \) is a primary element of \( L \), then \( q \) is a 2-absorbing primary element of \( L \).

2. If \( \sqrt{q} \) is a weakly primary element of \( L \), then \( q \) is a weakly 2-absorbing primary element of \( L \).

Proof. (1) Suppose that \( abc \leq q \) for some \( a, b, c \in L \) and \( ab \not\leq q \). Since \( (ac)(bc) = abc^2 \leq q \leq \sqrt{q} \) and \( \sqrt{q} \) is a primary element of \( L \), we have \( bc \leq \sqrt{q} \) or \( ac \leq \sqrt{q} \). Hence \( q \) is a 2-absorbing primary element of \( L \).

(2) Suppose that \( 0 \neq abc \leq q \) for some \( a, b, c \in L \) and \( ab \not\leq q \). Suppose that \( ab \not\leq \sqrt{q} \). Since \( \sqrt{q} \) is a weakly primary element of \( L \), we have \( c \leq \sqrt{q} \), and thus \( ac \leq \sqrt{q} \). Suppose that \( ab \leq \sqrt{q} \). Since \( 0 \neq abc \leq q \) and \( ab \leq \sqrt{q} \), we have \( 0 \neq ab \in \sqrt{q} \). Since \( \sqrt{q} \) is a weakly primary element of \( L \) and \( 0 \neq ab \leq \sqrt{q} \), we have \( a \leq \sqrt{q} \) or \( b \leq \sqrt{q} \). Thus \( ac \leq \sqrt{q} \) or \( bc \leq \sqrt{q} \). Thus \( q \) is a weakly 2-absorbing primary element of \( L \).

Definition 2.8 Let \( q \) be a 2-absorbing primary element of \( L \). Then \( p = \sqrt{q} \) is a 2-absorbing element by Theorem 2.2. We say that \( q \) is a \( p \)-2-absorbing primary element of \( L \).

Theorem 2.9 Let \( q_1 \) is a \( p_1 \)-primary element of \( L \) and \( q_2 \) is a \( p_2 \)-primary element of \( L \) for some prime elements \( p_1 \) and \( p_2 \) of \( L \). Then the following statements hold.

1. \( q_1 q_2 \) is a 2-absorbing primary element of \( L \).

2. \( q_1 \land q_2 \) is a 2-absorbing primary element of \( L \).

Proof. (1) Suppose that \( abc \leq q_1 q_2 \) for some \( a, b, c \in L \), \( ac \not\leq \sqrt{q_1 q_2} \), and \( bc \not\leq \sqrt{q_1 q_2} \). Then \( a, b, c \not\leq \sqrt{q_1 q_2} \). As \( \sqrt{q_1 q_2} = p_1 \land p_2 \), \( \sqrt{q_1 q_2} \) is a 2-absorbing element of \( L \) by [18]. Since \( ac, bc \not\leq \sqrt{q_1 q_2} \), we have \( ab \leq \sqrt{q_1 q_2} \). We show that \( ab \leq q_1 q_2 \). Since \( ab \leq \sqrt{q_1 q_2} \leq p_1 \), we may assume that \( a \leq p_1 \). Since \( a \not\leq \sqrt{q_1 q_2} = p_1 \land p_2 \) and \( ab \leq \sqrt{q_1 q_2} \leq p_2 \), we conclude that \( a \not\leq p_2 \) and \( b \leq p_2 \). Since \( b \leq p_2 \) and \( b \not\leq \sqrt{q_1 q_2} \), we have \( b \not\leq p_1 \). If \( a \leq q_1 \) and \( b \leq q_2 \), then \( ab \leq q_1 q_2 \), so we are done. Thus assume that \( a \not\leq q_1 \). Since \( q_1 \) is a \( p_1 \)-primary element of \( L \) and \( a \not\leq q_1 \), we have \( bc \leq p_1 \). Since \( b \leq p_2 \) and \( bc \leq p_1 \), we have \( bc \leq \sqrt{q_1 q_2} \), which is a contradiction. Thus \( a \leq q_1 \). Similarly, if \( b \not\leq q_2 \), we conclude \( ac \leq \sqrt{q_1 q_2} \), which is again a contradiction. So \( a \leq q_1 \) and \( b \leq q_2 \) and thus \( ab \leq q_1 q_2 \).

(2) Let \( q = q_1 \land q_2 \). Then \( \sqrt{q} = p_1 \land p_2 \) is a 2-absorbing element of \( L \). Suppose that \( abc \leq q \) for some \( a, b, c \in L \), \( ac \not\leq \sqrt{q} \), and \( bc \not\leq \sqrt{q} \). Then \( a, b, c \not\leq \sqrt{q} = p_1 \land p_2 \) and \( ab \leq \sqrt{q} \leq p_1 \). We show that \( ab \leq q \). Since \( ab \leq \sqrt{q} \leq p_1 \), we may assume that \( a \leq p_1 \). Since \( a \not\leq \sqrt{q} \) and \( ab \leq \sqrt{q} \leq p_2 \), we conclude that \( a \not\leq p_2 \) and \( b \leq p_2 \). Since \( b \leq p_2 \) and \( b \not\leq \sqrt{q} \), we get \( b \not\leq p_1 \). If \( a \leq q_1 \) and \( b \leq q_2 \), then \( ab \leq q \) and we are done. So suppose that \( a \not\leq q_1 \). Since \( q_1 \) is a \( p_1 \)-primary element of \( L \) and \( a \not\leq q_1 \), we have \( bc \leq p_1 \). Since \( b \leq p_2 \) and \( bc \leq p_1 \), we have \( bc \leq \sqrt{q} \),
a contradiction. Hence we have \( a \leq q_1 \). By the similar argument, we conclude \( a \leq q_1 \) and \( b \leq q_2 \). Thus \( ab \leq q \).

As a consequence of Theorem 2.9, we have the following corollary.

**Corollary 2.10** Let \( p_1, p_2 \) be prime elements of \( L \). If \( p_1^n \) is a \( p_1 \)-primary element of \( L \) and \( p_2^m \) is a \( p_2 \)-primary element of \( L \) for some positive integers \( n, m \), then \( p_1^n p_2^m \) and \( p_1^n \wedge p_2^m \) are \( 2 \)-absorbing primary elements of \( L \).

**Theorem 2.11** Let \( q_1, q_2, \ldots, q_n \) be \( p \)-\( 2 \)-absorbing primary elements of \( L \) for some \( 2 \)-absorbing element \( p \) of \( L \). Then \( q = \bigcap_{i=1}^{n} q_i \) is a \( p \)-\( 2 \)-absorbing primary element of \( L \).

**Proof.** Let \( a, b, c \in L \) with \( abc \leq q \). Suppose that \( ab \not\leq q \). Then \( ab \not\leq q_i \) for some \( i \in \{1, 2, \ldots, n\} \). It implies either \( bc \leq \sqrt{q_i} = p \) or \( ac \leq \sqrt{q_i} = p \). Since \( \sqrt{q} = \bigcap_{i=1}^{n} \sqrt{q_i} = p \), we are done.

**Definition 2.12** Let \( q \) be a weakly \( 2 \)-absorbing primary element of \( L \). We say \( (a, b, c) \) is a triple-zero of \( q \) if \( abc = 0 \), \( ab \not\leq q \), \( bc \not\leq \sqrt{q} \), and \( ac \not\leq \sqrt{q} \).

Note that if \( q \) is a weakly \( 2 \)-absorbing primary element of \( L \) that is not \( 2 \)-absorbing primary, then there exists a triple-zero \( (a, b, c) \) of \( q \) for some \( a, b, c \in L \).

**Theorem 2.13** Let \( q \) be a weakly \( 2 \)-absorbing primary element of \( L \) and suppose that \( (a, b, c) \) is a triple-zero of \( q \) for some \( a, b, c \in L \). Then

1. \( abq = bcq = acq = 0 \),
2. \( aq^2 = bq^2 = cq^2 = 0 \).

**Proof.** (1) Suppose that \( abq \neq 0 \). Then there exists a compact element \( x \leq q \) such that \( abx \neq 0 \). Hence \( 0 \neq ab(c \lor x) \leq q \). Since \( ab \not\leq q \) and \( q \) is weakly \( 2 \)-absorbing primary, we have \( a(c \lor x) \leq \sqrt{q} \) or \( b(c \lor x) \leq \sqrt{q} \). So \( ac \leq \sqrt{q} \) or \( bc \leq \sqrt{q} \), a contradiction. Thus \( abx = 0 \), and so \( abq = 0 \). Similarly, it can be easily verified that \( bcq = acq = 0 \).

(2) Suppose that \( aq_1 q_2 \neq 0 \) for some compact elements \( q_1, q_2 \leq q \). Hence from (1) we have \( 0 \neq a(b \lor q_1)(c \lor q_2) = aq_1 q_2 \leq q \). It implies either \( a(b \lor q_1) \leq q \) or \( a(c \lor q_2) \leq \sqrt{q} \) or \( (b \lor q_1)(c \lor q_2) \leq \sqrt{q} \). Thus \( ab \leq q \) or \( ac \leq \sqrt{q} \) or \( bc \leq \sqrt{q} \), a contradiction. Therefore \( aq^2 = 0 \). Similarly, one can easily show that \( bq^2 = cq^2 = 0 \).

**Theorem 2.14** If \( q \) is a weakly \( 2 \)-absorbing primary element of \( L \) that is not \( 2 \)-absorbing primary, then \( q^3 = 0 \).
Proof. Suppose that $q$ is a weakly 2-absorbing primary element that is not a 2-absorbing primary element of $L$. Then there exists $(a, b, c)$ a triple-zero of $q$ for some $a, b, c \in L$. Assume that $q^3 \neq 0$. Hence $q_1q_2q_3 \neq 0$, for some compact elements $q_1, q_2, q_3 \leq q$. By Theorem 2.13, we obtain $(a \lor q_1)(b \lor q_2)(c \lor q_3) = q_1q_2q_3 \neq 0$. This implies that $(a \lor q_1)(b \lor q_2) \leq q$ or $(a \lor q_1)(c \lor q_3) \leq \sqrt{q}$ or $(b \lor q_2)(c \lor q_3) \leq \sqrt{q}$. Thus we have $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$, a contradiction. Thus $q^3 = 0$. 

Corollary 2.15 If $q$ is a weakly 2-absorbing primary element of $L$ that is not 2-absorbing primary, then $\sqrt{q} = \sqrt{0}$.

Theorem 2.16 Let $q_1, q_2, \ldots, q_n$ be weakly 2-absorbing primary elements of $L$ that are not 2-absorbing primary. Then $q = \bigwedge_{i=1}^n q_i$ is a weakly 2-absorbing primary element of $L$.

Proof. Since $q_i$’s are weakly 2-absorbing primary that are not 2-absorbing primary, we get $\sqrt{q_i} = \sqrt{0}$ for each $1 \leq i \leq n$ by Corollary 2.15. So the result is obtained easily similar to the argument in the proof of Theorem 2.11.

Theorem 2.17 Suppose that $0$ has a triple-zero $(a, b, c)$ for some $a, b, c \in L$ such that $ab \not\leq \sqrt{0}$. Let $q$ be a weakly 2-absorbing primary element of $L$. Then $q$ is not a 2-absorbing primary element of $L$ if and only if $q \leq \sqrt{0}$.

Proof. Suppose that $q$ is not a 2-absorbing primary element of $L$. Then $q \leq \sqrt{0}$ by Corollary 2.15. Conversely, suppose that $q \leq \sqrt{0}$. By hypothesis, we conclude that $ab \not\leq q$, $ac \not\leq \sqrt{0}$, and $bc \not\leq \sqrt{0}$. Thus $(a, b, c)$ is a triple-zero of $q$. Hence $q$ is not a 2-absorbing primary element of $L$.

Recall that $L$ is said to be reduced if $\sqrt{0} = 0$.

Corollary 2.18 Let $L$ be a reduced lattice and $q \neq 0$ be a proper element of $L$. Then $q$ is a weakly 2-absorbing primary element if and only if $q$ is a 2-absorbing primary element of $L$.

Theorem 2.19 Let $m$ be a maximal element of $L$ and $q$ be a proper element of $L$. If $q$ is a 2-absorbing primary element of $L$, then $q_m$ is a 2-absorbing primary element of $L_m$.

Proof. Let $a, b, c \in L$ such that $a_n b_m c_m \leq q_m$. Then $abc \leq q_m$, so $uabc \leq q$ for some $u \not\leq m$. Hence we get either $uabc \leq q$ or $bc \leq \sqrt{q}$ or $uac \leq \sqrt{q}$. Since $(\sqrt{q})_m = \sqrt[q_m]{}$ by [15], and $u_m = 1_m$, it follows either $a_n b_m \leq q_m$ or $b_m c_m \leq \sqrt[q_m]{q_m}$ or $a_n c_m \leq \sqrt[q_m]{q_m}$.

Recall that for any $a \in L$, $L/a = \{b \in L : a \leq b\}$ is a multiplicative lattice with multiplication $c \circ d = cd \lor a$. For more details, the reader is referred to [2].
Lemma 1 Let $a$ and $q$ be proper elements of $L$ with $a \leq q$. If $q$ is a 2-absorbing primary element of $L$, then $\overline{q}$ is a weakly 2-absorbing primary element of $L/a$.

Proof. The proof is clear. \hfill \blacksquare

Theorem 2.20 Let $L = L_1 \times L_2$, where $L_1$ and $L_2$ are $C$-lattices. Then a proper element $q$ is a 2-absorbing primary element of $L$ if and only if it has one of the following three forms.

1. $q = (q_1, 1_{L_2})$ for some 2-absorbing primary element $q_1$ of $L_1$,
2. $q = (1_{L_1}, q_2)$ for some 2-absorbing primary element $q_2$ of $L_2$,
3. $q = (q_1, q_2)$ for some primary element $q_1$ of $L_1$ and some primary element $q_2$ of $L_2$.

Proof. If $q = (q_1, 1_{L_2})$ for some 2-absorbing primary element $q_1$ of $L_1$ or $q = (1_{L_1}, q_2)$ for some 2-absorbing primary element $q_2$ of $L_2$, then it is clear that $q$ is a 2-absorbing primary element of $L$. Hence assume that $q = (q_1, q_2)$ for some primary element $q_1$ of $L_1$ and some primary element $q_2$ of $L_2$. Then $q_1' = (q_1, 1_{L_2})$ and $q_2' = (1_{L_1}, q_2)$ are primary elements of $L$. Hence $q_1' \land q_2' = (q_1, q_2) = q$ is a 2-absorbing primary element of $L$ by Theorem 2.9.

Conversely, suppose that $q$ is a 2-absorbing primary element of $L$. Then $q = (q_1, q_2)$ for some element $q_1$ of $L_1$ and some element $q_2$ of $L_2$. Suppose that $q_2 = 1_{L_2}$. Since $q$ is a proper element of $L$, $q_1 \neq 1_{L_1}$. Let $L' = L/\{0\} \times L_2$. Then $\overline{q} = (q_1, 1_{L_2})$ is a 2-absorbing primary element of $L'$ by Lemma 1. Now, we show that $q_1$ is a 2-absorbing primary element of $L_1$. Let $abc \leq q_1$ for some $a, b, c \in L_1$. Hence $(a, 1_{L_2})(b, 1_{L_2})(c, 1_{L_2}) = (abc, 1_{L_2}) \leq \overline{q}$, which implies that $(a, 1_{L_2})(b, 1_{L_2}) \leq \overline{q}$ or $(b, 1_{L_2})(c, 1_{L_2}) \leq \sqrt{q}$ or $(a, 1_{L_2})(c, 1_{L_2}) \leq \sqrt{q}$. It means that either $ab \leq q_1$ or $bc \leq \sqrt{q_1}$ or $ac \leq \sqrt{q_1}$. Thus $q_1$ is a 2-absorbing primary element of $L_1$.

If $q_1 = 1_{L_1}$, then $q_2$ can be obtained as a 2-absorbing primary element of $L_2$ by the similar way. Hence assume that $q_1 \neq 1_{L_1}$ and $q_2 \neq 1_{L_2}$. Then $\sqrt{q} = (\sqrt{q_1}, \sqrt{q_2})$. On the contrary, suppose that $q_1$ is not a primary element of $L_1$. Then there are $a, b \in L_1$ such that $ab \leq q_1$ but neither $a \leq q_1$ nor $b \leq q_1$. Let $x = (a, 1), y = (1, 0),$ and $z = (b, 1)$. Then $xyz = (ab, 0) \leq q$ implies that either $xy = (a, 0) \leq q$ and $xz = (ab, 1) \leq \sqrt{q}$ and $yz = (b, 0) \leq \sqrt{q}$, a contradiction. Therefore $q_1$ is a primary element of $L_1$. Similarly it can be easily seen that $q_2$ is a primary element of $L_2$, as needed. \hfill \blacksquare

Theorem 2.21 Let $L_1$ and $L_2$ be $C$-lattices, $q$ be a proper element of $L_1$, and $L = L_1 \times L_2$. Then the following statements are equivalent.

1. $(q, 1_{L_2})$ is a weakly 2-absorbing primary element of $L$.
2. $(q, 1_{L_2})$ is a 2-absorbing primary element of $L$.
3. $q$ is a 2-absorbing primary element of $L_1$. 

270 F. ÇALLIALP, E. YETKIN, U. TEKIR
Let $q \neq 0$, that $q$ is a 2-absorbing primary element of $L$. Thus $a, b, c$ exist such that $a, b, c, 1L = (1L)$. Hence we have \( (a, 1L)(b, 1L) = (ab, 1L) \) or \( (a, 1L)(c, 1L) = (ac, 1L) \). It follows that $ab \leq q$ or $bc \leq \sqrt{q}$, a contradiction. Thus $q$ is a 2-absorbing primary element of $L$.

(3)⇒(1) Let $q$ be a 2-absorbing primary element of $L$. Then it can be easily shown that $(q, 1L)$ is a 2-absorbing primary element of $L$, therefore (1) holds.

**Theorem 2.22** Let $L_1$ and $L_2$ be $C$-lattices, $q_1, q_2$ be nonzero elements of $L_1$ and $L_2$, respectively, and let $L = L_1 \times L_2$. If $(q_1, q_2)$ is a proper element of $L$, then the following statements are equivalent.

1. $(q_1, q_2)$ is a weakly 2-absorbing primary element of $L$.
2. $q_1 = 1L_1$ and $q_2$ is a 2-absorbing primary element of $L_1$ or $q_2 = 1L_2$ and $q_1$ is a 2-absorbing primary element of $L_1$ or $q_1, q_2$ are primary elements of $L_1$ and $L_2$, respectively.
3. $(q_1, q_2)$ is a 2-absorbing primary element of $L$.

**Proof.** (1)⇒(2) Assume that $(q_1, q_2)$ is a weakly 2-absorbing primary element of $L$. If $q_1 = 1L_1$ $(q_2 = 1L_2)$, then $q_2$ is a 2-absorbing primary element of $L_2$ $(q_1$ is a 2-absorbing primary element of $L_1)$ by Theorem 2.21. So we may assume that $q_1 \neq 1L_1$ and $q_2 \neq 1L_2$. Let $a, b \in L_2$ such that $ab \leq q_2$ and let $x \in L_2$ with $0 \neq x \leq q_1$. Then $0 \neq (x, 1)(1, a)(1, b) = (x, ab) \leq (q_1, q_2)$. Since $q_1$ is proper, $(1, a)(1, b) = (1, ab) \leq \sqrt{(q_1, q_2)}$. Hence we have $(x, 1)(1, a) = (x, a) \leq (q_1, q_2)$. Thus $q_2$ is a primary element of $L_2$. Similarly, it can be easily shown that $q_1$ is a primary element of $L_1$.

(2)⇒(3) The proof is by Theorem 2.20.

(3)⇒(1) It is clear.
Proof. Suppose that $q$ is a nonzero weakly 2-absorbing primary element of $L$ that is not 2-absorbing primary element. Then $q = (q_1, q_2)$ for some elements $q_1$, $q_2$ of $L_1$ and $L_2$ respectively. Assume that $q_1 \neq 0$ and $q_2 \neq 0$. Then $q$ is a 2-absorbing primary element of $L$ by Theorem 2.22, a contradiction. Therefore $q_1 = 0$ or $q_2 = 0$. Without loss of generality we may assume that $q_2 = 0$. We show that $q_2 = 0$ is a primary element of $L$. Let $a, b \in L$ such that $ab \leq q_2$, and let $x \in L$ such that $0 \neq x \leq q_1$. Since $0 \neq (x, 1)(1, a)(b, 1) = (x, ab) \leq q$ and $(1, a)(1, b) = (1, ab) \notin \sqrt{q}$, we obtain $(x, a) = (x, 1)(1, a) \leq q$ or $(x, b) = (x, 1)(1, b) \leq \sqrt{q}$, and so $a \leq q_2$ or $b \leq \sqrt{q_2}$. Thus $q_2 = 0$ is a primary element of $L$. Next, we show that $q_1$ is a weakly primary element of $L$. Let $0 \neq ab \leq q_1$, for some $a, b \in L_1$. Since $0 \neq (a, 1)(b, 1)(1, 0) \leq (q_1, 0)$ and $(ab, 1) \notin (q_1, 0)$, we conclude $(a, 0) = (a, 1)(1, 0) \leq \sqrt{(q_1, 0)} = \sqrt{q}$ or $(b, 0) = (b, 1)(1, 0) \leq \sqrt{(q_1, 0)} = \sqrt{q}$. Thus $a \leq q_1$ or $b \leq \sqrt{q_1}$, and therefore $q_1$ is a weakly primary element of $L$.

Now, we show that $q_1$ is not primary. Suppose that $q_1$ is a primary element of $L$. Since $q_2 = 0$ is a primary element of $L_2$, we conclude that $q = (q_1, q_2)$ is a 2-absorbing primary element of $L$ by Theorem 2.20, a contradiction. Thus $q_1$ is a weakly primary element of $L_1$ that is not primary.

Conversely, suppose that (1) holds. Assume that $(0, 0) \neq (a, a')(b, b')(c, c') \leq q = (q_1, 0)$. Since $a'b'c' = 0$ and $(0, 0) \neq (a, a')(b, b')(c, c') \leq (q_1, 0)$, we conclude that $abc \neq 0$. Assume $(a, a')(b, b') \notin q$. We consider three cases.

Case one: Suppose that $ab \notin q_1$, but $a'b' = 0$. Since $q_1$ is a weakly primary element of $L_1$, we have $c \leq \sqrt{q_1}$. Since $q_2 = 0$ is a primary element of $L_2$, we have $a' = 0$ or $b' \leq \sqrt{q_2}$. Thus $(a, a')(c, c') \leq \sqrt{q}$ or $(b, b')(c, c') \leq \sqrt{q}$.

Case two: Suppose that $ab \notin q_1$ and $a'b' \neq 0$. Then $(c, c') \leq (\sqrt{q_1}, \sqrt{0}) = \sqrt{q}$. Thus $(a, a')(c, c') \leq \sqrt{q}$ or $(b, b')(c, c') \leq \sqrt{q}$.

Case three: Suppose that $ab \leq q_1$, but $a'b' \neq 0$. Since $0 \neq ab \leq q_1$ and $q_1$ is a weakly primary element of $L_1$, we have $a \leq q_1$ or $b \leq \sqrt{q_1}$. Since $a'b' \neq 0$ and $q_2 = 0$ is a primary element of $L_2$, we have $c' \leq \sqrt{q_2}$. Thus $(a, a')(c, c') \leq \sqrt{q}$ or $(b, b')(c, c') \leq \sqrt{q}$. Hence $q$ is a weakly 2-absorbing primary element of $L$. Since $q_1$ is not a primary element of $L_1$, $q$ is not a 2-absorbing primary element of $L$ by Theorem 2.22.

Theorem 2.24 Let $L = L_1 \times L_2 \times \ldots \times L_n$, where $2 < n < \infty$, and $L_1, L_2, \ldots, L_n$ are $C$-lattices and let $q$ be a nonzero proper element of $L$. Then the following statements are equivalent.

$(1)$ $q$ is a weakly 2-absorbing primary element of $L$.

$(2)$ $q$ is a 2-absorbing primary element of $L$.

$(3)$ Either $q = (q_t)_{t=1}^n$ such that for some $k \in \{1, 2, \ldots, n\}$, $q_k$ is a 2-absorbing primary element of $L_k$, and $q_t = 1_{L_t}$ for every $t \in \{1, 2, \ldots, n\} \setminus \{k\}$ or $q = (q_t)_{t=1}^n$ such that for some $k, m \in \{1, 2, \ldots, n\}$, $q_k$ is a primary element of $L_k$, $q_m$ is a primary element of $L_m$, and $q_t = 1_{L_t}$ for every $t \in \{1, 2, \ldots, n\} \setminus \{k, m\}$.
Proof. (1) $\Leftrightarrow$ (2) Since $q$ is a proper element of $L$, we have $q = (q_1, \ldots, q_n)$, where every $q_i$'s are element of $L_i$, and $q_j \neq 1_{L_j}$ for some $j \in \{1, \ldots, n\}$. Suppose that $q = (q_1, q_2, \ldots, q_n) \neq 0$ is a weakly 2-absorbing primary element of $L$. Then there is a compact element $0 \neq (a_1, a_2, \ldots, a_n) \leq q$. Hence $0 \neq (a_1, a_2, \ldots, a_n) = (a_11, 1, 1, \ldots, 1)(a_2, 1, 1, \ldots, 1) \cdots (1, 1, \ldots, a_n) \leq q$ implies there is a $j \in \{1, \ldots, n\}$ such that $b_j = 1_{L_j}$ and $(b_1, \ldots, b_n) \leq \sqrt{q} = (\sqrt{q_1}, \ldots, \sqrt{q_n})$, where $b_1, \ldots, b_n \in \{a_1, \ldots, a_n\}$. Hence $\sqrt{b_j} = 1_{L_j}$, and so $q_j = 1_{L_j}$. Thus $\sqrt{q} \neq 0$, and hence by Corollary 2.15, $q$ is a 2-absorbing primary element. The converse is obvious.

(2) $\Leftrightarrow$ (3) We use induction on $n$. If $n = 2$, then we are done by Theorem 2.22. Hence let $3 \leq n < \infty$ and assume that the result is satisfied when $S = L_1 \times \cdots \times L_{n-1}$. Thus $L = S \times L_n$. Theorem 2.22 implies that $q$ is a 2-absorbing primary element of $L$ if and only if either $q = (s, 1_{L_n})$ for some 2-absorbing primary element $s$ of $S$ or $q = (1_s, t)$ for some 2-absorbing primary element $t$ of $L_n$ or $q = (s, t)$ for some primary element $s$ of $S$ and some primary element $t$ of $L_n$. Since a proper element $s$ of $S$ is a primary element of $S$ if and only if $s = (q_k)^{i-1}$ such that for some $k \in \{1, 2, \ldots, n - 1\}$, we conclude that $q_k$ is a primary element of $L_k$, and $q_t = 1_{L_t}$ for every $t \in \{1, 2, \ldots, n - 1\} \setminus \{k\}$. So this completes the proof of the theorem.

\section{2-absorbing primary elements in some special lattices}

\textbf{Theorem 3.25} Suppose that $\sqrt{0}$ is a prime (primary) element of $L$. Let $q$ be a proper element of $L$. Then $q$ is a weakly 2-absorbing primary element of $L$ if and only if $q$ is a 2-absorbing primary element of $L$.

\textbf{Proof.} Suppose that $q$ is a weakly 2-absorbing primary element of $L$. Assume that $abc \leq q$ for some $a, b, c \in L$. If $0 \neq abc \leq q$, then $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$. Hence assume that $abc = 0$ and $ab \notin q$. Since $abc = 0 \leq \sqrt{0}$ and $\sqrt{0}$ is a prime element of $L$, we conclude that $a \leq \sqrt{0}$ or $b \leq \sqrt{0}$ or $c \leq \sqrt{0}$. Since $\sqrt{0} \leq \sqrt{q}$, we conclude that $ac \leq \sqrt{0} \leq \sqrt{q}$ or $bc \in \sqrt{0} \leq \sqrt{q}$. Thus $q$ is a 2-absorbing primary element of $L$. The converse is clear.

Recall that $L$ is called \textit{quasilocal} if it has exactly one maximal element.

\textbf{Theorem 3.26} Let $L$ be a quasilocal lattice with maximal element $\sqrt{0}$. The following statements hold.

\begin{enumerate}
\item Every element of $L$ is a weakly 2-absorbing primary element of $L$.
\item A proper element $q$ of $L$ is a weakly 2-absorbing primary element if and only if $q$ is a 2-absorbing primary element.
\end{enumerate}

\textbf{Proof.} It is obvious by Theorem 3.25.

\textbf{Theorem 3.27} Let $L_1, L_2$ and $L_3$ be $C$-lattices and let $L = L_1 \times L_2 \times L_3$. Then every proper element of $L$ is a weakly 2-absorbing primary element of $L$ if and only if $L_1$, $L_2$ and $L_3$ are fields.
Proof. Suppose that every proper element of $L$ is a weakly 2-absorbing primary element of $L$. Without loss of generality, we may assume that $L_1$ is not a field. Then there exists a nonzero proper element $q$ of $L_1$. Thus $a = (q, 0, 0)$ is a weakly 2-absorbing primary element of $L$, which contradicts with Theorem 2.24.

Conversely, suppose that $L_1$, $L_2$, $L_3$ are fields. Then every nonzero proper element of $L$ is a 2-absorbing element by Theorem 2.24. Since 0 is always weakly 2-absorbing primary, the proof is completed.

Theorem 3.28 Suppose that every proper element of $L$ is a weakly 2-absorbing primary element. Then $L$ has at most three incomparable prime elements.

Proof. Assume that there are $p_1, p_2, p_3$ and $p_4$ incomparable prime elements of $L$. Let $q = p_1 \land p_2 \land p_3$. Hence $\sqrt{q} = \sqrt{p_1} \land \sqrt{p_2} \land \sqrt{p_3}$. Thus $\sqrt{q}$ is not a 2-absorbing element of $L$ by Theorem 2.6. So $q$ is not a 2-absorbing primary element of $L$ by Theorem 2.2. Hence $q^3 = 0$ by Theorem 2.14. Thus $q^3 = p_1^3p_2^3p_3^3 = 0 < p_4$ implies that $p_1 < p_4$ or $p_2 < p_4$ or $p_3 < p_4$, a contradiction. Thus $L$ has at most three incomparable prime elements.

In view of Theorem 3.28, we have the following result.

Corollary 3.29 Suppose that every proper element of $L$ is a weakly 2-absorbing primary element. Then $L$ has at most three maximal elements.

Theorem 3.30 Let $L$ be a principally generated domain that is not a field. Then the following statements are equivalent.

(1) $L$ is a principal element domain.

(2) Every maximal element is strong compact and a nonzero proper element $q$ of $L$ is a 2-absorbing primary element of $L$ if and only if either $q = m^n$ for some maximal element $m$ of $L$ and some positive integer $n$ or $q = m_1^n m_2^n$ for some maximal elements $m_1, m_2$ of $L$ and some positive integers $n, k$.

(3) Every maximal element is strong compact and a nonzero proper element $q$ of $L$ is a 2-absorbing primary element of $L$ if and only if either $q = p^n$ for some prime element $p$ of $L$ and some positive integer $n$ or $q = p_1^n p_2^n$ for some prime elements $p_1, p_2$ of $L$ and some positive integers $n, k$.

Proof. (1) $\Rightarrow$ (2). Let $L$ be a principal element domain. Then every maximal element is strong compact by [16, Theorem 2]. Suppose $q$ is a nonzero 2-absorbing primary element of $L$ that is not maximal. Then $q = m_1^{n_1} m_2^{n_2} \cdots m_k^{n_k}$ for some distinct maximal elements $m_1, ..., m_k$ of $L$ and some integers $n_1, ..., n_k \geq 1$. Since every nonzero prime element of $L$ is maximal and $\sqrt{q}$ is either a maximal element of $L$ or $q_1 \land q_2$ for some maximal elements $q_1, q_2$ of $L$ by Theorem 2.6, we conclude that either $q = m^n$ for some maximal element $m$ of $L$ and some $n \geq 1$ or $q = m_1^n m_2^n$ for some maximal elements $m_1, m_2$ of $L$ and some $n, m \geq 1$. Conversely, suppose that $q = m^n$ for some maximal element $m$ of $L$ and some positive integer $n \geq 1$. 

\begin{align*}
\sqrt{q} &= \sqrt{m^n} \\
&= \sqrt{m} \land \sqrt{m} \cdots \sqrt{m} \\
&= m_1^n m_2^n \\
&= q
\end{align*}
or \( q = m_1^n m_2^k \) for some maximal elements \( m_1, m_2 \) of \( L \) and some integers \( n, k \geq 1 \). Then \( q \) is a 2-absorbing primary element of \( L \) by Theorem 2.9 and Corollary 2.10. (2)\( \Rightarrow \) (3) It is clear.

(3)\( \Rightarrow \) (1) Suppose that \( m \) is a maximal element of \( L \) and \( q \in L \) with \( m^2 \leq q \leq m \). Then \( q \) is an \( m \)-primary element. Hence \( q \) is a 2-absorbing primary element. From the hypothesis (3), either \( q = m \) or \( q = m^2 \), so there is no element \( a \in L \) such that \( m^2 < a < m \) which shows that \( m \) is simple. Therefore, by [16, Theorem 2], \( L \) is a principal element domain.

Suppose that \( L \) is principally generated. Then \( L \) is a Dedekind domain if and only if \( L \) is a principal element lattice by Theorem 2.7 in \([3]\). So we have the following result as a consequence of Theorem 3.30.

**Corollary 3.31** Let \( L \) be a principally generated domain. If \( L \) is a Dedekind domain, then \( 1_L \neq q \in L \) is 2-absorbing primary if and only if \( q = p^n \) for some prime element \( p \) of \( L \), a positive integer \( n \) or \( q = p_1^n p_2^m \) for some prime elements \( p_1, p_2 \) of \( L \), some positive integers \( n, m \).

**Acknowledgements.** This work is supported by the Scientific Research Project Program of Marmara (BAPKO).

**References**


Accepted: 06.12.2014