T-SYSTEMS IN TERNARY SEMIGROUPS

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Abstract. In this paper, we introduce the notions of right T-system transitive, T-homomorphism, semispace in ternary semigroups. We characterize different classes of ternary semigroups by the properties of their right T-system and T-homomorphism.

Keywords: ternary semigroup, right T-system, fixed element, transitive, irreducible, T-homomorphism, semispace.

1. Introduction

The theory of ternary algebraic systems was introduced by Lehmer [4] in 1932, but earlier such structures was studied by Kasner [3] who give the idea of n-ary algebras. Lehmer [4] investigated certain algebraic systems called triplexes which turn out to be commutative ternary groups. Ternary semigroups are universal algebras with one associative ternary operation. Anjaneyulu [1] introduce S-semispace and obtain an isomorphism theorem of semigroup of S-homomorphism on semispaces and deduce the well known Ljapin's [5] theorem on the semigroup of transformations over a set. In this paper we introduce right T-system and T-semispace and study some properties of these T-systems.

To present the main theorems, we first recall the definition of a ternary semigroup which is important here.

Definition. A nonempty set T is called a *ternary semigroup* [3] if there exists a ternary operation $T \times T \times T \to T$ written as $(x_1, x_2, x_3) \to [x_1x_2x_3]$, satisfying the following identity, for any $x_1, x_2, x_3, x_4, x_5 \in T$,

$$[(x_1x_2x_3)\, x_4x_5] = [x_1\, (x_2x_3x_4)\, x_5] = [x_1x_2\, (x_3x_4x_5)]\,.$$

Example. Let $T = \{-i, 0, i\}$. Then T is a ternary semigroup under the multiplication over complex number while T is not a semigroup under complex number multiplication.

2. T-systems in Ternary semigroups

Definition 2.1. Let T be a ternary semigroup. A non empty set M is called a *right T-system* provided there exists a mapping $(m, n, s) \to mns$ of $M \times M \times T \to M$ such that $mn(stu) = m(nst)u = (mns)tu$ for all $m, n \in M$ and $s, t, u \in T$. We denote a right T-system M by M_T.

Let M_T be a right T-system. Then an element $m \in M$ is called a *fixed element* of M_T provided $mmt = m$ for all $t \in T$.

If M_T is a right T-system, then we denote the set

$$F_M = \{m \in M : mnt = m \text{ for all } t \in T\}$$

and F_M is read as the set of fixed (*invariant*) elements of an operand M_T over a ternary semigroup T.

A right T-system M_T is said to be a *transitive* provided for any $m, n, p \in M$, there exists a $t \in T$ such that $mnt = p$.

Let M_T be a right T-system. Then a non empty subset N of M is called a T-subsystem of M_T provided $NNT \subseteq N$, that is, for all $m, n \in N$ and $t \in T$, $mnt \in N$.

A right T-system M_T is said to be *irreducible* provided $MMT \not\subseteq FM$ and the only subsystem of M of cardinality greater than one is M itself.

Theorem 2.2. *Let M_T be a right T-system with $FM = \phi$, that is, M_T has no fixed elements. Then M_T is a transitive T-system if and only if M_T is an irreducible.*

Proof. Let M_T is a transitive T-system. Suppose that if possible M_T is not irreducible. Then $MMT \subseteq FM \Rightarrow$ for all $m \in M, t \in T$, $mmt = m$ and hence M_T is not transitive. We have the contradiction. Therefore $MMT \not\subseteq FM$ implies that M_T is an irreducible.

Conversely, suppose that M_T is an irreducible T-system, i.e., $MMT \subseteq FM \Rightarrow$ for $m \in M$, $mmt \neq m$, for all $t \in T \Rightarrow m, n, p \in M$, there exists a $t \in T$ such that $mnt = p$. Therefore, M_T is a transitive T-system.

Definition 3.3. Let M_T and N_T be two right T-systems. A mapping $f : M \to N$ is called a *T-homomorphism from M_T into N_T* provided $f(mnt) = f(m)\,nt$, for all $m \in M$ and $n, t \in T$.

We denote the set of all T-homomorphism from M_T into N_T by $H_T(M, N)$ and the set of all T-homomorphism from M_T into itself by $H_T(M)$ or simply H.

Definition 3.4. An unital T-system M_T is said to be a *T-semispace* or, simply, a *semispace* provided T is a ternary group such that $mns = mnt$, for some $m, n \in M$, and $s, t \in T$ implies $s = t$. We call T, a *centralizer* of M.

It can be observed that a semispace is a vector set with $FM = \phi$ in the sense of Hoehnke [2].

Let M_T be any semispace. Then the transitive relation on M_T is an equivalence relation and the corresponding equivalence classes as T-equivalence classes. Also, each equivalence class is a transitive T-system and hence an irreducible T-system.

Let $\{C_\alpha\}_{\alpha\in\Delta}$ be the family of T-equivalence classes. By the choice axiom, there exist $\{W_\alpha\}_{\alpha\in\Delta}$ such that $W_\alpha \in C_\alpha$.

In what follows, we fix the family of elements $\{W_\alpha\}_{\alpha\in\Delta}$ and, for simplicity, we write α instead of W_α for each $\alpha \in \Delta$, i.e., we consider Δ as a subset of M.

Let $\alpha \in \Delta$, we define a mapping S_α on M as follows. Let $m \in M$. Then $m = \beta st$, for some $\beta \in \Delta$ and $s, t \in T$. Write for $r \in T$, $ms_\alpha = (\beta st)\,s_\alpha = \alpha st$. Now, clearly, s_α is a T-homoorphism.

Theorem 3.5. *For every $X \in H$, range of X is a union of T-equivalence classes.*

Proof. Let $n \in$ range of X. Then there exist an element $m \in M$ such that $mX = n$. If $n \in C_\alpha$, then $n = \alpha pt$, for some $p \in M$ and $t \in T$.

Let $q \in C_\alpha$. Then $q = \alpha ps$, for some $s \in T$. Consider $(mt^{-1}s)\,X = (mX)\,t^{-1}s = nt^{-1}s = \alpha ptt^{-1}s = \alpha ps = q$. So $q \in$ range of X. Thus range of X is a union of T-equivalence classes.

Definition 3.6. Let $X \in H$. The cardinality of the set of all T-equivalence classes in the range of X is called the *rank* of X.

It is clear that the rank of X is greater than or equal to 1 for all $X \in H$ and, for each $\alpha \in \Delta$, T_α has rank 1. We denote the set of all S-homomorphisms of rank 1 by $\bigcup$. We note that $\bigcup$ does not depend on Δ.

Write

$$V = \left\{S \in \bigcup : \alpha SS = \alpha \text{ for some } \alpha \in \Delta\right\}.$$

We, now, characterize the idempotents of rank 1 in H.

Theorem 3.7. *V is the set of all idempotents of rank 1 in H.*

Proof. Let $S \in V$. So $\alpha S = \alpha$, for some $\alpha \in \Delta$. Since S has rank 1, the range of S is C_α. Let $m, s \in M$. Then $m = \beta st$ for some $\beta \in \Delta$ and $t \in T$. Assume

$\beta S = \alpha pq$ for some $p, q \in T$. Now, $mS^3 = (\beta st)\,S^3 = (\beta S)\,(stS^2) = (\alpha pq)\,stS^2 = (\alpha S)\,pqstS = \alpha pqstS = (\alpha S)\,pqst = \alpha pqst = (\beta S)\,st = (\beta st)\,S = mS$. Since this is true for all $m \in M$, S is an idempotent.

Conversely, suppose that $S \in \bigcup$ is an idempotent. Suppose range of S is C_α. If $\alpha T = \alpha st$ for some $s, t \in T$, then $\alpha st = \alpha S = \alpha S^3 = (\alpha st)\,SS = (\alpha S)\,stS = (\alpha st)\,stS = (\alpha S)\,stst = \alpha s^3 t^3$. So $s = t = e$ where e is the identity of T. Hence $\alpha S = \alpha$ for some $\alpha \in \Delta$. Therefore, $S \in V$.

In the following theorem, we exhibit a class of primitive idempotents of rank 1 in H.

Theorem 3.8. *For each $\alpha \in \Delta$, the T-homomorphism T_α is a primitive idempotent in H.*

Proof. Let $\alpha \in \Delta$. Clearly S_α is an idempotent in H. Suppose S is an idempotent in H such that $SS_\alpha = S_\alpha S = S$. Let $m, s \in M$. Then $m = \beta st$ for some $\beta \in \Delta$ and $t \in T$. Now, since $mS \in M$ and range of S = rang of $S_\alpha = C_\alpha$, we have $mS = \alpha pq$, for some $p, q \in T$. Now, $(\alpha S)\,st = (\alpha st)\,S = (\beta st)\,S_\alpha S = (\beta st)\,S = (\beta st)\,S^3 = (mS)\,SS = (\alpha pq)\,SS = (\alpha SS)\,pq = (\alpha S)\,pqS = (\alpha pq)\,S = (\alpha S)\,pq$. Since M_T is semispace, it follows that $s = p, t = q$. Therefore,, $mS = \alpha pq = \alpha St = mS_\alpha$. Since this is true for all $m \in M$. We have $S = S_\alpha$. Therefore S_α is a primitive idempotent.

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