UNIFORMLY STABILITY OF IMPULSIVE DELAYED LINEAR SYSTEMS WITH IMPULSE TIME WINDOWS

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Abstract. In this paper, we formulate a new kind of mathematical model of impulsive delayed linear system, which is called *impulsive delayed linear system with impulse time windows*. By constructing a Lyaponov function, we obtain some conditions for the uniformly stability of the system. An example is also given to illustrate the efficiency of the results.

Keywords: uniformly stability, delayed system, impulsive control system, impulse time windows.

1. Introduction

Impulsive control is a control paradigm based on impulsive differential equations. In recent years, many researchers have studied impulsive systems and impulsive control, for example, [1]–[5].

Time delay phenomenon is very common in electric circuit systems. Many researchers have done outstanding works in this area. For instance, Zhang and Sun [3] have studied the stability of impulsive linear differential equations with time delay, Zhou and Wu [4] have given some conditions to ensure the exponential stability of impulsive delayed linear differential equations, Liu et al. [5] have obtained the stability criteria for impulsive systems with time delay, Su et al. [6] have researched the delay-dependent robust H_∞ control for uncertain time-delay systems, Wu et al. [7] have studied the stability and dissipativity analysis of static neural networks with time delay, Shin and Cui in [8] have shown the computing time delay and its effects on real-time control systems, Knospe and Roozbehani [9] have studied the stability of linear systems with interval time delays excluding zero, Zhang et al. [10] have designed a fuzzy controller for nonlinear impulsive fuzzy systems with time delay, Michiels, Van Assche and Niculescu [11] have researched the stabilization of time-delay systems with a controlled time-varying delay and applications.

Impulsive control can provide an efficient method for some cases in which the systems cannot endure continuous disturbance. For the traditional impulsive control system, the impulses are assumed to put at fixed time or the occurrence of the impulses is determined by the state of the system. In the latter situation, the time of the occurrence can also be calculated. How can we input impulses if we don't know or we cannot calculate the the exact occurrence time, but we know that the occurrence time is limited to a small time interval? Can we find some conditions to ensure the system's stability? In this paper, we will answer these questions.

We introduce a delayed impulsive control system with its occurrence time of impulses is limited to a small time interval, which is named by *impulsive delayed linear system with impulse time windows.*

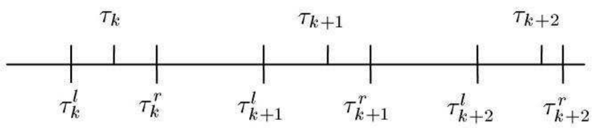

Figure 1: In an impulsive control system with impulse time windows, the occurrence time τ_k $(k = 0, 1, 2, ...)$ of impulses are unknown, but the impulse time windows $[\tau_k^l, \tau_k^r)$ are known, i.e., τ_k^l, τ_k^r $(k = 0, 1, 2, ...)$ are known

From Figure 1, we know that every occurrence time of impulses can be chosen randomly in a small *impulse time window.* So the system is more complicated than the traditional one. To our knowledge, there are seldom papers dealt with delayed impulsive control systems with impulse time windows.

The rest of the paper is organized as follows. In Section 2, we formulate the problem and introduce some notions and definitions. We then obtain, in Section 3, several conditions to ensure that the system is uniformly stable. In Section 4, we give a numerical example. Finally, we conclude our results.

2. Problem statement and preliminaries

Consider the impulsive delayed linear system with impulse time windows

$$\text{(2.1)} \qquad \begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau), & t \geq t_0, \quad t \neq \tau_k \in [\tau_k^l, \tau_k^r), \\ \Delta x(t) = x(t) - x(t^-) = Cx(t^-), & t = \tau_k, \quad k \in \mathbf{Z}^+, \end{cases}$$

where $x \in R^n$, $A, B, C \in R^{n\times n}$, B is nonsingular, $x(t^+) = \lim_{s\to t^+} x(s)$, $x(t^-) = \lim_{s\to t^-} x(s)$, $[\tau_k^l, \tau_k^r)$ $(k = 0, 1, 2, \cdots)$ are impulse time windows and τ_k $(k = 0, 1, 2, \cdots)$ are unknown time points where the impulses occur. We assume that

$$0 = t_0 = \tau_0^l = \tau_0 = \tau_0^r \leq \tau_1^l \leq \tau_1 < \tau_1^r \leq \cdots \leq \tau_k^l \leq \tau_k < \tau_k^r \leq \cdots,$$

and

$$\lim_{k\to\infty} \tau_k = \infty.$$

Obviously, $x(t) = 0$ is a solution of (2.1), which is called *the zero solution*.

Let $PC([-\tau, 0], R^n)$ is a class of piecewise continuous functions $\phi : [-\tau, 0] \to R^n$ and there is at most a finite number of discontinuous points $\hat{t}$, at which both $\phi(\hat{t}^+)$ and $\phi(\hat{t}^-)$ exist and $\phi(\hat{t}^+) = \phi(\hat{t})$.

For $\psi \in PC([-\tau, 0], R^n)$, the norm of ψ is defined by

$$|\psi| = \sup_{-\tau\leq s\leq 0} ||\psi(s)||,$$

where $||\cdot||$ denotes the norm of vector in R^n.

Define

$$PC(\rho) = \{\phi \in PC([-\tau, 0], R^n) : |\phi| < \rho\},$$

for any $\rho > 0$.

For given $\sigma \geq t_0$ and $\varphi \in PC([-\tau, 0], R^n)$, the initial value problem of (2.1) is

$$\text{(2.2)} \qquad \begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau), & t \geq \sigma, \ t \neq \tau_k \in [\tau_k^l, \tau_k^r), \\ \Delta x(t) = x(t) - x(t^-) = Cx(t^-), & t = \tau_k, \ k \in \mathbf{Z}^+, \\ x(\sigma + t) = \varphi(t) & t \in [-\tau, 0]. \end{cases}$$

Definition 1. The zero solution of (2.1) is stable if for any $\sigma \geq t_0$ and $\varepsilon > 0$ there is a $\delta = \delta(\sigma, \varepsilon) > 0$ such that for $t \geq \sigma$ and $\varphi \in PC(\delta)$ we have that

$$||x(t, \sigma, \varphi)|| < \varepsilon.$$

The zero solution of (2.1) is said to be uniformly stable if δ is independent of σ.

Definition 2. [1] For $(t, x) \in (\tau_{i-1}, \tau_i] \times R^n$, we define

$$D^+V(t,x) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h, x+h\dot{x}) - V(t,x)]$$

and

$$D\_V(t,x) = \lim_{h\to 0^-} \inf \frac{1}{h}[V(t+h, x+h\dot{x}) - V(t,x)].$$

Through the rest of the paper, I stands for the identity matrix.

3. Theoretical analysis

Theorem 1. *If there exists a symmetric and positive definite matrix $P \in \mathbf{R}^{n\times n}$, such that, for $k = 0, 1, 2, \cdots$, we have that*

$$\lambda_3(\tau_{k+1}^r - \tau_k^l) < -\ln \lambda_7,$$

where $\lambda_7 = \max\{\lambda_5, \lambda_6\}$, $\lambda_5 \in (0,1)$ is the largest eigenvalue of $P^{-1}(I+C)^T P(I+C)$. $\lambda_6 \in (0,1)$ is the largest eigenvalue of $(B^TB)^{-1}(I+C)^T B^T B(I+C)$ and λ_3 is the largest eigenvalue of $P^{-1}(A^TP + PA + B^TB + P^TP)$, then the zero solution of (1) *is uniformly stable.*

Proof. Let $\lambda_1 > 0$ and $\lambda_2 > 0$ be the minimum eigenvalue and maximum eigenvalue of P, respectively. Let $\lambda_4 > 0$ is the maximum eigenvalue of B^TB. For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that $\delta < \sqrt{\frac{\lambda_1\lambda_7}{\lambda_2 + \tau\lambda_4}}\varepsilon$.

Choose the Lyapunov function as

$$V(t, x(t)) = x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(s)B^TBx(s)ds,$$

then $\lambda_1||x(t)||^2 \le V(t, x(t)) \le \lambda_2||x(t)||^2 + \tau\lambda_4 \sup\limits_{-\tau\le s\le 0} ||x(t+s)||^2$.

If $t \ne \tau_k$, $k = 1, 2, \cdots$, we have that

$$\begin{aligned}
D^+V(t, x(t)) &= (x^T(t))'Px(t) + x^T(t)Px'(t) \\
&\quad + x^T(t)B^TBx(t) - x^T(t-\tau)B^TBx(t-\tau) \\
&= x^T(t)(A^TP + PA + B^TB)x(t) + 2x^T(t-\tau)B^TPx(t) \\
&\quad - x^T(t-\tau)B^TBx(t-\tau) \\
&\le x^T(t)(A^TP + PA + B^TB)x(t) + x^T(t-\tau)B^TBx(t-\tau) \\
&\quad + x^T(t)P^TPx(t) - x^T(t-\tau)B^TBx(t-\tau) \\
&= x^T(t)(A^TP + PA + B^TB + P^TP)x(t) \\
&\le \lambda_3 x^T(t)Px(t) \\
&\le \lambda_3(x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(s)B^TBx(s)ds) \\
&= \lambda_3 V(t, x(t)).
\end{aligned}$$

For any $\sigma \geq t_0$ and $\varphi \in PC(\delta)$, set $x(t) = x(t, \sigma, \phi)$ be the solution of (2.1) through (σ, φ).

Suppose that $\sigma \in [\tau_{m-1}^l, \tau_m^l)$ is valid for some $m \in \mathbf{Z}^+$.

Two cases are possible:

Case 1. If $\tau_{m-1} < \sigma < \tau_m^l$, then we have the fact that

$$V(t, x(t)) \leq \frac{\lambda_2 + \tau\lambda_4}{\lambda_7}\delta^2, \ \sigma \leq t < \tau_m. \tag{3.1}$$

Subcase 1 If $t = \sigma$, then

$$\begin{aligned} V(t, x(t)) &= V(\sigma, x(\sigma)) \\ &= V(\sigma, \varphi(0)) \\ &\leq \lambda_2||\varphi(0)||^2 + \tau\lambda_4 \sup_{-\tau \leq s \leq 0} ||x(s)||^2 \\ &\leq (\lambda_2 + \tau\lambda_4)|\varphi|^2 \\ &\leq (\lambda_2 + \tau\lambda_4)\delta^2 \\ &< \frac{\lambda_2 + \tau\lambda_4}{\lambda_7}\delta^2. \end{aligned}$$

Subcase 2 If $\sigma < t < \tau_m$ and suppose that (3.1) is not valid for $t \in (\sigma, \tau_m)$, then there exists $\hat{s} \in (\sigma, \tau_m)$, such that

$$V(\hat{s}, x(\hat{s})) > \frac{\lambda_2 + \tau\lambda_4}{\lambda_7}\delta^2 > (\lambda_2 + \tau\lambda_4)\delta^2 \geq V(\sigma, x(\sigma)).$$

From the continuity of $V(t, x(t))$ in (σ, τ_m), we know that there is a $s_1 \in (\sigma, \hat{s})$ such that

$$\begin{aligned} V(s_1, x(s_1)) &= \frac{\lambda_2 + \tau\lambda_4}{\lambda_7}\delta^2, \\ V(t, x(t)) &\leq \frac{\lambda_2 + \tau\lambda_4}{\lambda_7}\delta^2, \quad \sigma < t \leq s_1, \\ D^+V(s_1, x(s_1)) &\geq 0. \end{aligned}$$

Form the fact that $V(\sigma, x(\sigma)) \leq (\lambda_2 + \tau\lambda_4)\delta^2 < \frac{\lambda_2 + \tau\lambda_4}{\lambda_7}\delta^2$, we know that there exists an $s_2 \in [\sigma, s_1)$ such that

$$\begin{aligned} V(s_2, x(s_2)) &= (\lambda_2 + \tau\lambda_4)\delta^2, \\ V(t, x(t)) &\leq (\lambda_2 + \tau\lambda_4)\delta^2, \quad s_2 \leq t \leq s_1, \\ D^+V(s_2, x(s_2)) &\geq 0. \end{aligned}$$

From $D^+V(t, x(t)) \leq \lambda_3 V(t, x(t))$ we know that $\frac{D^+V(t, x(t))}{V(t, x(t))} \leq \lambda_3$. Thus

$$\int_{s_2}^{s_1} \frac{D^+V(t, x(t))}{V(t, x(t))}dt \leq \int_{s_2}^{s_1} \lambda_3 dt \leq \int_{\tau_{m-1}^l}^{\tau_m^r} \lambda_3 dt = \lambda_3(\tau_m^r - \tau_{m-1}^l) < -\ln \lambda_7.$$

At the same time,

$$\begin{aligned}\int_{s_2}^{s_1} \frac{D^+V(t,x(t))}{V(t,x(t))}dt &= \int_{V(s_2,x(s_2))}^{V(s_1,x(s_1))} u^{-1}du = \int_{(\lambda_2+\tau\lambda_4)\delta^2}^{\frac{\lambda_2+\tau\lambda_4}{\lambda_7}\delta^2} u^{-1}du \\ &= \ln\left(\frac{\lambda_2+\tau\lambda_4}{\lambda_7}\delta^2\right) - \ln((\lambda_2+\tau\lambda_4)\delta^2) = -\ln\lambda_7.\end{aligned}$$

So, it is a contradiction. Hence, (3.1) is valid for $t \in (\sigma, \tau_m)$.

Next, we will prove that, for any $k = 0, 1, 2, \cdots$, the following is valid

$$V(t,x(t)) \leq \frac{\lambda_2+\tau\lambda_4}{\lambda_7}\delta^2, \quad \tau^l_{m+k} \leq t < \tau^l_{m+1}. \tag{3.2}$$

Since

$$\begin{aligned}V(\tau_m, x(\tau_m)) &= V(\tau_m, (I+C)x(\tau_m^-)) \\ &= x^T(\tau_m^-)(I+C)^T P(I+C)x(\tau_m^-) \\ &\quad + \int_{-\tau}^{0} x^T(s+\tau_m^-)(I+C)^T B^T B(I+C)x(s+\tau_m^-)ds \\ &\leq \lambda_5 x^T(\tau_m^-)Px(\tau_m^-) + \lambda_6 \int_{-\tau}^{0} x^T(s+\tau_m^-)B^T Bx(s+\tau_m^-)ds \\ &\leq \lambda_7 V(\tau_m^-, x(\tau_m^-)) \\ &\leq (\lambda_2+\tau\lambda_4)\delta^2,\end{aligned}$$

then similarly to the proof of Case 1, we can easily prove that (3.2) is valid.

Thus we obtain that

$$V(t,x(t)) \leq \frac{\lambda_2+\tau\lambda_4}{\lambda_7}\delta^2, \ t \geq \sigma. \tag{3.3}$$

Case 2 If $\tau^l_{m-1} \leq \sigma \leq \tau_{m-1}$, then similar to the Case 1, we can prove that

$$V(t,x(t)) \leq \frac{\lambda_2+\tau\lambda_4}{\lambda_7}\delta^2, \ \sigma \leq t < \tau_{m-1}.$$

And finally we can obtain that (3.3) is valid.

Hence, from (3.3) we know that

$$\lambda_1||x(t)||^2 \leq V(t,x(t)) \leq \frac{\lambda_2+\tau\lambda_4}{\lambda_7}\delta^2, \ t \geq \sigma,$$

which implies that

$$||x(t)|| \leq \sqrt{\frac{\lambda_2+\tau\lambda_4}{\lambda_7}}\delta < \varepsilon.$$

Therefore, the zero solution of (2.1) is uniformly stable and we complete the proof. ∎

4. Numerical example

Consider the impulsive delayed linear system with impulse time windows as following.

$$
(4.1)\qquad \begin{cases} \begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix}, \\ \qquad\qquad t \geq t_0,\ t \neq \tau_k \in [\tau_k^l, \tau_k^r), \\ \begin{pmatrix} x_1(\tau_k) \\ x_2(\tau_k) \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} x_1(\tau_k^-) \\ x_2(\tau_k^-) \end{pmatrix}, \qquad k \in \mathbf{Z}^+. \end{cases}
$$

In the previous system, $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ and $I+C = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

Set $P = I$, then

$$
P^{-1}(I+C)^T P(I+C) = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix},
$$

$$
(B^T B)^{-1}(I+C)^T B^T B(I+C) = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix},
$$

$$
P^{-1}(A^T P + PA + B^T B + P^T P) = \begin{pmatrix} 4 & 2 \\ 2 & 7 \end{pmatrix}.
$$

Thus $\lambda_7 = \max\{\lambda_5, \lambda_6\} = \lambda_5 = \lambda_6 = 1/4$, $\lambda_3 = 8$. Apply Theorem 1,

$$
(\tau_{k+1}^r - \tau_k^l) < -\frac{\ln \lambda_7}{\lambda_3} = \frac{\ln 2}{4}.
$$

So, the zero solution of system (4.1) is uniformly stable, if

$$
(\tau_{k+1}^r - \tau_k^l) < \frac{\ln 2}{4}.
$$

5. Conclusions

In this paper, we have studied the uniformly stability of impulsive delayed linear systems with impulse time windows. We have obtained some conditions to ensure that the systems are uniformly stable. An example is also given to illustrate the efficiency of the results.

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