

## ENGEL FUZZY SUBGROUPS

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**Abstract.** In this paper we introduce and study Engel fuzzy subgroups. We will proceed by introduce and study soluble and nilpotent fuzzy subgroups. In particular, we show that if  $x \in L_3(\mu)$  and  $\mu(x^{p^n}) = \mu(e)$  for some integer  $n \geq 2$ , then  $\mu$  is fuzzy soluble.

*Keywords:* fuzzy subgroups, nilpotent fuzzy subgroup, Engel fuzzy subgroup, soluble fuzzy subgroup.

### 1. Introduction

Let  $G$  be an arbitrary group and  $x, y \in G$ . Define inductively the  $n$ -commutator

$$[x, {}_0 y] = x, [x, {}_1 y] = x^{-1}y^{-1}xy$$

and, for all  $n > 0$ ,

$$[x, {}_n y] = [[x, {}_{n-1} y], y].$$

**Definition 1.1.** A group  $G$  is called an *Engel group* if for each ordered pair  $(x, y)$  of elements in  $G$  there exists positive integer  $n(x, y)$  such that  $[x, {}_n y] = e$ ;  $[x, y] = x^{-1}y^{-1}xy$ .

Suppose  $n = n(x, y)$  can be chosen independently of  $x, y$  then we say that  $G$  is an  $n$ -Engel group. In this definition we have used bracket from the left. But since

$$[y, {}_n x] = [{}_n x^{-1}, y]^{x^n},$$

it does not matter whether we use bracketing from the right or from the left. If  $n = 1$  then the 1-Engel group is abelian. Levi [3] proved that a group  $G$  is a 2-Engel group if and only if the normal closure  $x^G$  of arbitrary element  $x$  is abelian. Furthermore, we have 2-Engel groups are nilpotent of class at most 3. Also, he has shown that every group of exponent 3 is a 2-Engel group. Heinken [6] shown that every 3-Engel group  $G$  is nilpotent of class at most 4 if  $G$  has no element of orders 2, or 5. L. Kappe and W. Kappe [7] gave a characterization for 3-Engel groups which is analogous to Levi's theorem on 2-Engel groups. They shown that the following are equivalent:

- 1)  $G$  is a 3-Engel group.
- 2)  $x^G$  is a 2-Engel group for all  $x \in G$ .
- 3) for all  $x \in G$ ,  $x^G$  is nilpotent of class at most 2.

We do not have a corresponding characterization for 4-Engel groups. Traustason [8] studied 4-Engel groups. The origin of Engel groups lies in the theory of lie algebras. In fact, they are a group theoretic analog of Engel lie algebras. According to Engel's theorem every finite dimensional Engel lie algebra over a field is nilpotent. In 1936 Zorn [5] proved a corresponding theorem for Engel groups.

**Zorn's Theorem.** *A finite Engel group is nilpotent.*

**Definition 1.2.** [9] Let  $\mu$  be a fuzzy subset of a semigroup  $G$ . Let

$$Z(\mu) = \{x \in G; \mu(xy) = \mu(yx) \text{ and } \mu(xyz) = \mu(yxz) \text{ for all } y, z \in G\}.$$

Then  $\mu$  is called *commutative* in  $G$  if  $Z(\mu) = G$ .

We recall the notion of the ascending central series of a fuzzy subgroup and a nilpotent fuzzy subgroup of a group [9]. Let  $\mu$  be a fuzzy subgroup of a group  $G$ . Let  $Z^0(\mu) = \{e\}$  and  $\pi_0$  be the natural homomorphism of  $G$  onto  $G/Z^0(\mu)$ . Suppose that  $Z^i(\mu)$  has been defined and that  $Z^i(\mu)$  is a normal subgroup of  $G$  for  $i \in N \cup \{0\}$ . Let  $\pi_i$  be the natural homomorphism of  $G$  onto  $G/Z^i(\mu)$ . Define  $Z^{i+1}(\mu) = \pi_i^{-1}(Z(\pi_i(\mu)))$ . Then  $Z^{i+1}(\mu) \supseteq Ker(\pi_i) = Z^i(\mu)$  for  $i = 0, 1, \dots$ . The normality of  $Z^{i+1}(\mu)$  in  $G$  is proved.

**Definition 1.3.** Let  $\mu$  be a fuzzy subgroup of a group  $G$ . The *ascending central series* of  $\mu$  is defined to be the ascending chain of normal subgroups of  $G$ ,

$$Z^0(\mu) \subseteq Z^1(\mu) \subseteq \dots$$

**Definition 1.4.** A fuzzy subgroup  $\mu$  of a group  $G$  is called *nilpotent* if there exist a nonnegative integer such that  $Z^m(\mu) = G$ . The smallest such integer is called the class of  $\mu$ .

In [9] we have the following main results, that will be used throughout this paper.

**Theorem 1.5.** Let  $\mu$  be a fuzzy subgroup of a group  $G$ . If  $G$  is nilpotent of class  $m$ , then  $\mu$  is nilpotent of class  $n$  for some nonnegative integer  $n \leq m$ .

**Theorem 1.6.** Let  $\lambda$  be a nilpotent fuzzy subgroup of  $G$ . If  $\mu$  is a fuzzy subgroup of  $G$  such that  $\mu \subseteq \lambda$ , then  $\mu$  is nilpotent.

**Theorem 1.7.** Let  $\mu$  be a fuzzy subgroup of a group  $G$ . Then for all  $x, y \in G$ ,  $\mu(x) \neq \mu(y)$  implies  $\mu(xy) = \mu(x) \wedge \mu(y)$ .

**Theorem 1.8.** Let  $\mu$  be a fuzzy subgroup of a group  $G$ . Let  $i \in \mathbb{N}$ . If  $xyx^{-1}y^{-1} \in Z^{i-1}(\mu)$  for all  $y \in G$ , then  $x \in Z^i(\mu)$ .

**Theorem 1.9.** Let  $\mu$  be a fuzzy subgroup of a group  $G$ . Let  $T = \{x \in G; \mu(xyx^{-1}y^{-1}) = \mu(e) \text{ for all } y \in G\}$ . Then  $T = Z(\mu)$ .

## 2. Engel fuzzy subgroups

In this section we introduce the concept of Engel fuzzy subgroups and investigate some basic properties of Engel fuzzy subgroups.

**Definition 2.1.** Let  $G$  be a group and  $\mu : G \rightarrow [0, 1]$  be a fuzzy subgroup. Then  $\mu : G \rightarrow [0, 1]$  is called an *n-Engel fuzzy subgroup* if for all  $x, y \in G$ ,  $\mu[x,{}_n y] = \mu(e)$ , where  $e$  is identity element of  $G$ .

**Example 2.2.** Let  $D_3 = \langle a, b; a^3 = b^2 = e, ba = a^2b \rangle$  be the dihedral group with six element. Define a fuzzy subgroup  $\mu$  of  $D_3$  by the following:

$$\mu(x) = \begin{cases} t_0 & \text{if } x \in \langle a \rangle \\ t_1 & \text{if } x \notin \langle a \rangle, \end{cases}$$

where  $t_0 > t_1$ . It is easy to see that  $\mu$  is an 1-Engel fuzzy subgroup while  $D_3$  is not an Engel group, since  $[a,{}_1 b] = a \neq e$ .

**Theorem 2.3.** Let  $\mu$  be a fuzzy subgroup of a group  $G$ . If the non-empty  $\alpha$ -level cut  $\mu_t$  is Engel group, for all  $t \in [0, 1]$ , then  $\mu$  is an Engel fuzzy subgroup of  $G$ . If  $\mu(x) = \mu(e)$ , implies that  $x = e$ , then the converse of the theorem is true.

**Proof.** Let  $x, y \in G$ , and  $t = \min\{\mu(x), \mu(y)\}$ . Then  $\mu(x) \geq t$  and  $\mu(y) \geq t$ , so  $x, y \in \mu_t$ . Now, by hypotheses,  $[x,{}_n y] = e$ . Therefore,  $\mu[x,{}_n y] = \mu(e)$ . The converse is clear. ■

**Theorem 2.4.** [9] *Let  $\mu$  be a fuzzy subgroup of a group  $G$ . Then  $\mu(xy x^{-1}y^{-1}) = \mu(e)$ , for all  $x, y \in G$ , if and only if  $\mu$  is commutative in  $G$ .*

Now, as a consequence of this theorem, we have that:

**Theorem 2.5.** *Let  $\mu : G \rightarrow [0, 1]$  be a fuzzy subgroup. Then  $\mu$  is commutative if and only if  $\mu$  is 1-Engel fuzzy subgroup. Moreover every 1-Engel fuzzy subgroup is nilpotent of class at most 1.*

**Theorem 2.6.** *Let  $\mu$  be a fuzzy subgroup of a group  $G$ . Then  $\mu$  is nilpotent of class at most 3 if  $G$  is a 2-Engel group.*

**Proof.** It is the immediate result of Theorem 1.5 and Levi's theorem. ■

**Theorem 2.7.** *Let  $G$  and  $H$  be two groups and  $f : G \rightarrow H$  be a group homomorphism. If  $\mu$  is an  $n$ -Engel fuzzy subgroup of  $H$ , then  $f^{-1}(\mu)$  is an  $n$ -Engel fuzzy subgroup of  $G$ .*

**Proof.** Clearly,  $f^{-1}(\mu)$  is a fuzzy subgroup of  $G$ . Let  $x, y \in G$  and  $e, e'$  be the identity elements of  $G, H$ , respectively. Then

$$f^{-1}\mu[x, {}_n y] = \mu[f[x, {}_n y]] = \mu[f(x), {}_n f(y)] = \mu(e') = f^{-1}\mu(e). \quad \blacksquare$$

**Theorem 2.8.** *Let  $\mu$  be a  $n$ -Engel fuzzy subgroup of  $G$  and  $H$  be a group. Suppose that  $f : G \rightarrow H$  is an onto homomorphism. Then  $f(\mu)$  is an  $n$ -Engel fuzzy subgroup of  $H$ .*

**Proof.** Clearly,  $f(\mu)$  is a fuzzy subgroup of  $H$ . Let  $u, v \in H$  and  $e, e'$  be the identity elements of  $G, H$ , respectively. Then  $u = f(x), v = f(y)$  for some  $x, y \in G$ . Then

$$\begin{aligned} f(\mu)[u, {}_n v] &= \sup\{\mu(z), z \in f^{-1}[u, {}_n v]\} \\ &\geq \sup\{\mu[x, {}_n y], u = f(x), v = f(y)\} \\ &= \mu(e) \\ &= (f\mu)(e') \end{aligned}$$

This completes the proof. ■

**Theorem 2.9.** *Let  $\mu, \eta$  be two Engel fuzzy subgroups then  $\mu \cap \eta$  and  $\mu \times \eta$  are Engel fuzzy subgroups too.*

**Proof.**

$$\begin{aligned} \mu \times \eta[(x_1, y_1), {}_n (x_2, y_2)] &= \mu \times \eta([x_1, {}_n x_2], [y_1, {}_n y_2]) \\ &= \min\{\mu([x_1, {}_n x_2]), \eta([y_1, {}_n y_2])\} \\ &= \min\{\mu(e_1), \eta(e_2)\} \\ &= \mu \times \eta(e_1, e_2). \end{aligned}$$

Also

$$(\mu \cap \eta)[x, {}_n y] = \min\{\mu[x, {}_n y], \eta[x, {}_n y]\} = \min\{\mu(e), \eta(e)\} = (\mu \cap \eta)(e)$$

This completes the proof. ■

**Theorem 2.10.** *Let  $\mu$  be a normal  $(n - 1)$ -Engel fuzzy subgroup of  $G$ , then  $\mu$  is an  $n$ -Engel fuzzy subgroup.*

**Proof.** Let  $x, y \in G$ . Then  $\mu[x_n y] = \mu[[x_{n-1} y], y] = \mu([x_{n-1} y]^{-1}[x_{n-1} y]^y) \geq \min\{\mu([x_{n-1} y]^{-1}), \mu([x_{n-1} y]^y)\}$ . Since  $\mu$  is a normal  $(n - 1)$ -Engel fuzzy subgroup so  $\mu[x_{n-1} y] = \mu(e)$ . Therefore  $\mu$  is an  $n$ -Engel fuzzy subgroup. ■

**Theorem 2.11.** *If  $G$  is an  $n$ -Engel group then  $G/\mu$  is  $n$ -Engel, where  $G/\mu = \{x\mu, x \in G\}$ ,  $(x\mu)o(y\mu) = (xy)\mu$ ,  $(x\mu)^{-1} = x^{-1}\mu$ .*

**Proof.** First let  $n = 1$  so

$$[x\mu, y\mu] = (x^{-1}\mu)o(y^{-1}\mu)o(x\mu)o(y\mu) = [x, y]\mu.$$

And, by hypotheses,  $[x\mu, y\mu] = e\mu$ . Therefore,  $G/\mu$  is an 1-Engel group.

Now, by induction on  $n$ , we have

$$\begin{aligned} [x\mu_n y\mu] &= [[x\mu_{n-1} y\mu], y\mu] \\ &= ([x\mu_{n-1} y\mu]^{-1})o(y^{-1}\mu)o([x\mu_{n-1} y\mu])o(y\mu) \\ &= ([x_{n-1} y]^{-1}\mu)o(y^{-1}\mu)o([x_{n-1} y]\mu)o(y\mu) \\ &= [[x_{n-1} y], y]\mu \\ &= [x_n y]\mu \\ \implies [x\mu_n y\mu] &= e\mu \end{aligned}$$

This completes the proof. ■

**Theorem 2.12.** *Let  $\mu$  be a normal fuzzy subgroup. Then  $\mu$  is an  $n$ -Engel fuzzy subgroup if and only if  $G/\mu$  is an  $n$ -Engel group.*

**Proof.** By (1.3.11) [9],  $\mu(x) = \mu(e)$  if  $x\mu = e\mu$  for a normal fuzzy subgroup  $\mu$ . Also, if  $\mu(x) = \mu(e)$ , then, for all  $z \in G$ , we have

$$(x\mu)z = \mu(x^{-1}z) \geq \min\{\mu(x), \mu(z)\} = \mu(z) = (e\mu)z.$$

Therefore,  $x\mu = e\mu$ . If  $G/\mu$  is  $n$ -Engel, then using the method of the last theorem

$$[x\mu_n y\mu] = [x_n y]\mu = e\mu$$

if and only if  $\mu([x_n y]) = \mu(e)$ . ■

**Theorem 2.13.** *Let  $\frac{G}{\mu}$  be a nilpotent group of class  $n$ . Then  $\mu$  is nilpotent of class  $n$ .*

**Proof.** Since  $\frac{G}{\mu}$  is nilpotent of class  $n$ , then

$$\frac{G}{\mu} = \{x\mu; [x\mu, y_1\mu, \dots, y_n\mu] = e\mu \text{ for all } y_1, \dots, y_n \in G\}.$$

Let  $x$  be an arbitrary element of  $G$ . Then  $x\mu \in \frac{G}{\mu}$  implies that, for all  $y_i\mu$ ,  $[x\mu, y_1\mu, \dots, y_n\mu] = e\mu$ . Consequently,  $[x, y_1, \dots, y_n]\mu = e\mu$ . Thus  $[x, y_1, \dots, y_{n-1}] \in Z(\mu)$ . By Theorem 1.8 we have  $[x, y_1, \dots, y_{n-2}] \in Z^2(\mu)$ . By a similar method,  $[x, y_1] \in Z^{n-1}(\mu)$ . Thus  $x \in Z^n(\mu)$ . Consequently,  $Z^n(\mu) = G$ . This completes the result. ■

**Theorem 2.14.** *Let  $\mu$  be a normal finite Engel fuzzy subgroup. Then  $\mu$  is nilpotent.*

**Proof.** By 2.12,  $\frac{G}{\mu}$  is an Engel group. Now, Zorn's theorem implies that  $\frac{G}{\mu}$  is nilpotent of class, say  $n$ . The result follows by the previous theorem. ■

**Theorem 2.15.** *Let  $\mu$  be a normal fuzzy subgroup. Then  $\eta = \mu|_{y^G}$  is commutative for all  $y$  if and only if  $\mu$  is a 2-Engel fuzzy subgroup.*

**Proof.** By hypotheses,  $Z(\mu|_{y^G}) = y^G$ . So, by 1.9,  $\mu|_{y^G} [t, s] = \mu|_{y^G} (e) = \mu(e)$  for all  $s, t \in y^G$ . Therefore,

$$\mu[[x, y], y] = \mu[y^{-x}y, y] = \mu|_{y^G} ([y^{-x}y, y]) = \mu|_{y^G} (e) = \mu(e).$$

Conversely, let  $\mu$  be a 2-Engel fuzzy subgroup so

$$\begin{aligned} \mu|_{x^G} ([x, x^y]) &= \mu([x, x^y]) = \mu([x, x[x, y]]) \\ &= \mu([x, [x, y]]) \\ &= \mu(e). \end{aligned}$$

Also

$$\begin{aligned} \mu|_{x^G} ([x^y x^z, x^s]) &= \mu|_{x^G} ([x^y, x^s]^{x^z} [x^z, x^s]) \\ &\geq \mu|_{x^G} ([x, x^{s y^{-1}}]^{y x^z}) \wedge \mu|_{x^G} ([x, x^{s z^{-1}}]^z) \\ &= \mu(e). \end{aligned}$$

This completes the proof. ■

**Theorem 2.16.** *Let  $\mu$  be a normal 2-Engel fuzzy subgroup and  $x, y, z, t$ , be elements of  $G$ . Then the followings are equivalent:*

- (1)  $\mu$  is 2-Engel,
- (2)  $\mu|_{x^G}$  is commutative,
- (3)  $\mu[x, y, z] = \mu[z, x, y]$ .

**Proof.** By the last theorem, it is enough to show that parts (2) and (3) are equivalent. Let  $A = x^G$ . If part (2) holds, then  $\mu[a_1, a_2] = \mu(e)$  for all  $a_1, a_2 \in A$ . Now, since  $\mu[x, y] \geq \mu(x)$ , then

$$\mu[a, y, y^{-1}] = \mu[[a, y], y]^{-y^{-1}} \geq \mu[[a, y], y] \wedge \mu[[a, y], y] = \mu(e),$$

which implies that

$$\begin{aligned} \mu(e) &= \mu([a, yz, z^{-1}y^{-1}]) \\ &= \mu([a, z, y^{-1}][a, z, y^{-1}, [a, y]^z][[a, y][a, y, z], y^{-1}] \\ &\quad [a, z, z^{-1}][[a, y]^z, z^{-1}][[a, z, z^{-1}][[a, y]^z, z^{-1}], y^{-1}]) \\ &= \mu([a, z, y^{-1}][a, z, y^{-1}, [a, y]^z][a, y, y^{-1}]^{[a, y, z]}[a, y, z, y^{-1}] \\ &\quad [a, z, z^{-1}][a, y, z^{-1}]^{[a, y, z]}[a, y, z, z^{-1}]). \end{aligned}$$

By part (2), we have  $\mu[a, z, y^{-1}, [a, y]^z] = \mu(e) = \mu[a, y, z, y^{-1}]$ . Also  $e\mu = [a, y, y^{-1}]\mu$ ,  $e\mu = [a, y, y]\mu$ . Consequently,  $\mu[a, y, z] = \mu[a, z, y^{-1}]$ , which implies that  $\mu[a, y, z] = \mu[a, z, y^{-1}]$ . Thus  $\mu[x, y, z] = \mu[x, z, y^{-1}] = \mu[x, z, y]^{-y^{-1}} = \mu[x, z, y][x, z, y, y^{-1}]$ . Now, if  $\mu[x, z, y] = \mu[x, z, y, y^{-1}]$  then  $\mu[x, y, z] \geq \mu(e)$  which implies that  $\mu[x, y, z] = \mu[x, z, y] = \mu(e)$ . Also, if  $\mu[x, z, y] \neq \mu[x, z, y, y^{-1}]$ , then, by Theorem 1.7 and  $\mu[a, y, y^{-1}] = \mu(e)$ , we have  $\mu[x, y, z] = \mu[x, z, y]$ .

On the other hand,  $\mu[[x, z]^{-1}, y] = \mu[[x, z, y]^{-[x, z]^{-1}}]$ . Therefore, since  $\mu$  is normal, we have  $\mu[[x, z]^{-1}, y] = \mu[x, z, y]$ . Hence  $\mu[x, y, z] = \mu[z, x, y]$ .

If (3) is satisfied, then  $\mu[x^b, x] = \mu([x, x]^{[x, b]}[x, b, x]) = \mu[x, b, x] = \mu[x, x, b] = \mu(e)$ . Thus (2) holds. ■

**Theorem 2.17.** *Let  $\mu$  be a fuzzy subgroup of  $G$ . Then the following are equivalent:*

- (1)  $\mu$  is a fuzzy 3-Engel subgroup,
- (2)  $\mu|_{x^G}$  is a 2-Engel fuzzy subgroup for all  $x \in G$ ,
- (3) for all  $s, t \in x^G$ ,  $[t, s] \in Z(\mu|_{x^G})$ .

**Proof.** (2)  $\rightarrow$  (1)  $\mu[x, y, y, y] = \mu[y^{-x}y, y, y] = \mu|_{y^G}([y^{-x}y, y, y]) = \mu|_{y^G}(e) = \mu(e)$ .

(3)  $\rightarrow$  (2) Since for all  $t, s \in x^G$ ,  $[t, s] \in Z(\mu|_{x^G})$ , so  $\mu|_{x^G}([t, s], r) = \mu(e)$  for all  $r \in x^G$ . Hence, the result follows.

(1)  $\rightarrow$  (3) Since  $\mu$  is 3-Engel then  $\frac{G}{\mu}$  is 3-Engel. Now, by Kappe's theorem,  $(x\mu)^{\frac{G}{\mu}}$  is nilpotent of class 2. Thus, for all  $g_1, g_2, g_3$  in  $G$ , we have:

$$\begin{aligned} [(x\mu)^{g_1\mu}, (x\mu)^{g_2\mu}, (x\mu)^{g_3\mu}] = e\mu &\implies [x^{g_1}, x^{g_2}, x^{g_3}]\mu = e\mu \\ &\implies \mu[x^{g_1}, x^{g_2}, x^{g_3}] = \mu(e) \\ &\implies [x^{g_1}, x^{g_2}] \in Z(\mu|_{x^G}). \end{aligned} \quad \blacksquare$$

### 3. Right and left Engel fuzzy subgroups

In this section, we will define right and left fuzzy Engel elements. Also we get some results which are similar to theorems of right and left Engel elements.

**Definition 3.1.** Let  $\mu : G \rightarrow [0, 1]$  be a fuzzy subgroup. Then we call  $x \in G$  a *right fuzzy  $n$ -Engel element* if  $\mu[x_n, y] = \mu(e)$  for all  $y \in G$ . The set of all right(left) fuzzy  $n$ -Engel elements is called a *right(left) fuzzy  $n$ -Engel set*. We denote the set of all right(left) fuzzy  $n$ -Engel elements by  $R_n(\mu)$ ,  $(L_n(\mu))$ .

**Theorem 3.2.** *Let  $\mu$  be a normal fuzzy subgroup. Then*

$$L_2(\mu) = \{x \in G, \mu|_{xG} \text{ is commutative}\}.$$

**Proof.** Let  $a, b \in G$ . Then

$$\begin{aligned} x \in L_2(\mu) &\iff \mu[ab^{-1}, x, x] = \mu(e) \\ &\iff \mu[x^{-ab^{-1}}x, x] = \mu(e) \\ &\iff \mu([x^{-ab^{-1}}, x]^x[x, x]) = \mu(e) \\ &\iff \mu([x^{-ab^{-1}}, x]^x) = \mu(e) \\ &\iff \mu([x^{ab^{-1}}, x]^{-ab^{-1}}) = \mu(e) \\ &\iff \mu([x^{ab^{-1}}, x]) = \mu(e) \\ &\iff \mu([x^a, x^b]^{b^{-1}}) = \mu(e) \\ &\iff \mu([x^a, x^b]) = \mu(e). \end{aligned}$$

By Theorem 2.4  $\mu|_{xG}$  is commutative. This completes the proof. ■

**Corollary 3.3.** *Let  $\mu$  be a normal fuzzy subgroup. Then*

$$L_2(\mu) = \{x \in G, \mu|_{xG} \text{ is nilpotent}\}.$$

**Proof.** It is clear by Theorem 3.2 and the definition of commutative and nilpotent fuzzy subgroups.

**Theorem 3.4.** *Let  $\mu$  be a normal fuzzy subgroup of  $G$ ,  $a, g \in G$  and  $a \in R_n(\mu)$ . Then  $a^g \in R_n(\mu)$ .*

**Proof.**  $\mu[a_{,n}x] = \mu(e)$  for all  $x \in G$ . Thus

$$\begin{aligned} \mu[a^g_{,n}x] &= \mu([[a^g, x]_{,n-1}x]) \\ &= \mu([[a, x^{g^{-1}}]^g_{,n-1}x]) \\ &= \mu([a, x^{g^{-1}}_{,n-1}x^{g^{-1}}]^g) \\ &= \mu([a_{,n}x^{g^{-1}}]) = \mu(e). \end{aligned}$$

This completes the proof. ■

**Theorem 3.5.** *Let  $\mu$  be a normal fuzzy subgroup. Then  $(R_n(\mu))^{-1} \subseteq L_{n+1}(\mu)$ .*

**Proof.**

$$\begin{aligned} \mu[x_{,n+1}g] &= \mu([[x, g]_{,n}g]) \\ &= \mu([x^{-1}g^{-1}xg_{,n}g]) \\ &= \mu([(g^{-1})^xg_{,n}g]) \\ &= \mu([[(g^{-1})^xg, g]_{,n-1}g]) \\ &= \mu([[(g^{-1})^x, g]^g_{,n-1}g]) \\ &= \mu([(g^{-1})^x_{,n}g]^g) \\ &= \mu([(g^{-1})^x_{,n}g]). \end{aligned}$$



If  $g \in (R_n(\mu))^{-1}$ , then  $g^{-1} \in (R_n(\mu))$ . Therefore, by Theorem 3.4,  $(g^{-1})^x \in (R_n(\mu))$ . Hence  $\mu[x_{,n+1}g] = \mu[(g^{-1})^x_{,n}g] = \mu(e)$ . This completes the proof. ■

**Theorem 3.6.** *Let  $\mu$  be a fuzzy subgroup of  $G$ . Then  $G = R(\mu)$  if and only if  $G = L(\mu)$ , where  $R(\mu)$  ( $L(\mu)$ ) is the set of all right (left) Engel fuzzy elements.*

**Proof.** Let  $g \in G = R(\mu)$ . Then  $g^{-1} \in G = R(\mu)$ . By Theorem 3.6  $g \in L(\mu)$ . Conversely, let  $G = L(\mu)$ . Then, for all  $x \in G = L(\mu)$  is a left Engel fuzzy subgroup. Thus,  $\forall x, \forall g, \mu[g_{,n}x] = \mu(e)$ . Thus for all  $g, g$  is a right Engel fuzzy element. Now by Theorem 3.5 for all  $g, g^{-1}$  is a left Engel fuzzy element. Thus, for all  $g \in G$ , we have  $\mu[x_{,n+1}g^{-1}] = \mu(e), \forall x$ , which implies that  $x$  is a right Engel fuzzy element. Therefore,  $G = L(\mu) \subseteq R(\mu)$ . Consequently,  $G = R(\mu)$ . ■

**Remark 3.7.** If  $\mu$  is an  $n$ -Engel fuzzy subgroup of  $G$ , then every element of  $G$  is both left and right  $n$ -Engel fuzzy element.

**Theorem 3.8.** *Suppose  $\mu$  be a normal fuzzy subgroup. Then  $x \in L_3(\mu)$  if and only if  $\mu|_{\langle x, x^y \rangle}$  is nilpotent of class at most 2,  $x, y \in G$ .*

**Proof.** Since  $[y^\epsilon, {}_3x] = [x^{-1}, [x^{-1}, [x^{-1}, y^\epsilon]]]^{x^3}$ , where  $\epsilon \in \{-1, 1\}$ , we have

$$\begin{aligned} x \in L_3(\mu) &\iff \mu([y, {}_3x]) = \mu([y^{-1}, {}_3x]) = \mu(e) \\ &\iff \mu([x^{-1}, [x^{-1}, [x^{-1}, y]]) = \mu([x^{-1}, [x^{-1}, [x^{-1}, y^{-1}]]) = \mu(e). \end{aligned}$$

Therefore,

$$\begin{aligned} \mu([x^{-x^{-y}}, x^{-1}]) &= \mu([(x^{-1})^{x^{-y}}, x^{-1}]) \\ &= \mu([x^{-1}[x^{-1}, x^{-y}], x^{-1}]) \\ &= \mu([x^{-1}, x^{-1}]^{[x^{-1}, x^{-y}]}[[x^{-1}, x^{-y}], x^{-1}]) \\ &= \mu([x^{-1}, x^{-1}[x^{-1}, y]], x^{-1}) \\ &= \mu([x^{-1}, [x^{-1}, y]][x^{-1}, x^{-1}]^{[x^{-1}, y]}, x^{-1}) \\ &= \mu([x^{-1}, [x^{-1}, x^{-1}, y]]^{-1}) \\ &= \mu([x^{-1}, x^{-1}, x^{-1}, y]) \\ &= \mu(e). \end{aligned}$$

Thus

$$\mu(e) = \mu([x^{-x^{-y}}, x^{-1}]) = \mu([(x^{x^{-y}})^{-1}, x^{-1}]) = \mu([(x^{x^{-y}}), x^{-1}]^{-x^{-x^{-y}}}) = \mu([(x^{x^{-y}}), x^{-1}]).$$

On the other hand

$$\begin{aligned} \mu([x^{-y}, x^{-1}], x^{-1}) &= \mu([x^y x x^{-y} x^{-1}, x^{-1}]) \\ &= \mu([(x^{x^{-y}})x^{-1}, x^{-1}]) \\ &= \mu([(x^{x^{-y}}), x^{-1}]^{x^{-1}}[x^{-1}, x^{-1}]). \end{aligned}$$

Since  $\mu$  is normal and  $\mu([(x^{x^{-y}}), x^{-1}]) = \mu(e)$ , therefore,

$$\mu([x^{-y}, x^{-1}], x^{-1}) = \mu(e). \tag{I}$$

Also

$$\begin{aligned}
 \mu([x^{-1}, [x^{-1}, [x^{-1}, y^{-1}]]) = \mu(e) &\implies \mu([x^{-1}, [x^{-1}, y^{-1}], x^{-1}]^{-1}) = \mu(e) \\
 &\implies \mu([x^{-1}, [x^{-1}, y^{-1}], x^{-1}]) = \mu(e) \\
 &\implies \mu([x^{-x^{-y^{-1}}}, x^{-1}]) = \mu(e) \\
 &\implies \mu([x^{-1}[x^{-1}, x^{-y^{-1}}], x^{-1}]) = \mu(e) \\
 &\implies \mu([x^{-1}, x^{-y^{-1}}, x^{-1}]) = \mu(e) \\
 &\implies \mu([x^{-y^{-1}}, x^{-1}]^{-1}, x^{-1}) = \mu(e) \\
 &\implies \mu([x^{-y^{-1}}, x^{-1}], x^{-1}]^{-[x^{-y^{-1}}, x^{-1}]^{-1}}) = \mu(e) \\
 &\implies \mu([x^{-y^{-1}}, x^{-1}], x^{-1}) = \mu(e). \tag{II} \\
 &\implies \mu([x^{-1}, x^{-y}, x^{-y}]) = \mu(e). \tag{III}
 \end{aligned}$$

Now, we can show that  $\mu|_{\langle x, x^y \rangle}$  is nilpotent of class at most 2, since, by Theorem 1.8, if for all  $z \in \langle x, x^y \rangle$ ,  $[x^{-1}, z] \in Z(\mu|_{\langle x, x^y \rangle})$ , then  $x^{-1} \in Z^2(\mu|_{\langle x, x^y \rangle})$ . Similarly,  $x^y \in Z^2(\mu|_{\langle x, x^y \rangle})$ . Therefore,  $Z^2(\mu|_{\langle x, x^y \rangle}) = \langle x, x^y \rangle$ . Hence  $\mu|_{\langle x, x^y \rangle}$  is nilpotent of class at most 2. ■

**Corollary 3.9.** *Let  $\mu$  be a normal fuzzy subgroup of  $G$ . Then  $L_3(\mu) = \{x \in G, \mu|_{\langle x, x^y \rangle}$  is nilpotent of class at most 2 for all  $y \in G\}$ .*

**Theorem 3.10.** *Let  $\mu$  be a normal fuzzy subgroup. Then  $R_2(\mu) \subseteq L_2(\mu)$ .*

**Proof.** Let  $a \in R_2(\mu)$ . Then, for all  $x \in G$ , we have

$$\begin{aligned}
 \mu[a, ax, ax] = \mu(e) &\implies \mu[[a, a]^x[a, x], ax] = \mu(e) \\
 &\implies \mu[[a, x], ax] = \mu(e) \\
 &\implies \mu([a, x, a]^x[a, x, x]) = \mu(e).
 \end{aligned}$$

i) If  $\mu([a, x, a]^x) = \mu([a, x, x])$ , then  $\mu([a, x, x]) = \mu(e)$  implies that  $\mu([a, x, a]^x) = \mu(e)$ . Then by hypotheses  $\mu([a, x, a]) = \mu(e)$ . But  $\mu(e) = \mu([a, x, a]) = \mu([x, a]^{-1}, a) = \mu([x, a, a]^{-[x, a]^{-1}})$ . Since  $\mu$  is normal, then  $\mu(e) = \mu([x, a, a]^{-[x, a]^{-1}}) = \mu([x, a, a]^{-1}) = \mu([x, a, a])$ . Therefore,  $a \in L_2(\mu)$ .

ii) if  $\mu([a, x, a]^x) \neq \mu([a, x, x])$ , then by Theorem 1.7  $\mu([a, x, a]^x) = \mu([a, x, x]) = \mu(e)$ . Now, by the similar method of part (i), the result follows. ■

**Theorem 3.11.** *Let  $\mu$  be a normal fuzzy subgroup,  $x \in L_3(\mu)$  and  $p$  be a prime number. If  $\mu(x^{p^n}) = \mu(e)$  for some integer  $n \geq 2$ , then  $x^{p^{n-1}} \in L_2(\mu)$ .*

**Proof.** Let  $y$  be an arbitrary element of  $G$ . By the proof of Theorem 3.8,  $\mu([x_1, x_2, x_3]) = \mu(e)$  for all  $x_i \in \langle x, x^y \rangle$ . Thus

$$\mu([(x^{-y})^{p^{n-1}}, x^{p^{n-1}}]) = \mu([(x^{-y})^{p^{n-2}}, x^{p^{n-1}}]) = \mu([(x^{-y})^{p^{n-2}}, x^{p^{n-1}}]^p a), \tag{I}$$

in which  $\mu(a) = \mu(e)$ .

i) If  $\mu([(x^{-y})^{p^{n-2}}, x^{p^{n-1}}]^p) = \mu(a) = \mu(e)$ , then

$$\mu([(x^{-y})^{p^{n-1}}, x^{p^{n-1}}]) \geq \mu([(x^{-y})^{p^{n-2}}, x^{p^{n-1}}]^p) \wedge \mu(a) = \mu(e).$$

Therefore, since  $\mu([y, x^{p^{n-1}}, x^{p^{n-1}}]) = \mu([(x^{-y})^{p^{n-1}}, x^{p^{n-1}}])$ , then  $x^{p^{n-1}} \in L_2(\mu)$ .

ii) If  $\mu([(x^{-y})^{p^{n-2}}, x^{p^{n-1}}]^p) \neq \mu(a)$ , then, by Theorem 1.7, we have

$$\mu([(x^{-y})^{p^{n-1}}, x^{p^{n-1}}]) = \mu([(x^{-y})^{p^{n-2}}, x^{p^{n-1}}]^p).$$

Similarly,  $\mu([(x^{-y})^{p^{n-2}}, (x^{p^{n-1}})^p]) = \mu([(x^{-y})^{p^{n-2}}, x^{p^{n-1}}]^p b)$ , in which  $\mu(b) = \mu(e)$ . If  $\mu([(x^{-y})^{p^{n-2}}, x^{p^{n-1}}]^p) = \mu(b) = \mu(e)$  then  $\mu([(x^{-y})^{p^{n-1}}, x^{p^{n-1}}]) = \mu(e)$ . Also, if  $\mu([(x^{-y})^{p^{n-2}}, x^{p^{n-1}}]^p) \neq \mu(b)$ , then, by Theorem 1.7, we have

$$\mu([(x^{-y})^{p^{n-2}}, (x^{p^{n-1}})^p]) = \mu([(x^{-y})^{p^{n-2}}, (x^{p^{n-1}})^p]).$$

Thus  $\mu([(x^{-y})^{p^{n-1}}, x^{p^{n-1}}]) = \mu([(x^{-y})^{p^{n-2}}, (x^{p^{n-1}})^p]) = \mu([(x^{-y})^{p^{n-2}}, x^{p^n}]) \geq \mu(x^{p^n}) = \mu(e)$ . Hence  $x^{p^{n-1}} \in L_2(\mu)$ . ■

**Definition 3.12.** Let  $\mu$  be a fuzzy subgroup of  $G$ .

$$1 \trianglelefteq H_0 \dots \trianglelefteq H_n = H$$

has a fuzzy commutative factor, if  $\mu|_{\frac{H_{i+1}}{H_i}}$  is commutative, it means that  $\mu[xH_i, yH_i] = \mu(H_i)$  for all  $x, y \in H_{i+1}$ ,  $0 \leq i \leq n$ .

**Definition 3.13.** Let  $\mu$  be a fuzzy subgroup of  $G$ .  $\mu$  is called fuzzy soluble if there exist  $H \subseteq G$  such that its normal series have fuzzy commutative factors.

**Example 3.14.** Each fuzzy subgroup of a soluble group is a fuzzy soluble.

**Theorem 3.15.** Let  $\mu, \nu$  be two normal fuzzy subgroups of  $G$ ,  $\mu \subseteq \nu$  and  $\mu(e) = \nu(e)$ . If  $\mu$  is a fuzzy soluble, then  $\nu$  is.

**Proof.** By hypotheses there is  $H \subseteq G$  such that its normal series have fuzzy commutative factors with respect to  $\mu$ . Now it is easy to see that this normal series have fuzzy commutative factors with respect to  $\nu$ . ■

**Theorem 3.16.** Let  $\mu$  be a normal fuzzy subgroup and  $p$  be a prime number. If  $x \in L_3(\mu)$  and  $\mu(x^{p^n}) = \mu(e)$  for some integer  $n \geq 2$ , then  $\mu$  is fuzzy soluble.

**Proof.** By Theorem 3.11,  $x^{p^{n-1}} \in L_2(\mu)$ . Therefore, using Theorem 3.3,  $\mu|_{((x^{p^{n-1}})^G)}$  is commutative. Also  $\mu|_{\frac{\langle x^{p^1} \rangle^G}{\langle x^{p^2} \rangle^G}}$  is commutative, since by the same manipulation

of Theorem 3.15, for  $a, b \in G$ , we have  $\mu([(x^a)^p \langle x^{p^2} \rangle^G, (x^b)^p \langle x^{p^2} \rangle^G]) = \mu([(x^a) \langle x^{p^2} \rangle^G, (x^b) \langle x^{p^2} \rangle^G]^p (a \langle x^{p^2} \rangle^G))$ , in which

$$\mu(a \langle x^{p^2} \rangle^G) = \mu(e \langle x^{p^2} \rangle^G). \tag{*}$$

Now

1) if  $\mu(a \langle x^{p^2} \rangle^G) = \mu(e \langle x^{p^2} \rangle^G) = [(x^a) \langle x^{p^2} \rangle^G, (x^b) \langle x^{p^2} \rangle^G]^p$ , then

$\mu[(x^a)^p < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G] = \mu(e < x^{p^2} >^G)$ . Therefore, we get the result.

2) If  $\mu([(x^a) < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]^p) \neq (a < x^{p^2} >^G)$ , then we have

$$\mu([(x^a)^p < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]) = \mu([(x^a) < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]^p). \quad (**)$$

Now, we have two cases:

(i) If  $\mu([(x^a) < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]^p) = \mu(e < x^{p^2} >^G)$ , then by (\*\*) we have  $\mu([(x^a)^p < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]) = \mu(e < x^{p^2} >^G)$ .

(ii) If  $\mu[(x^a) < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]^p \neq \mu(e < x^{p^2} >^G)$ , then by the same manipulation of Theorem 3.11, and (I) in the proof of Theorem 3.11, we have  $\mu[(x^a) < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]^p = \mu[(x^a) < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]^p (b < x^{p^2} >^G) = \mu[(x^a) < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G]^p$  in which  $\mu(b < x^{p^2} >^G) = \mu(e < x^{p^2} >^G)$ . Therefore, by (\*\*), we have  $\mu[(x^a)^p < x^{p^2} >^G, (x^b)^p < x^{p^2} >^G] = \mu[(x^a) < x^{p^2} >^G, e < x^{p^2} >^G] = e < x^{p^2} >^G$ .

Similarly,  $\mu \mid_{\frac{\langle x^{p^{n-i}} \rangle_G}{\langle x^{p^{n-(i-1)}} \rangle_G}}$  is commutative, which implies that  $1 \trianglelefteq (\langle x^{p^{n-1}} \rangle)^G \trianglelefteq \dots \trianglelefteq \langle x^p \rangle^G$  is a series of normal subgroups of  $G$  with fuzzy commutative factors. Thus  $\mu$  is soluble. ■

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