ON COMMUTING TRACES OF GENERALIZED BIDERIVATIONS
OF PRIME RINGS

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Abstract. In this paper, we prove some theorems on symmetric generalized biderivations of a ring, which extend a result of Vukman [9, Theorem 1] and a result of Bresar [3, Theorem 4.1].

Keywords: prime rings, Symmetric generalized biderivations, cocommuting mappings.

2010 Mathematics Subject Classification: 16W25, 16R50, 16N60.

1. Introduction

Throughout the paper all ring will be associative. We shall denote by $Z(R)$ the centre of ring $R$ and by $C$ the extended centroid of $R$, which is the centre of the two sided Martindale quotients ring $Q$ (we refer the reader [3] for more details). A ring $R$ is said to be prime (resp. semiprime) if $aRb = (0)$ implies that either $a = 0$ or $b = 0$ ( resp. $aRa = (0)$ implies that $a = 0$). We shall write for any pair of elements $x, y \in R$ the commutator $xy - yx$ and $x \circ y$ stands for the skew commutator $xy + yx$. We make extensive use of the basic commutator identities (i) $[x, yz] = [x, y]z + y[x, z]$ and (ii) $[xy, z] = [x, z]y + x[y, z]$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A derivation $d$ is inner if there exists an element $a \in R$ such that $d(x) = [a, x]$ for all $x \in R$. A mapping $D : R \times R \to R$ is said to be symmetric if $D(x, y) = D(y, x)$, for all $x, y \in R$. A mapping $f : R \to R$ defined
by \( f(x) = D(x, x) \), where \( D : R \times R \rightarrow R \) is a symmetric mapping, is called the trace of \( D \). It is obvious that in the case \( D : R \times R \rightarrow R \) is a symmetric mapping which is also biadditive (i.e. additive in both arguments). The trace \( f \) of \( D \) satisfies the relation \( f(x + y) = f(x) + f(y) + 2D(x, y) \), for all \( x, y \in R \). A biadditive symmetric mapping \( D : R \times R \rightarrow R \) is called a symmetric biderivation if \( D(xy, z) = D(x, z)y + xD(y, z) \) for all \( x, y, z \in R \). Obviously, in this case the relation \( D(x, yz) = D(x, y)z + yD(x, z) \) is also satisfied for all \( x, y, z \in R \).

Typical examples are mapping of the form \((x, y) \mapsto \lambda [x, y] \) where \( \lambda \in C \). We shall call such maps inner biderivations. In \([6]\) it was shown that every biderivation \( D \) of a noncommutative prime ring \( R \) is of the form \( D(x, y) = \lambda [x, y] \) for some \( \lambda \in C \). Further Bresar extended this result for semiprime rings. Some results on biderivations can be found in\([2]\), \([6]\) and \([8]\).

G. Maksa \([8]\) introduced the concept of a symmetric biderivation (see also \([9]\), where an example can be found). It was shown in \([8]\) that symmetric biderivations are related to general solution of some functional equations. Some results on symmetric biderivations in prime and semiprime rings can be found in \([5]\), \([11]\) and \([12]\). The notion of additive commuting mappings is closely connected with the notion of biderivations. Every commuting additive mapping \( f : R \rightarrow R \) gives rise to a biderivation on \( R \). Namely linearizing \([x, f(x)] = 0 \) for all \( x, y \in R \) \((x, y) \mapsto [f(x), y] \) is a biderivation (moreover, all derivations appearing are inner).

The notion of generalized symmetric biderivations was introduced by Nurcan \([1]\). More precisely, a generalized symmetric biderivation is defined as follows: Let \( R \) be a ring and \( D : R \times R \rightarrow R \) be a biadditive map. A biadditive mapping \( \Delta : R \times R \rightarrow R \) is said to be generalized biderivation if for every \( x \in R \), the map \( y \mapsto \Delta(x, y) \) is a generalized derivation of \( R \) associated with function \( y \mapsto D(x, y) \) as well as if for every \( y \in R \), the map \( x \mapsto \Delta(x, y) \) is a generalized derivation of \( R \) associated with function \( x \mapsto D(x, y) \) for all \( x, y \in R \). It also satisfies \( \Delta(x, yz) = \Delta(x, y)z + yD(x, z) \) and \( \Delta(xy, z) = \Delta(x, z)y + xD(y, z) \) for all \( x, y, z \in R \). For example consider a biderivation \( \Delta \) of \( R \) and biadditive a function \( \alpha : R \times R \rightarrow R \) such that \( \alpha(x, yz) = \alpha(x, y)z \) and \( \alpha(xy, z) = \alpha(x, y)z \) for all \( x, y, z \in R \). Then \( \Delta + \alpha \) is a generalized \( \Delta \)-biderivation of \( R \).

An additive mapping \( h : R \rightarrow R \) is called left (resp. right) multiplier of \( R \) if \( h(xy) = h(x)y \) (resp. \( h(xy) = xh(y) \)) for all \( x, y \in R \). A biadditive mapping \( D : R \times R \rightarrow R \) is said to be a left (resp. right) bi-multiplier of \( R \) if \( D(x, yz) = D(x, y)z \) (resp. \( D(xz, y) = xD(z, y) \)) for all \( x, y, z \in R \).

In this paper, we prove some theorems on symmetric generalized biderivations of a ring which extend a result of Vukman \([9, \text{Theorem 1}]\) and a result of Bresar \([3, \text{Theorem 4.1}]\).

2. Generalized biderivations on prime rings

The result proved in this section generalizes Theorem 1 in \([11]\). More precisely, we consider the case when the ring \( R \) is prime and replace symmetric biderivations with symmetric generalized biderivations.
In [11], Vukman proved the following result: Let $R$ be a noncommutative prime ring of characteristic different from two and $D: R \times R \rightarrow R$ be a symmetric biderivation with trace $f$. If $f$ is commuting on $R$, then $d = 0$. Vukman [10, Theorem 2] further generalized the result by proving that let $R$ be a noncommutative prime ring of characteristic different from two. Suppose there exists a symmetric biderivation $D: R \times R \rightarrow R$ with trace $f$ such that the mapping $x \mapsto [f(x), x]$ is centralizing on $R$. In this case $D = 0$.

**Theorem 2.1.** Let $R$ be a prime ring of characteristic different from two and $I$ be a nonzero left ideal of $R$. If $\Delta$ is a symmetric generalized biderivation with associated biderivation $D$ such that $[\Delta(x, x), x] = 0$ for all $x \in I$, then either $R$ is commutative or $\Delta$ acts as a left bimultiplier on $I$.

**Proof.** Suppose that

\[ [\Delta(x, x), x] = 0, \quad \text{for all } x \in I. \]  

Linearization of (2.1) yields that

\[ [\Delta(x, x), x] + [\Delta(x, y), y] + [\Delta(x, y), x] + [\Delta(x, y), y] + [\Delta(y, x), x] + [\Delta(y, y), x] + [\Delta(y, y), y] = 0, \quad \text{for all } x, y \in I. \]

Since $\Delta$ is symmetric and using (2.1), we obtain

\[ [\Delta(x, x), y] + 2[\Delta(x, y), x] + 2[\Delta(x, y), y] + [\Delta(y, y), x] = 0, \quad \text{for all } x, y \in I. \]

Substituting $-y$ for $y$ in (2.3), we have

\[ -[\Delta(x, x), y] - 2[\Delta(x, y), x] + 2[\Delta(x, y), y] + [\Delta(y, y), x] = 0, \quad \text{for all } x, y \in I. \]

Adding (2.3) and (2.4) and using char $R \neq 2$, we find

\[ 2[\Delta(x, y), y] + [\Delta(y, y), x] = 0, \quad \text{for all } x, y \in I. \]

Replace $x$ by $xz$ in (2.5) to get

\[ 2\Delta(x, y)[z, y] + 2[\Delta(x, y), y]z + 2x[D(z, y), y] + 2[x, y]D(z, y) \]

\[ + [\Delta(y, y), x]z + x[\Delta(y, y), z] = 0, \quad \text{for all } x, y, z \in I. \]

In view of (2.5), (2.6) gives that

\[ \Delta(x, y)[z, y] + 2x[D(z, y), y] + 2[x, y]D(z, y) + x[\Delta(y, y), z] = 0, \quad \text{for all } x, y, z \in I. \]

Substitute $y$ for $z$ to obtain

\[ 2x[D(y, y), y] + 2[x, y]D(y, y) = 0, \quad \text{for all } x, y \in I. \]
Since char $R$ not two, we have
\[(2.9)\quad x[D(y,y), y] + [x, y]D(y, y) = 0, \text{ for all } x, y \in I.
\]
Substitute $rx$ for $x$ in (2.9) and using (2.9), we obtain
\[(2.10)\quad [r, y]xD(y, y) = 0, \text{ for all } x, y \in I, \text{ for all } r \in R.
\]
Replace $r$ by $rs$ in (2.10), we find
\[(2.11)\quad [r, y]RxD(y, y) = 0, \text{ for all } x, y \in I, \text{ for all } r \in R.
\]
Primeness of $R$ yields that either $[y, r] = 0$ or $xD(y, y) = 0$ for all $x, y \in I$. If $[y, r] = 0$ for all $y \in I$ and $r \in R$, then $I$ is contained in $Z(R)$. Since $I$ is a central ideal of $R$, we have $R$ is commutative by [10]. On the other hand, we have $xD(y, y) = 0$ for all $x, y \in I$. Linearization in $y$ yields that $xD(y, z) + xD(z, y) = 0$ for all $x, y, z \in I$. Since $D$ is symmetric and using char $R \neq 2$, we get $xD(y, z) = 0$ for all $x, y, z \in I$, i.e. $\Delta$ acts as a left bimultiplier on $I$.

**Corollary 2.1.** Let $R$ be a prime ring of characteristic different from two and $I$ be a nonzero left ideal of $R$. If $\Delta$ is a symmetric generalized biderivation with associated biderivation $D$ such that $\Delta(x, y) + [x, y] \in Z(R)$ for all $x, y \in I$, then either $R$ is commutative or $\Delta$ acts as a left bimultiplier on $I$.

**Corollary 2.2.** Let $R$ be a prime ring of characteristic different from two and $I$ be a nonzero left ideal of $R$. If $\Delta$ is a symmetric generalized biderivation with associated biderivation $D$ such that $\Delta(x, y) + x \circ y \in Z(R)$ for all $x, y \in I$, then either $R$ is commutative or $\Delta$ acts as a left bimultiplier on $I$.

**Theorem 2.2.** Let $R$ be a prime ring of characteristic different from two and $I$ be a nonzero left ideal of $R$. If $\Delta$ is a symmetric generalized biderivation with associated biderivation $D$ such that $\Delta(x, x) \circ x = 0$ for all $x \in I$, then either $R$ is commutative or $\Delta$ acts as a left bimultiplier on $I$.

**Proof.** By assumption, we have
\[(2.12)\quad \Delta(x, x) \circ x = 0 \text{ for all } x \in I.
\]
Linearization of (2.12) yields that
\[(2.13)\quad \Delta(x, x)x + \Delta(y, y)x + 2\Delta(x, y)x + \Delta(x, x)y + \Delta(y, y)y
+ 2\Delta(x, y)y + x\Delta(y, y) + y\Delta(x, x) + 2y\Delta(x, y) = 0 \text{ for all } x, y \in I.
\]
In view of (2.12), (2.13), gives that
\[(2.14)\quad \Delta(y, y)x + 2\Delta(x, y)x + \Delta(x, x)y + 2\Delta(x, y)y + x\Delta(y, y)
+ 2x\Delta(x, y) + y\Delta(x, x) + 2y\Delta(x, y) = 0 \text{ for all } x, y \in I.
\]
Substituting \(-y\) for \(y\) in (2.14), we have
\[
\Delta(y, y)x - 2\Delta(x, y)x - \Delta(x, x)y + 2\Delta(x, y)y + x\Delta(y, y)
- 2x\Delta(x, y) - y\Delta(x, x) + 2y\Delta(x, y) = 0 \quad \text{for all } x, y \in I.
\]

Adding (2.14) and (2.15) and using the fact that \(\text{char} R \neq 2\), we get
\[
\Delta(y, y)x + 2\Delta(x, y)y + x\Delta(y, y) + 2y\Delta(x, y) = 0, \quad \text{for all } x, y \in I.
\]

Replacing \(x\) by \(xu\) in (2.16), we have
\[
\Delta(y, y)xu + 2\Delta(x, y)uy + xu\Delta(y, y) + 2yu\Delta(x, y) = 0, \quad \text{for all } x, y, u \in I.
\]

Right multiplying (2.16) by \(u\) and then subtracting from (2.17), we obtain
\[
2\Delta(u, y)[u, y] + 2x\Delta(u, y)y + xu\Delta(y, y) + 2yu\Delta(x, y) = 0, \quad \text{for all } x, y, u \in I.
\]

Substituting \(u\) by \(y\) in (2.18), we get
\[
2x\Delta(y, y)y + x[y, \Delta(y, y)] + 2yu\Delta(y, y) = 0, \quad \text{for all } x, y, u \in I.
\]

Replacing \(rx\) for \(x\) in (2.19) and using it, we obtain
\[
2rx\Delta(y, y)y + x[y, \Delta(y, y)] + 2yr\Delta(y, y) = 0, \quad \text{for all } x, y, r \in R.
\]

Left multiplying (2.19) by \(r\) and then subtracting from (2.20), we get
\[
2[y, r]\Delta(y, y) = 0, \quad \text{for all } x, y \in I \quad \text{and} \quad \text{for all } r \in R.
\]

This implies that \(2[y, r]\Delta(y, y) = 0\) for all \(x, y \in I\) and for all \(r \in R\). Since \(\text{char} R \neq 2\) we have \([y, r]\Delta(y, y) = 0\) for all \(x, y \in I\) and for all \(r \in R\). Primeness of \(R\) yields that either \([y, r] = 0\) or \(\Delta(y, y) = 0\) for all \(x, y \in I\) and for all \(r \in R\). Arguing in the similar manner as in the proof of Theorem 2.1, we get the result.

**Theorem 2.3.** Let \(R\) be a \(2, 3\) and \(5\)-torsion free semiprime ring, \(I\) an additive subgroup of \(R\) such that \(x^2 \in I\) for all \(x \in I\) and \(\Delta : R \times R \to R\) be a symmetric generalized biderivation associated with biderivation \(D\) with the trace \(f\) of \(\Delta\). If \(f\) is centralizing on \(I\), then \(f\) is commuting on \(I\).

**Proof.** Let \(x \in I\) and take \(t = [f(x), x]\), where \(f(x) = \Delta(x, x)\). Then \(t \in Z(R)\).

By our hypothesis, we have
\[
[f(x), x] \in Z(R) \quad \text{for all } x \in I.
\]
Replacing \( x \) by \( x + y \) in (2.22), we have

\[
(2.23) \quad [f(x), x] + [f(x), y] + [f(y), x] + [f(y), y] + [\Delta(x, y), x] + [\Delta(y, x), y] + [\Delta(y, x), y] \in Z(R) \quad \text{for all } x, y \in I.
\]

Putting \( x = -x \) in (2.23) and using (2.22), we get

\[
(2.24) \quad [f(x), y] + 2[\Delta(x, y), x] \in Z(R) \quad \text{for all } x, y \in I.
\]

Substituting \( x^2 \) for \( y \) in (2.24), we have

\[
(2.25) \quad [f(x), x^2] + [\Delta(x, x)x + D(x, x), x] \in Z(R) \quad \text{for all } x \in I.
\]

We have \([f(x), x^2] = [f(x), x]x + x[f(x), x] = 2tx\).

Since \([\Delta(x, x^2), x] = 2tx + 2x[D(x, x), x]\), the last expression reduces to

\[
(2.26) \quad 2x[D(x, x), x] + 2tx \in Z(R) \quad \text{for all } x \in I.
\]

Since \( R \) is 2-torsion free, we get \( x[D(x, x), x] + 2tx \in Z(R) \).

Let \( z = x[D(x, x), x] + 2tx \in Z(R) \). This implies that

\[
(z - 2tx) = x[D(x, x), x].
\]

Replacing \( x \) by \( x^2 \) in our hypothesis, we can write

\[
[f(x^2), x^2] = [\Delta(x^2, x^2), x^2] = [\Delta(x^2, x)x + xD(x^2, x), x^2]
\]

\[
= [\Delta(x^2, x), x^2]x + x[D(x^2, x), x^2]x
\]

\[
= [\Delta(x, x)x + xD(x, x), x^2]x + x[D(x, x)x + xD(x, x), x^2]
\]

\[
= [\Delta(x, x), x]x^3 + x[\Delta(x, x), x]x^2 + 2x^2[D(x, x), x]x
\]

\[
+ 2x[D(x, x), x]x^2 + x^3[D(x, x), x] + x^2[D(x, x), x]x
\]

\[
= 2tx^3 + 2x(z - 2tx)x + 2(z - 2tx)x^2 + x^2(z - 2tx) + x(z - 2tx)x
\]

\[
= -10tx^3 + 6zx^2
\]

This implies that \(-10tx^3 + 6zx^2 \in Z(R)\). Commuting both sides with \( f(x) \), we get \([f(x), -10tx^3 + 6zx^2] = 0\), i.e.,

\[
-10t[f(x), x^2] + 6z[f(x), x^2] = -10t[f(x), x]x^2 - 10tx[f(x), x^2] + 6z[f(x), x]x + 6zx[f(x), x]
\]

\[
= -10t^2x^2 - 10tx[f(x), x]x - 10tx^2[f(x), x] + 12ztx
\]

\[
= -30t^2x^2 + 12ztx = 0.
\]

Again commuting with \( f(x) \), we have

\[
-30t^2[f(x), x^2] + 12zt[f(x), x] = -30t^2[f(x), x]x - 30t^2x[f(x), x] + 12zt^2
\]

\[
= -60t^3x + 12zt^2 = 0.
\]
Repeating the same argument, we finally arrive at \(-60t^4 = 0\). Since \(R\) is 2, 3 and 5 torsion free, we get \(t^4 = 0\). Since the center of a semiprime ring contains no nonzero nilpotent elements, we conclude that \(t = 0\). This completes the proof.

3. Cocommuting biderivations

In this section, we consider the case in which the mappings \(\mu, \phi : R \rightarrow R\) satisfy \(\mu(x)x + x\phi(x) = 0\) for all \(x \in R\). Bresar [3, Theorem 4.1] proved that if \(R\) is a prime ring, \(I\) a nonzero left ideal of \(R\) and \(\alpha\) and \(\beta\) are nonzero derivations of \(R\) satisfying \(\alpha(x)x - x\beta(x) \in Z(R)\) for all \(x \in I\), then \(R\) is commutative. Argac [1, Theorem 3.5] proved a result for generalized derivation of \(R\).

We extend the aforementioned results by proving the following theorem for a biderivation of \(R\).

**Theorem 3.1.** Let \(R\) be a prime ring of characteristic not two, \(I\) a nonzero left ideal of \(R\) and \(D, G\) be symmetric biderivations of \(R\) with trace \(f\) and \(g\) respectively. If \(D(x, x)x + xG(x, x) = 0\) for all \(x \in I\), then either \(R\) is commutative or \(G\) acts as a left bimultiplier on \(I\). Moreover, in the last case either \(D = 0\) or \(I[I, I] = 0\).

**Proof.** By hypothesis, we have

\[(3.1) \quad f(x)x + xg(x) = 0 \text{ for all } x \in I,\]

where \(f(x) = D(x, x)\) and \(g(x) = G(x, x)\). Linearization of (3.1) yields that

\[(3.2) \quad f(y)x + f(x)y + 2D(x, y)x + 2D(x, y)y + xg(y) + yg(x) + 2xG(x, y) + 2yG(x, y) = 0, \text{ for all } x, y \in I.\]

Substituting \(-y\) for \(y\) in (3.2), we get

\[(3.3) \quad f(y)x - f(x)y - 2D(x, y)x + 2D(x, y)y + xg(y) - yg(x) - 2xG(x, y) + 2yG(x, y) = 0, \text{ for all } x, y \in I.\]

Adding (3.1) and (3.2), we obtain

\[(3.4) \quad 2f(y)x + 4D(x, y)y + 2xg(y) + 4yG(x, y) = 0, \text{ for all } x, y \in I.\]

Since \(\text{char } R\) is not two, we have

\[(3.5) \quad f(y)x + 2D(x, y)y + xg(y) + 2yG(x, y) = 0, \text{ for all } x, y \in I.\]

Replacing \(x\) by \(xz\) in (3.5), we obtain

\[(3.6) \quad f(y)xz + 2D(x, y)zy + 2xD(z, y)y + xzg(y) + 2yG(x, y)z + 2yG(z, y) = 0, \text{ for all } x, y, z \in I.\]
Comparing (3.5) and (3.6), we obtain
\begin{equation}
-2D(x, y)yz - xg(y)z + 2D(x, y)zy + 2xD(z, y)y + xzg(y) + 2yxG(z, y) = 0, \text{ for all } x, y, z \in I.
\end{equation}

This implies that
\begin{equation}
2D(x, y)[z, y] + x[z, g(y)] + 2xD(z, y)y + 2yxG(z, y) = 0, \quad \text{for all } x, y, z \in I.
\end{equation}

Substituting $rx$ for $x$ in (3.8), we get
\begin{equation}
2rD(x, y)[z, y] + 2D(r, y)x[z, y] + rx[z, g(y)] + 2rxyG(z, y) = 0, \quad \text{for all } x, y, z \in I, \text{ for all } r \in R.
\end{equation}

Comparing (3.8) and (3.9), we get
\begin{equation}
2D(r, y)x[z, y] + 2yrxG(z, y) - 2ryxG(z, y) = 0, \quad \text{for all } x, y, z \in I, \text{ for all } r \in R.
\end{equation}

Since $R$ is of characteristic not two, we obtain
\begin{equation}
D(r, y)x[z, y] + [y, r]xG(z, y) = 0, \quad \text{for all } x, y, z \in I, \text{ for all } r \in R.
\end{equation}

Replacing $y$ by $z$ in (3.12), we obtain
\begin{equation}
[z, r]xg(z) = 0, \quad \text{for all } x, z \in I, \text{ for all } r \in R
\end{equation}

Substituting $rx$ for $x$ in (3.12), we get
\begin{equation}
[z, r]Rxg(z) = 0, \quad \text{for all } x, z \in I, \text{ for all } r \in R
\end{equation}

Primeness of $R$ yields that either $[z, r] = 0$ or $xg(z) = 0$. If $[z, r] = 0$ for all $z \in I$ and $r \in R$, then $R$ is commutative by [10]. Suppose $xg(z) = 0$ for all $x, z \in I$. Linearization in $z$ yields that
\begin{equation}
0 = xG(z, y) + xG(y, z) = 2xG(y, z)
\end{equation}

and using $R$ is not of characteristic two, we get
\begin{equation}
xG(y, z) = 0 \quad \text{for all } x, y, z \in I.
\end{equation}

This implies that
\begin{equation}
G(x, yz) = G(x, y)z.
\end{equation}

Hence $G$ acts as left multiplier. Since $xG(y, z) = 0$ for all $x, y, z \in I$ and using (3.11), we arrive at
\begin{equation}
D(r, y)x[z, y] = 0, \quad \text{for all } x, y, z \in I, r \in R.
\end{equation}
Replace $r$ by $rs$ in (3.14) to get
\begin{equation}
D(r, y)Rx[y, z] = 0, \text{ for all } x, y, z \in I, r \in R.
\end{equation}

Primeness of $R$ implies that either $D(r, y) = 0$ or $x[y, z] = 0$ for all $x, y, z \in I$. Later yields that $I[I, I] = 0$ as $D \neq 0$.

Proceeding on the same parallel lines, we can prove the following:

**Theorem 3.2.** Let $R$ be a prime ring of characteristic not two, $I$ a nonzero right ideal of $R$ and $D, G$ are symmetric biderivations of $R$ with trace $f$ and $g$ respectively. If $D(x, x) + xG(x, x) = 0$ for all $x \in I$, then either $R$ is commutative or $D$ acts as a left bimultiplier on $I$. Moreover in the last case either $G = 0$ or $I[I, I] = 0$.

**Acknowledgment.** The authors would like to express their thanks to the referees for the careful reading of the paper and several helpful suggestions.

**References**


Accepted: 08.10.2014