ON CONNECTIONS BETWEEN VECTOR SPACES AND HYPERCOMPOSITIONAL STRUCTURES

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Abstract. During his sort life, F. Marty, through three articles of his, introduced the notion of hypergroup. W. Prenowitz utilized this structure in the study of Geometry. This paper contributes to the methodology of connecting vector spaces with hypergroups. Convexity is presented in hypercompositional algebra terms and we get to the theorems of Kakutani, Stone, Helly, Randon, Carathéodory and Steinitz, through more general theorems which are valid in hypergroups.

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1. Hypergroups and the theorems of Kakutani and Stone

In 1934, F. Marty, in order to study problems in non-commutative algebra, such as cosets determined by non-invariant subgroups, generalized the notion of the group, thus defining the *hypergroup* [19], [20], [21], an algebraic structure in which the result of the composition of two elements is not an element, but a set of elements. More specifically, an *operation* or *composition* in a non-void set H is a function from $H \times H$ to H, while a *hyperoperation* or *hypercomposition* is a function from $H \times H$ to the powerset $P(H)$ of H. An algebraic structure that satisfies the axioms

(i) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for every $a, b, c \in H$ (associative axiom), and

(ii) $a \cdot H = H \cdot a = H$ for every $a \in H$ (reproductive axiom),

is called *group* if "$\cdot$" is a composition, and *hypergroup* if "$\cdot$" is a hypercomposition [36], [37].

Proposition 1.1. *If a non-void set H is endowed with a composition which satisfies the associative and the reproductive axiom, then H has a bilateral neutral element and any element in H has a bilateral symmetric.*

Proof. Let $x \in H$. Because of reproductive axiom $x \in xH$. Therefore, there exists $e \in H$ such that $xe = x$. Next, let y be an arbitrary element in H. Per reproductive axiom there exists $z \in H$ such that $y = zx$. Consequently, $ye = (zx)e = z(xe) = zx = y$. Hence e is a right neutral element. In an analogous way, there exists a left neutral element e'. Then, the equality $e = e'e = e'$ is valid. Therefore, e is the bilateral neutral element of H. In addition, because of reproductive axiom $e \in xH$. Thus, there exists $x' \in H$, such that $e = xx'$. Hence, any element in H has a right symmetric. Similarly, any element in H has a left symmetric and it is easy to prove that these two symmetric elements coincide. ∎

Remark. An analogous proposition to Proposition 1.1 is not valid when H is endowed with a hypercomposition. In hypergroups there exist different types of neutral elements [34], [53] (e.g., scalar [4], [45], strong [17], [30], [41] etc.). There also exist special types of hypergroups which have a neutral element and each one of their elements has one symmetric element (e.g., canonical hypergroups [45], quasicanonical hypergroups [27], fortified join hypergroups [41], fortified transposition hypergroups [17]) or more symmetric elements (e.g., transposition polysymmetrical hypergroups [30], canonical polysymmetric hypergroups [48], M-polysymmetric hypergroups [33]).

Both equations $a = xb$ and $a = bx$ have a unique solution in groups. On the contrary, in the case of hypergroups, the analogous relations $a \in xb$ and $a \in bx$ do not have unique solutions. Thus, F. Marty in [19] defined the two induced hypercompositions (right and left division) that derive from the hypercomposition of the hypergroup:

$$\frac{a}{|b} = \{x \in H \mid a \in xb\} \qquad \text{and} \qquad \frac{a}{b|} = \{x \in H \mid a \in bx\}.$$

If H is a group, then $\frac{a}{|b} = ab^{-1}$ and $\frac{a}{b|} = b^{-1}a$. It is obvious that if "." is commutative, then the right and the left division coincide. For the sake of notational simplicity, a/b or $a : b$ is used to denote the right division (or right hyperfraction) as well as the division in commutative hypergroups and $b \setminus a$ or $a..b$ is used to denote the left division (or left hyperfraction) [16], [22], [25].

Consequences of axioms (i) and (ii) are [22], [25]:

(i) $ab \neq \emptyset$, for all a, b in H,

(ii) $a/b \neq \emptyset$ and $a \setminus b \neq \emptyset$, for all a, b in H,

(iii) $H = H/a = a/H$ and $H = a \setminus H = H \setminus a$, for all a in H.

Proposition 1.2. [16], [22], [25] *In any hypergroup*

(i) $(a/b)/c = a/(cb)$ *and* $c \setminus (b \setminus a) = (bc) \setminus a$ *(mixed associativity),*

(ii) $(b \setminus a)/c = b \setminus (a/c)$,

(iii) $b \in (a/b) \setminus a$ *and* $b \in a/(b \setminus a)$.

A hypercomposition in a non-void set H is called *closed* if the two participating elements are always included in the result, i.e., if $a, b \in ab$ for all $a, b \in H$. For example, if H is a non-void set and $ab = \{a, b\}$ for all $a, b \in H$ or, if $(H, \cdot)$ is a semigroup and $ab = \{a, b, a \cdot b\}$ for all $a, b \in H$, then these are closed hypercompositions. A hypercomposition is called *right closed* if $a \in ba$ for all $a, b \in H$ and left closed if $a \in ab$ for all $a, b \in H$. A hypercomposition is called *right open* if $a \notin ba$ for all $a, b \in H$ with $b \neq a$. The definition of the *left open* hypercomposition is similar. Obviously, a hypercomposition is *open*, if it is both right and left open.

Proposition 1.3. *The hypercomposition in a hypergroup H is right closed if and only if $a/a = H$ for all $a \in H$, while it is left closed if and only if $a \setminus a = H$ for all $a \in H$.*

Proof. Suppose that the hypercomposition is right closed. Then $a \in xa$ for all $x \in H$. Hence $x \in a/a$ for all $x \in H$. Therefore, $H = a/a$. Conversely now. Let $H = a/a$ for all $a \in H$. Then $a \in ba$ for all $a, b \in H$. Thus the hypercomposition is right closed. ∎

Proposition 1.4. *The hypercomposition in a hypergroup H is right open if and only if $a/a = a$ for all $a \in H$, while it is left open if and only if $a \setminus a = a$ for all $a \in H$.*

Proof. Suppose that the hypercomposition is right open. Let a be an arbitrary element of H. Then $a \notin ba$ for all $b \in H$ with $b \neq a$. Hence $b \notin a/a$ for all $b \in H$ with $b \neq a$. Moreover, because of the reproductive axiom, $a \in Ha$, thus $a \in aa$. Therefore, $a = a/a$. Conversely now. Let $a/a = a$ for all $a \in H$. Then $b \notin a/a$ for all $b \in H$ with $b \neq a$. So $a \notin ba$, for all $b \in H$ with $b \neq a$, i.e., the hypercomposition is right open. ∎

Proposition 1.5. *If the hypercomposition in a hypergroup H is right or left open, then all its elements are idempotent.*

Proof. Suppose that the hypercomposition is right open and that for some $a \in H$ there exists $b \neq a$, such that $b \in aa$. Then, $a/b \subseteq a/aa$. Because of Propositions 1.2(i) and 1.4, $a/(aa) = (a/a)/a = a/a = a$. Thus, $a/b = a$. Therefore, $a \in ab$, which contradicts the assumption. Hence, $aa = a$ for all $a \in H$. ∎

A non-empty subset K of H is called *semi-subhypergroup* when it is stable under the hypercomposition, i.e., it has the property $xy \subseteq K$ for all $x, y \in K$.

Proposition 1.6. *If A, B are semi-subhypergroups of a commutative hypergroup H, then AB is a semi-subhypergroup of H as well.*

K is a subhypergroup of H, if it satisfies the axiom of reproduction, i.e. if the equality $xK = Kx = K$ is valid for all $x \in K$. This means that when K is a subhypergroup, the relations $a \in bx$ and $a \in yb$ can always be solved in K. The non-void intersection of two subhypergroups, although stable under the hypercomposition, usually is not a subhypergroup, since the reproduction is not always valid. In other words the solutions of the relation $a \in yb$ and $a \in bx$ do not lie in the intersection when a and b are elements of the intersection. This led (from the very early steps of hypergroup theory) to the consideration of more special types of subhypergroups. One of them is the *closed subhypergroup*. A subhypergroup K of H is called *left closed* with respect to H, if for any two elements a and b in K all possible solutions of the relation $a \in yb$ lie in K. This means that K is left closed if and only if $a/b \subseteq K$, for all $a, b \in K$. Similarly, K is right closed when all possible solutions of the relation $a \in bx$ lie in K or, equivalently, if $b \setminus a \subseteq K$ for all $a, b \in K$ [24], [25], [37]. Finally, K is *closed* when it is both right and left closed. The non-void intersection of two closed subhypergroups is a closed subhypergroup.

It has been proven ([24], [25]) that the set of the semi-subhypergroups (resp., the set of the closed subhypergroups) which contain a non-void subset E is a complete lattice. Hence, given a non-empty subset E of a hypergroup H, the minimum semi-subhypergroup (in the sense of inclusion) which contains E can be assigned. This semi-subhypergroup is denoted by $[E]$ and it is called the generated by E semi-subhypergroup of H. Similarly, $\langle E \rangle$ is the generated by E closed subhypergroup of H. For notational simplicity, if $E = \{a_1, ..., a_n\}$, $[E] = [a_1, ..., a_n]$ and $\langle E \rangle = \langle a_1, ..., a_n \rangle$ are used instead.

F. Marty's life was short, as he died in a military mission during World War II and [19], [20], [21] are the only works on hypergroups he left behind. However, several papers by other authors began to appear shortly thereafter and until now hundreds of papers have been written on this issue (e.g. see [4], [9]). Moreover since the hypergroup is a very general structure, it was progressively enriched with further axioms, more or less powerful, thus leading to a significant number of special hypergroups – e.g., [4], [9], [11], [16], [17], [18], [28], [29], [30], [33], [41], [45], [47], [52]. Thus, W. Prenowitz enriched hypergroups with an axiom, in order to use them in the study of geometry. More precisely, he introduced into the commutative hypegroup, the *transposition axiom*:

$$a/b \cap c/d \neq \emptyset \text{ implies } ad \cap bc \neq \emptyset \text{ for all } a, b, c, d \in H$$

and named this new hypergroup *join space* [54], [55], [56], [57], [58], [59]. W. Prenowitz utilized this structure in the study of Geometry. Prenowitz was followed by others, such as J. Jantosciak [15], [58],V.W. Bryant, R.J. Webster [2], D. Freni [12], [13] etc. Material from the above mentioned authors, as well as from previous work of the author of this paper, is used in this study in order to make it self-contained.

At this point, it is worth mentioning that a big number of researchers dealt with the further study of the certain hypergroup which W. Prenowitz introduced (see, e.g., [1], [3], [5], [6], [7], [8], [10], [14], [28], [29], [30], [35], [65]).

It is also worth mentioning that the generalization of the vector spaces, which are associated directly with the algebraic study of geometry, attracted the interest of many researchers. So, J. Mittas [46], [50] and M. Scafati-Tallini [60]-[64] presented their approach to the generalization of the vector spaces in the hypercompositional algebra.

Later on, J. Jantosciak generalized the transposition axiom in an arbitrary hypergroup as follows:

$$b \setminus a \cap c/d \neq \emptyset \ \text{ implies } \ ad \cap bc \neq \emptyset \text{ for all } a, b, c, d \in H.$$

He named this particular hypergroup *transposition hypergroup* [16]. For the sake of terminology unification, join spaces are also called *join hypergroups*. It has been proven that these hypergroups also comprise a useful tool in the study of languages and automata [31], [38], [40], [43] and a constructive origin for the development of other, new hypercompositional structures [32], [39], [42], [44], [50], [51].

Proposition 1.7. [24], [29] *The following are true in any join hypergroup:*

(i) $a(b/c) \cup b(a/c) \cup a/(c/b) \cup b/(c/a) \subseteq ab/c$,

(ii) $(a/b)(c/d) \cup (a/d)(c/b) \cup (a/b)/(d/c) \cup (a/d)/(b/c) \cup (c/d)/(b/a)$
$\cup\, (c/b)/(d/a) \subseteq ac/bd$.

Corollary 1.1. *If A, B are semi-subhypergroups of a join hypergroup H, then A/B is a semi-subhypergroup of H.*

Proposition 1.8. *Let V be a vector space over an ordered field F. Then V, when endowed with the hypercomposition*

$$ab = \{\kappa a + \lambda b \mid \kappa, \lambda > 0,\ \kappa + \lambda = 1\},$$

becomes a join hypergroup (join space).

This hypergroup, which was derived from the vector space and is connected with it, was named *attached hypergroup* of V [24], [25]. Observe that the hypercomposition of the attached hypergroup is an open hypercomposition. In [49], one can find some other hypergroups annexed to vector spaces and in [23], [26] more hypecompositional structures connected to vector spaces. A direct consequence of the above proposition is that the convex sets of V are the semi-subhypergroups of the attached hypergroup H_V, while the subspaces of V are the closed subhypergroups of this hypergroup [24], [25].

The following two theorems result in two known propositions of vector spaces, thus showing the importance of the connection of vector spaces with hypergroups,which is achieved through the attached hypergroup.

Theorem 1.1. *Let A, B be two disjoint semi-subhypergroups in a join hypergroup and let x be an idempotent element not in the union $A \cup B$. Then $[A \cup \{x\}] \cap B = \emptyset$ or $[B \cup \{x\}] \cap A = \emptyset$.*

Proof. Suppose that $[A \cup \{x\}] \cap B \neq \emptyset$ and $[B \cup \{x\}] \cap A \neq \emptyset$. Since x is idempotent, the equalities $[A \cup \{x\}] = Ax$ and $[B \cup \{x\}] = Bx$ are valid. Thus, there exists $a \in A$ and $b \in B$, such that $ax \cap B \neq \emptyset$ and $bx \cap A \neq \emptyset$. Hence,

$x \in B/a$ and $x \in A/b$. Thus, $B/a \cap A/b \neq \emptyset$. Next, by application of the transposition axiom, we arrive at $Bb \cap Aa \neq \emptyset$. However, $Bb \subseteq B$ and $Aa \subseteq A$, since A, B are semi-subhypergroups. Therefore, $A \cap B \neq \emptyset$, which contradicts the theorem's assumption. ■

Corollary 1.2. *Let H be a join hypergroup endowed with an open hypercomposition. If A, B are two disjoint semi-subhypergroups of H and x is an element not in the union $A \cup B$, then $[A \cup \{x\}] \cap B = \emptyset$ or $[B \cup \{x\}] \cap A = \emptyset$.*

Corollary 1.3. (Kakutani's Lemma) *If A, B are disjoint convex sets in a vector space and x is a point not in their union, then either the convex envelope of $A \cup \{x\}$ and B or the convex envelope of $B \cup \{x\}$ and A are disjoint.*

Theorem 1.2. *Let H be a join hypergroup consisting of idempotent elements and suppose that A, B are two disjoint semi-subhypergroups in H. Then, there exist disjoint semi-subhypergroups M, N such that $A \subseteq M$, $B \subseteq N$ and $H = M \cup N$.*

Proof. Suppose that M and N are the maximum disjoint semi-subhypergroups such that $A \subseteq M$, $B \subseteq N$. If we assume that $M \cup N \subset H$, then there exists an element w in H, which does not belong to the union $M \cup N$. Therefore, per Theorem 1.1, either $[M \cup \{w\}] \cap N = \emptyset$ or $[N \cup \{w\}] \cap M = \emptyset$ is valid. This contradicts the hypothesis that M and N are the maximum disjoint semi-subhypergroups with the required property. Hence $H = M \cup N$. ■

Corollary 1.4. (Stone's Theorem) *If A, B are disjoint convex sets in a vector space V, there exist disjoint convex sets M and N, such that $A \subseteq M$, $B \subseteq N$ and $V = M \cup N$.*

2. Closed subhypergroups and Helly's theorem

As mentioned above, every vector subspace of a vector space V, considered as a subset of the attached hypergroup of V, is a closed subhypergroup of this hypergroup. Therefore, properties of vector subspaces can derive as corollaries of more general properties that are valid in closed subhypergroups. An interesting issue is the construction of closed subhypergroups from a finite set of elements.

Proposition 2.1. *Let H be a commutative hypergroup and $\{a_1, ..., a_n\} \subseteq H$. Then,*

$$[a_1, a_2, ..., a_n] = ([a_1] \cup [a_2] \cup \cdots \cup [a_n]) \cup ([a_1][a_2] \cup \cdots \cup [a_{n-1}][a_n]) \cup \cdots \cup ([a_1] \cdots [a_n]).$$

Proof. It is obvious that the right part of the above equality is a subset of the left part. Inversely, suppose that $x \in [a_{i_1}] \cdots [a_{i_m}]$ and $y \in [a_{j_1}] \cdots [a_{j_n}]$. Then, $xy \subseteq [a_{i_1}] \cdots [a_{i_m}][a_{j_1}] \cdots [a_{j_n}]$ and, through rearrangement of the indices, $xy \subseteq [a_{k_1}] \cdots [a_{k_r}]$. ■

Proposition 2.2. *Let H be a hypergroup and $a \in H$. Then,* $[a] = a^1 \cup a^2 \cup \cdots \cup a^k \cup \cdots$, *where* $a^1 = \{a\}$, $a^2 = aa$ *and* $a^i = aa^{i-1}$.

Proposition 2.3. *If the hypercomposition in a hypergroup H is right (resp. left) open, then* $a/[a] = a$ (*resp.* $[a] \setminus a = a$).

Proof. Because of mixed associativity and per Proposition 1.4, the equality $a/aa = (a/a)/a = a/a = a$ is valid. The rest follow throw induction. ∎

Proposition 2.4. [25] *In every commutative hypergroup H, the set* $\prod_{i=1}^{n}[a_i]$ *is a semi-subhypergroup of H, which absorbs every element of* $[a_1, ..., a_n]$.

An extensive presentation of properties of semi-subhypergroups of commutative hypergroups can be found in [25].

Definition 2.1. In a hypergroup H the elements $a_1, ..., a_n$ are called *correlated*, if there exist distinct integers $i_1, ..., i_k, j_1, ..., j_n$ that belong to $\{1, ..., n\}$, such that $[a_{i_1}, ..., a_{i_k}] \cap [a_{j_1}, ..., a_{j_m}] \neq \emptyset$. Otherwise, $a_1, a_2, ..., a_n$ are called *non-correlated*.

In a hypergroup endowed with an open hypercomposition, elements $a_1, ..., a_n$ are correlated, if there exist distinct integers $i_1, ..., i_k, j_1, ..., j_m \in \{1, ..., n\}$, such that $a_{i_1} \cdots a_{i_k} \cap a_{j_1} \cdots a_{j_m} \neq \emptyset$. As proven in [24], [25], in the case of the attached hypergroup H_V of a vector space V, a subset of H_V consists of correlated elements if and only if these elements are affinely dependent in V.

Proposition 2.5. *Let A be a semi-subhypergroup of a join hypergroup H. Then, A/A is a closed subhypergroup of H containing A.*

Proof. Let x, y be arbitrary elements in A/A. Then, there exist $a, b, c, d \in A$ such that $x \in a/b$ and $y \in c/d$. Per Proposition 1.7(ii):

$$xy \subseteq (a/b)(c/d) \subseteq ac/bd \subseteq A/A$$

and

$$x/y \subseteq (a/b)/(c/d) \subseteq ad/bc \subseteq A/A.$$

Hence, A/A is stable both under the hypercomposition and the induced hypercomposition. Next, $xA \subseteq A$ is valid for all $x \in A$. Hence $x \in A/A$ for all $x \in A$. Therefore, $A \subseteq A/A$. ∎

Proposition 2.6. *Let H be a join hypergroup and let $\{a_1, ..., a_n\} \subseteq H$. Then,*

$$\langle a_1, a_2, ..., a_n \rangle = [a_1] \cdots [a_n]/[a_1] \cdots [a_n].$$

Proof. Because of Proposition 2.4, $\prod_{i=1}^{n}[a_i] = [a_1]\cdots[a_n]$ is a semi-subhypergroup of H. Therefore, because of Proposition 2.5, $\prod_{i=1}^{n}[a_i]/\prod_{i=1}^{n}[a_i]$ is a closed subhypergroup of H. Since $[a_i]$ is a semi-subhypergroup, the inclusion $a_i[a_i] \subseteq [a_i]$ is valid. Hence, $a_i[a_1]\cdots[a_n] \subseteq [a_1]\cdots[a_n]$. Therefore, $a_i \in [a_1]\cdots[a_n]/[a_1]\cdots[a_n]$, $1 \le i \le n$. ∎

Corollary 2.1. *If H is a join hypergroup endowed with open hypercomposition and $\{a_1,...,a_n\} \subseteq H$, then $\langle a_1, a_2, ..., a_n\rangle = a_1\cdots a_n/a_1\cdots a_n$.*

Theorem 2.1. *Suppose that elements $a_1,...,a_n$ of a hypergroup H are correlated. Consider all the semi-subhypergroups of H generated from $n-1$ elements of the above. Then, the intersection of all these semi-subhypergroups is non-void.*

Proof. Since the elements are correlated, there are distinct integers $i_1,...,i_r, j_1,...,$ $j_s \in \{1,...,n\}$ such that $[a_{i_1},...,a_{i_r}]\cap[a_{j_1},...,a_{j_s}] \neq \emptyset$. But $[a_{i_1},...,a_{i_r}]$ or $[a_{j_1},...,a_{j_s}]$ is contained in any semi-subhypergroup which is generated by $n-1$ elements from $a_1,...,a_n$. Thus, the intersection of all these semi-subhypergroups contains the elements of $[a_{i_1},...,a_{i_r}] \cap [a_{j_1},...,a_{j_s}]$ and, therefore, is non-void. ∎

Theorem 2.2. *Suppose that H is a hypergroup in which every set of cardinality greater than n consists of correlated elements. If $(K_i)_{i\in I}$, $\operatorname{card} I > n$, is a finite family of semi-subhypergroups of H, in which the intersection of every n members is non-void, then all the semi-subhypergroups $(K_i)_{i\in I}$ have a non-void intersection.*

Proof. The theorem will be proven by induction. First, it will be shown that the intersection of every $n+1$ semi-subhypergroups is non-void. Without loss of generality, this will be proven for semi-subhypergroups K_i, $1 \le i \le n+1$. Thus, let $x_i \in \bigcap_{j\neq i} K_j$. Then, $x_1,...,x_{i-1},x_{i+1},...,x_{n+1} \in K_i$. Therefore, $[x_1,...,x_{i-1},x_{i+1},...,x_{n+1}] \subseteq K_i$. Since every $n+1$ elements of H are correlated, the elements $x_1,...,x_{n+1}$ are correlated. Because of Theorem 2.1, the semi-subhypergroups $[x_1,...,x_{i-1},x_{i+1},...,x_{n+1}]$, $1 \le i \le n+1$, have a non-void intersection. Consequently, the sets K_i, $1 \le i \le n+1$, also have a non-void intersection. Next, suppose that the intersection of the members of each set of $(\operatorname{card} I) - 1$ semi-subhypergroups is non-void. For each $i \in I$, we choose an element x_i of the intersection $\bigcap_{j\neq 1} K_j$. Then, sets $X_i = \{x_j,\ j \in I - \{i\}\} \subseteq K_i$ are constructed. These sets generate the semi-subhypergroups $[X_i]$, $i \in I$. Since every $n+1$ elements of H are correlated, elements $\{x_i,\ i \in I\}$ are correlated and, because of Theorem 2.1, semi-subhypergroups $[X_i]$, $i \in I$, have a non-void intersection. Consequently, semi-subhypergroups K_i, $i \in I$, have a non-void intersection. ∎

In the case of the attached hypergroup of a vector space V, a subset of H_V consists of correlated elements if and only if these elements are affinely dependent [24], [25]. Therefore, we have the corollary:

Corollary 2.2. (Helly's Theorem) *Let us consider a finite family* $(C_i)_{i\in I}$ *of convex sets in* $\mathbb{R}^d$*, with* $d+1 < \operatorname{card} I$*. Then, if any* $d+1$ *of the sets* C_i *have a non-empty intersection, all the sets* C_i *have a non-empty intersection.*

3. Dimension theory in hypergroups and Randon's theorem

During his study on join spaces, Prenowitz introduced a new axiom, which he named "exchange postulate":

$$\text{if } c \in \langle a, b\rangle \text{ and } c \neq a, \text{ then } \langle a, b\rangle = \langle a, c\rangle\,.$$

Consequently, join spaces that satisfy this axiom were named "exchange spaces" [55], [57], [59]. The above axiom enabled Prenowitz to develop a theory of linear independence and dimension of a type familiar to classical geometry. On the other hand, a generalization of this theory has been achieved by Freni, who developed the notions of independence, dimension, etc. in a hypergroup H that satisfies only the axiom:

$$x \in \langle A \cup \{y\}\rangle\,,\; x \notin \langle A\rangle \Longrightarrow y \in \langle A \cup \{x\}\rangle\,, \text{ for every } x, y \in H \text{ and } A \subseteq H.$$

Freni called these hypergroups *cambiste* [4], [12], [13].

A subset B of a hypergroup H is called free or independent if either $B = \emptyset$, or $x \notin \langle B - \{x\}\rangle$ for all $x \in B$, otherwise it is called non-free or dependent. B generates H, if $\langle B\rangle = H$, in which case B is a set of generators of H. If H has a finite set of generators, it is called a finite type hypergroup. A free set of generators is a basis of H. Among the results reached by Freni are:

Proposition 3.1. *Let* B *be a non-empty subset of a cambiste hypergroup* H*.* B *is a basis of* H *if and only if:*

(i) B *is a maximal free set, and*

(ii) B *is a minimal set of generators of* H*.*

Proposition 3.2. *Every cambiste hypergroup has at least one basis.*

Proposition 3.3. *All the bases of a cambiste hypergroup have the same cardinality.*

The *dimension* of a cambiste hypergroup H (denoted by $\dim H$) is the cardinality of any basis of H.

The dimension theory gives very interesting results in convexity hypergroups. A *convexity hypergroup* is a join hypergroup which satisfies the axioms:

(i) the hypercomposition is open,

(ii) $ab \cap ac \neq \emptyset$ implies $b = c$ or $b \in ac$ or $c \in ab$.

Prenowitz, defined this hyperstructure with equivalent axioms to the above, named it *convexity space* and used it, as did Bryant and Webster [2], for generalizing some of the theory of linear spaces. A direct consequence of Propositions 1.4 and 1.5 is the following propositions:

Proposition 3.4. *All the elements of a convexity hypergroup H are idempotent and, moreover, $a/a = a$ for all $a \in H$.*

Proposition 3.5. *The following are true in any convexity hypergroup:*

(i) $ab/ac = (b/c) \cup (ab/c) \cup (b/ac)$,

(ii) $ab/a = \{b\} \cup ab \cup (b/a)$,

(iii) $a/ab = a/b$.

Proof. (i) Let $x \in ab/ac$. Then, $ab \cap a(xc) \neq \emptyset$. Hence, $b \in xc$ or $b \in a(xc) = x(ac)$ or $xc \cap ab \neq \emptyset$. Therefore, $x \in b/c$ or $x \in b/ac$ or $x \in ab/c$. Thus, $ab/ac \subseteq (b/c) \cup (ab/c) \cup (b/ac)$. Next, for the opposite inclusion, suppose that:

(a) $x \in b/c$, then $b \in cx \Longrightarrow ab \in acx \Longrightarrow x \in ab/ac$.

(b) $x \in ab/c$, then $xc \cap ab \neq \emptyset$. Since the hypercomposition is open, $aa = a$ is valid. Therefore, $xac \cap ab \neq \emptyset$. Hence, $x \in ab/ac$.

(c) $x \in b/ac$, then $b \in acx \Longrightarrow ab \cap acx \neq \emptyset$. Hence, $x \in ab/ac$.

From (a), (b) and (c), the desired result follows. Therefore, (i) is valid.

(ii) According to Proposition 1.7(i), we have

$$a(b/a) \cup b(a/a) \cup a/(a/b) \cup b/(a/a) \subseteq ab/a.$$

Since the hypercomposition is open, $a/a = a$ is valid. Also, according to Proposition 1.2(iii), $b \in a/(a/b)$ is valid. From the above inclusion it follows that $ba \cup \{b\} \cup b/a \subseteq ab/a$. The opposite inclusion is easily proven and, therefore, (ii) follows.

(iii) is a direct consequence of mixed associativity.

Corollary 3.1. *If a, b with $a \neq b$ are two elements of a convexity hypergroup H, then $ab/ab = ab \cup a/b \cup b/a \cup \{a, b\}$.*

Remark. The above proposition supplies us with a simplification canon (rule) for hyperfractions in convexity hypergroups.

So, we are naturally led to the following definition:

Definition 3.1. A hyperfraction whose numerator and denominator consist of hyperproducts in which a common factor does not exist, will hereafter be called *irreducible hyperfraction.*

Proposition 3.6. *If K is a closed subhypergroup of a convexity hypergroup H, then $\langle\{x\} \cup K\rangle = K \cup xK/K \cup K/x$.*

Proof. Since x is idempotent, according to Proposition 1.6, xK is a semi-subhypergroup and, therefore, according to Proposition 2.5, xK/xK is a closed subhypergroup. Next, suppose that $t \in K$. Equality $xt = xt$, combined with Proposition 3.4, gives: $xxt = xt \Longrightarrow x \in xt/xt$ and $xtt = xt \Longrightarrow t \in xt/xt$. Hence, $\{x\}\cup K \subseteq xK/xK$. Proposition 3.5 is employed to conclude that $xK/xK = K/K \cup xK/K \cup K/xK$. Since K is closed, $K/K = K$ is valid and mixed associativity gives $K/xK = (K/K)/x = K/x$. Therefore, the proposition is established.

Proposition 3.7. *If $a_1, ..., a_n$ are elements of a convexity hypergroup, then the closed subhypergroup generated by these elements is the union of hyperproducts of the form $a_{i_1}, ..., a_{i_s}$ (where $1 \leq i_j \leq n$) and irreducible hyperfractions of the form $a_{i_1} \cdots a_{i_k}/a_{i_{k+1}} \cdots a_{i_r}$, $1 \leq k < r \leq n$.*

Proof. According to Corollary 2.1, $\langle a_1, a_2, ..., a_n\rangle = a_1a_2 \cdots a_n/a_1a_2 \cdots a_n$. Next, the previous proposition applies and yields the equality:

$$\begin{aligned} &a_1a_2 \cdots a_n/a_1a_2 \cdots a_n \\ &\quad = a_2 \cdots a_n/a_2 \cdots a_n \cup a_1a_2 \cdots a_n/a_2 \cdots a_n \cup a_2 \cdots a_n/a_1a_2 \cdots a_n. \end{aligned}$$

The previous proposition applies again repeatedly to rewrite the sets of the right-hand side of the above equality as union of hyperproducts and irreducible hyperfractions and so the proposition is established. ∎

Example. If a, b with $a \neq b$ are two elements in a convexity hypergroup H, then $[a, b] = ab$. Therefore, $\langle a, b\rangle = ab/ab$. Hence, according to Corollary 3.1, $\langle a, b\rangle = ab \cup a/b \cup b/a \cup \{a, b\}$.

Proposition 3.8. *Every convexity hypergroup is a cambiste hypergroup.*

Proof. Suppose that A is a subset of a convexity hypergroup H and that x, y are elements of H such that $x \in \langle A \cup \{y\}\rangle$, $x \notin \langle A\rangle$. Then, the previous proposition applies, yielding $\langle\{y\} \cup A\rangle = A \cup yA/A \cup A/y$. Thus, either $x \in yA/A$ or $x \in A/y$. Hence, $y \in xA/A$ or $y \in A/x$. Therefore, $y \in \langle A \cup \{x\}\rangle$.

Theorem 3.1. *Every $n + 1$ elements of an n-dimensional convexity hypergroup H are correlated.*

Proof. Let $A = \{a_1, ..., a_n, a_{n+1}\}$ be a subset of $n + 1$ elements of H. Without loss of generality, suppose that $a_{n+1} \in \langle a_1, ..., a_n\rangle$. Then, according to Proposition 3.7, either $a_{n+1} \in a_{i_1} \cdots a_{i_s}$ or $a_{n+1} \in a_{i_1} \cdots a_{i_k}/a_{i_{k+1}} \cdots a_{i_r}$, where $1 \leq i_j \leq n$, $1 \leq k < r \leq n$ and $i_j \neq i_\ell$, when $j \neq \ell$. In the first case, $[a_{n+1}] \cap [a_{i_1}, ..., a_{i_s}] \neq \emptyset$ and in the second case, $[a_{n+1}, a_{i_{k+1}}, ..., a_{i_r}] \cap [a_{i_1}, ..., a_{i_k}] \neq \emptyset$. ∎

One can easily see that the attached hypergroup of a vector space is a convexity hypergroup and, moreover, if the dimension of the attached hypergroup H_V of a vector space V is n, then the dimension of V is $n - 1$.

Corollary 3.2. (Radon's Theorem). *Any set of $d + 2$ points in R^d can be partitioned into two disjoint subsets, whose convex hulls intersect.*

4. Carathéodory-type theorems

Proposition 4.1. *Suppose that an element x of a convexity hypergroup belongs both to a hyperproduct $a_1 \cdots a_n$ and to an irreducible hyperfraction $a_{i_1} \cdots a_{i_k}/a_{i_{k+1}} \cdots a_{i_r}$, where $\{a_{i_1}, ..., a_{i_k}\}$ and $\{a_{i_{k+1}}, ..., a_{i_r}\}$ are non-empty subsets of $\{a_1, ..., a_n\}$. Then, x belongs to a hyperproduct with factors from a proper subset of $\{a_1, ..., a_n\}$.*

Proof. The proposition will be proven by induction on the denominator of the hyperfraction. Without loss of generality, suppose that $x \in a_{i_1} \cdots a_{i_k}/a_1$. Then, $a_1 \in a_{i_1} \cdots a_{i_k}/x$. Moreover, $x \in a_1 \cdots a_n$, hence, $a_1 \in x/a_2 \cdots a_n$. The above, in combination with the transposition axiom, lead to $x \in a_2 \cdots a_n$. Thus, the proposition is true, if the denominator of the hyperfraction consists of one element. Next, assume that the proposition holds true if the hyperproduct of the denominator has m factors. Without loss of generality, suppose that $x \in a_{i_1} \cdots a_{i_k}/a_1 \cdots a_m a_{m+1}$. Then, $a_1 \in a_{i_1} \cdots a_{i_k}/x a_2 \cdots a_m a_{m+1}$. Moreover, $a_1 \in x/a_2 \cdots a_n$. Therefore, the transposition axiom implies that $x a_2 \cdots a_m a_{m+1} \cap a_2 \cdots a_n \neq \emptyset$. Thus, $x \in a_2 \cdots a_n/a_2 \cdots a_m a_{m+1}$. Proposition 3.7 applies, yielding either $x \in a_{i_1} \cdots a_{i_s}$, where $\{a_{i_1}, ..., a_{i_s}\} \subset \{a_1, ..., a_n\}$, and so x is written in the desired form, or $x \in a_{i_1} \cdots a_{i_k}/a_{i_{k+1}} \cdots a_{i_r}$, where $\{a_{i_1}, ..., a_{i_k}\}$ and $\{a_{i_{k+1}}, ..., a_{i_r}\}$ are non-empty disjoint subsets of $\{a_1, ..., a_n\}$. In this latter case the number of factors of the denominator is less than $m+1$ and the induction hypothesis implies the result. ∎

Theorem 4.1. *If an element x of an n-dimensional convexity hypergroup H belongs to a hyperproduct of $n+1$ elements, then there exists a proper subset of these elements which contains x in their hyperproduct.*

Proof. Suppose that $a_1, ..., a_n, a_{n+1}$ are $n+1$ elements of H, such that $x \in a_1 \cdots a_n a_{n+1}$. Without loss of generality, suppose that $a_{n+1} \in \langle a_1, ..., a_n \rangle$. Then, according to Proposition 3.7, either $a_{n+1} \in a_{i_1} \cdots a_{i_s}$, or $a_{n+1} \in a_{i_1} \cdots a_{i_k}/a_{i_{k+1}} \cdots a_{i_r}$, where $1 \leq i_j \leq n$, $1 \leq k < r \leq n$ and $i_j \neq i_\ell$, when $j \neq \ell$. In the first case, $x \in a_1 \cdots a_n$. In the second case, Proposition 1.7(i) applies, yielding

$$x \in a_1 \cdots a_n a_{n+1} \Longrightarrow x \in a_1 \cdots a_n (a_{i_1} \cdots a_{i_k}/a_{i_{k+1}} \cdots a_{i_r}) \subseteq a_1 \cdots a_n/a_{i_{k+1}} \cdots a_{i_r}.$$

Hence, according to Proposition 3.7, either $x \in a_{i_1} \cdots i_{i_s}$ or $x \in a_{i_1} \cdots a_{i_k}/a_{i_{k+1}} \cdots a_{i_r}$, where $1 \leq i_j \leq n$, $1 \leq k < r \leq n$ and $i_j \neq i_\ell$, if $j \neq \ell$. In the former case, the theorem is proven. In the latter case, the theorem results by using Proposition 4.1. ∎

Corollary 4.1. (Carathéodory's Theorem) *Any convex combination of points in R^d is a convex combination of at most $d+1$ of them.*

Corollary 4.2. *Let S and T be two finite sets of elements in an n-dimensional convexity hypergroup H. If any semi-subhypergroup generated by $k+1$, $k \leq n$ elements of S is disjoint to any semi-subhypergroup generated by $\ell+1$, $\ell \leq n$ elements of T, then $[S] \cap [T] = \emptyset$.*

Proof. Suppose that $[S] \cap [T] \neq \emptyset$ and let $x \in [S] \cap [T]$. Proposition 2.1 yields $x \in s_1 \cdots s_i \cap t_1 \cdots t_j$, where $\{s_1, ..., s_i\} \subseteq S$ and $\{t_1, ..., t_j\} \subseteq T$. Then, per Theorem 4.1, there exists proper subsets of $\{s_1, ..., s_i\}$ and $\{t_1, ..., t_j\}$ not exceeding n elements, which contains x in their hyperproduct, i.e. $x \in s_1 \cdots s_p \cap t_1 \cdots t_q$, $p, q \leq n$. The contradiction obtained proves the validity of the corollary. ∎

Proposition 4.2. *Suppose that an element x of a convexity hypergroup belongs both to a hyperproduct $a_1 \cdots a_n$ and to an irreducible hyperfraction $ya_{i_1} \cdots a_{i_k}/a_{i_{k+1}} \cdots a_{i_r}$, where $\{a_{i_1}, ..., a_{i_k}\}$ and $\{a_{i_{k+1}}, ..., a_{i_r}\}$ are non-empty subsets of $\{a_1, ..., a_n\}$. Then, there is a hyperproduct containing x with factors from both y and a proper subset of $\{a_1, ..., a_n\}$.*

The proof of the above Proposition is similar to that of Proposition 4.1. Next, using techniques analogous to those used in proving Theorem 4.1, we are led to the following theorem:

Theorem 4.2. *In an n-dimensional convexity hypergroup H, if $A=\{a_1, ..., a_n, a_{n+1}\}$, $x \in a_1 \cdots a_n a_{n+1}$ and $y \in [a_1, ..., a_n, a_{n+1}]$, then there exists a subset B of A containing at most $n-1$ elements of A, such that x belongs to the hyperproduct of y by the elements of B.*

This theorem essentially asserts that one of the n factors of the hyperproduct of Theorem 4.1 may be chosen arbitrarily from the semi-subhypergroup which is generated by the $n+1$ elements, i.e. it can be any element of $[A]$. When this theorem is applied to the attached hypergroup of a vector space, it produces an obvious generalization of Carathéodory's Theorem. Moreover, from the above theorem follows the next theorem which is an extension of Carathéodory's Theorem.

Theorem 4.3. *In an n-dimensional convexity hypergroup H, if A is a subset of H, Y is a subset of $[A]$ and $\operatorname{card} Y \geq 2$, then there exists a subset B of A containing at most $(n-1)\operatorname{card} Y$ elements of A, such that $Y \subseteq [B]$.*

Proof. Let y be an arbitrary element of Y. For each $x \in Y$, let B_x be the subset of A, containing at most $n-1$ elements of A, such that x belongs to the hyperproduct of the fixed element y by the elements of B_x. Note that B_x exists because of Theorem 4.2. Consider the union $C = \bigcup_{x \in Y - \{y\}} B_x$. Then, $\operatorname{card} C \leq (n-1)(\operatorname{card} Y - 1)$ and $x \in [C \cup \{y\}]$ for each $x \neq y$. Next, consider an arbitrary element $b \in C$. According to the above theorem, there exists a subset B_y of A containing at most $n-1$ elements of A, such that y belongs to the hyperproduct of b by the elements of B_y. We define $B = C \cup B_y$. Then, $\operatorname{card} B \leq \operatorname{card} C + \operatorname{card} B_y \leq (n-1)\operatorname{card} Y$. ∎

Definition 4.1. An element a of a semi-subhypergroup S is called *interior element* of S if, for each $x \in S$, $x \neq a$, it exists $y \in S$, $y \neq a$, such that $a \in xy$.

Consequently to the above Definition 4.1, in the case of an n-dimensional cambiste hypergroup H, an element a of a semi-subhypergroup S of H, is interior element of S, if for every closed subhypergroup K, with $\dim K = n-1$ and $a \in K$, the intersections of S with the two disjoint classes K/x and K/y are non-void, i.e. $(K/x) \cap S \neq \emptyset$ and $(K/y) \cap S \neq \emptyset$.

Proposition 4.3. *Let H be a hypergroup endowed with an open hypercomposition and K a subhypergroup of H. Then any element of K is an interior element.*

Proposition 4.4. *Let H be a hypergroup endowed with an open hypercomposition, S a semi-subhypergroup of H and I the subset of the interior elements of S. Then I absorbs S, i.e. $IS \subseteq I$.*

Proof. Suppose that $a \in I$ and $b \in S$. Let r be an element of ab. In order to prove that r is an interior element, we have to show that for any $x \in S$ it exists $y \in S$ such that $r \in xy$. Since a is an interior element, there exists $z \in S$, such that $a \in xz$. Hence, $r \in ab \subseteq (xz)b = x(zb)$. But $zb \subseteq S$. So, there exists $y \in S$ such that $r \in xy$. ∎

Proposition 4.5. *Let H be a hypergroup endowed with an open hypercomposition, S a semi-subhypergroup of H and I the subset of the interior elements of S. Then I is a subhypergroup of H.*

Proof. Suppose that $a \in I$. Because of Proposition 4.4, $aI \subseteq I$. To prove the reverse inclusion, let $b \in I$. Since b is an interior element, there exists $z \in S$, such that $b \in az$. Per Proposition 1.5, $aa = a$, hence $az = (aa)z = a(az)$. Because of Proposition 4.4, $az \subseteq aS \subseteq I$. Thus, there exists $w \in I$, such that $b \in aw$. ∎

An almost direct consequence of Theorem 4.1 and Proposition 4.5 is the following proposition:

Proposition 4.6. *Let a be an interior element of a semi-subhypergroup S of an n-dimensional convexity hypergroup H. Then a is interior element of $[A]$, where A is a subset of S with* $\operatorname{card} A \leq (n+1)^2$.

This proposition states that any interior element of a semi-subhypergroup S of an n-dimensional convexity hypergroup is interior to a finitely generated semi-subhypergroup of S.

A refinement of this proposition is the following theorem:

Theorem 4.4. *Let a be an interior element of a semi-subhypergroup S of an n-dimensional convexity hypergroup H. Then a is interior element of a semi-subhypergroup of S, which is generated by at most $2n$ elements.*

Corollary 4.3. (Steinitz's Theorem) *Any point interior to the convex hull of a set E in R^d is interior to the convex hull of a subset of E, containing $2d$ points at the most.*

In [28], one can see that some of the above landmark theorems are also valid in other types of hypergroups.

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