# DIAMETERS OF SEMI-IDEAL BASED ZERO-DIVISOR GRAPHS FOR FINITE DIRECT PRODUCT OF POSETS

**K. Porselvi**
**B. Elavarasan**[1]

*Department of Mathematics*
*School of Science and Humanities*
*Karunya University*
*Coimbatore – 641 114*
*Tamilnadu*
*India*
*e-mails: porselvi94@yahoo.co.in*
*belavarasan@gmail.com*

**Abstract.** In this paper, we characterize the diameter of zero-divisor graph for direct product $P_1 \times P_2 \times ... \times P_n$ with respect to direct product $I_1 \times I_2 \times ... \times I_n$, where $I_1, I_2, ..., I_n$ are semi-ideals of posets $P_1, P_2, ..., P_n$, respectively.

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## 1. Preliminaries

Throughout this paper, $(P, \leq)$ denotes a poset with zero element 0 and the graph $G_I(P)$ denotes the semi-ideal based zero-divisor graph of a poset $P$ with respect to a semi-ideal $I$ of $P$. For $M \subseteq P$, let $(M)^l := \{x \in P : x \leq m \text{ for all } m \in M\}$ denotes the lower cone of $M$ in $P$. For $A, B \subseteq P$, we write $(A, B)^l$ instead of $(A \cup B)^l$. If $M = \{x_1, ..., x_n\}$ is finite, then we use the notation $(x_1, ..., x_n)^l$ instead of $(\{x_1, ..., x_n\})^l$. By a semi-ideal we mean a non-empty subset $I$ of $P$ such that if $b \in I$ and $a \leq b$, then $a \in I$. A proper semi-ideal $I$ of $P$ is called prime if for any $a, b \in P,\ (a, b)^l \subseteq I$ implies $a \in I$ or $b \in I$.

In [2], I. Beck introduced the concept of a zero-divisor graph of a commutative ring with identity, but this work was mostly concerned with coloring of rings. Later D.F. Anderson and P.S. Livingston in [1] studied the subgraph $\Gamma(R)$ of $G(R)$ whose vertices are the nonzero zero-divisors of $R$ and two distinct vertices $x$ and $y$ are joined by an edge if $xy = 0$. In [11], S.P. Redmond has generalized

[1]Corresponding author.

the notion of the zero-divisor graph. For a given ideal $I$ of a commutative ring $R$, he defined an undirected graph $\Gamma_I(R)$ with vertices $\{x \in R\backslash I : xy \in I$ for some $y \in R\backslash I\}$, where distinct vertices $x$ and $y$ are adjacent if and only if $xy \in I$. The zero-divisor graph of various algebraic structures have been studied by several authors ([4], [5], [6] and [7]).

In [9], Radomr Halas and Marek Jukl have introduced the concept of a graph structure of a posets, let $(P, \leq)$ be a poset with 0. Then the zero-divisor graph of $P$, denoted by $\Gamma(P)$, is an undirected graph whose vertices are just the elements of $P$ with two distinct vertices $x$ and $y$ are joined by an edge if and only if $L(x, y) = \{0\}$, and proved some interesting results related with clique and chromatic number of this graph structure. In [8], we have studied the semi-ideal-based zero-divisor graph of a poset $P$. Let $P$ be a poset and $I$ a semi-ideal of $P$. Then the graph of $P$ with respect to the semi-ideal $I$, denoted by $G_I(P)$, is the graph whose vertices are the set $\{x \in P\backslash I : (x, y)^l \subseteq I$ for some $y \in P\backslash I\}$ with distinct vertices $x$ and $y$ are adjacent if and only if $(x, y)^l \subseteq I$. If $I = \{0\}$, then $G_I(P) = G(P)$, and $I$ is a prime semi-ideal of $P$ if and only if $G_I(P) = \phi$. And investigated the interplay between the poset properties of $P$ and the graph-theoretic properties of $G_I(P)$.

The direct product of posets $P$ and $Q$ is the poset $P \times Q = \{(x, y) : x \in P, y \in Q\}$ such that $(x, y) \leq (x', y')$ in $P \times Q$ if $x \leq x'$ in $P$ and $y \leq y'$ in $Q$.

Throughout this paper, let us denote $I_1,\ I_2,\ ....,\ I_n$ be semi-ideals of posets $P_1,\ P_2,\ ...,\ P_n$, respectively and $P = P_1 \times P_2 \times ... \times P_n$, and $I = I_1 \times I_2 \times ... \times I_n$. For $j = 1, 2, ..., m$, if $|P_j| = n_j$, then we can observe that $|V(G_I(P))| \leq n_1 n_2 ... n_m - |I|$, if $P_j$ for each $j$ has a greatest element $e_j$, then $|V(G_I(P))| < n_1 n_2 ... n_m - |I|$. In this paper, we investigate the relationship between the diameter of $G_I(P)$ and properties of $P_i$ with respect to $V(G_{I_i}(P_i))$ for $i = 1, 2, ..., n$. The notations of graph theory are from [3], the notations of posets from [10].

## 2. Main results

**Lemma 2.1** *Let $I_1, I_2, ..., I_n$ be semi-ideals of posets $P_1, P_2, ..., P_n$, respectively, Then $I$ is a semi-ideal of $P$.*

**Proof.** It is trivial. ∎

The following example shows that $I$ is not necessarily to be a prime semi-ideal of $P$ even if $I_1, I_2, ..., I_n$ are prime semi-ideals of posets $P_1, P_2, ..., P_n$, respectively.

**Example 2.2** Let $P_1 = \{1, 2, 4\}$ and $P_2 = \{1, 3, 9\}$ be posets with respect to division. Then $I_1 = \{1\}$ and $I_2 = \{1\}$ are prime semi-ideals of $P_1$ and $P_2$, respectively. Here $I = I_1 \times I_2$ is a semi-ideal of $P = P_1 \times P_2$, but not prime semi-ideal.

**Theorem 2.3** *Let $I_1, I_2, ..., I_n$ be prime semi-ideals of posets $P_1, P_2, ..., P_n$, respectively. Then $V(G_I(P)) \cup I = \bigcup_{j=1}^{n} (P_1 \times P_2 \times ... \times I_j \times ... \times P_n)$ is a prime semi-ideal of $P$.*

**Proof.** Let $y = (y_1, ..., y_n)$, $y_i \notin I_i$ for all $i$. Suppose that $(y, x)^l \subseteq I$ for some $x \in V(G_I(P))$. Then $(y_i, x_i)^l \subseteq I_i$ for all $i$. Since $I_i's$ are prime semi-ideals of posets $P_i's$, we have $x_i \in I_i$ for all $i$. So $x \notin V(G_I(P))$, a contradiction. So, $V(G_I(P)) \cup I = \bigcup_{j=1}^{n}(P_1 \times P_2 \times ... \times I_j \times ... \times P_n)$.

Now, we claim that $V(G_I(P)) \cup I$ is a prime semi-ideal of $P$. Let $a = (a_1, a_2, ..., a_n)$, $b = (b_1, b_2, ..., b_n) \in P$. Suppose that $(a, b)^l \subseteq V(G_I(P)) \cup I$ and $a, b \notin V(G_I(P)) \cup I$. Then $a_i \notin I_i$ and $b_i \notin I_i$ for all $i$, which implies $(a_i, b_i)^l \nsubseteq I_i$ for all $i$. So there exists $t_i \in (a_i, b_i)^l$ such that $t_i \notin I_i$. Set $t = (t_1, t_2, ..., t_n)$. Then $t \in (a, b)^l \subseteq V(G_I(P)) \cup I$, a contradiction to $t_i \notin I_i$ for all $i$. ∎

**Theorem 2.4** *Let $I_1, I_2, ..., I_n$ be prime semi-ideals of posets $P_1, P_2, ..., P_n$, respectively. Then $G_I(P)$ is a n-partite graph. Moreover, if $V_1, V_2, ..., V_n$ are partitions of $V(G_I(P))$, then there exists an induced subgraph $K_{|X_1|,|X_2|,...,|X_n|}$, where $\phi \neq X_i \subseteq V_i$. Also there exist $|X_1||X_2|...|X_n|$ number of induced subgraphs $K_n's$ in $K_{|X_1|,|X_2|,...,|X_n|}$.*

**Proof.** Let

$$V_1 = \bigcup_{2 \le k \le n} ((P_1 \backslash I_1) \times P_2 \times ... \times I_k ... \times P_n),$$
$$V_2 = \bigcup_{3 \le k \le n} (I_1 \times (P_2 \backslash I_2) \times ... \times I_k ... \times P_n),$$
$$V_3 = \bigcup_{4 \le k \le n} (I_1 \times I_2 \times (P_3 \backslash I_3) \times ... \times I_k ... \times P_n),\ ...,$$
$$V_n = I_1 \times I_2 \times ... \times I_{n-1} \times (P_n \backslash I_n).$$

Then $V_1, V_2, ..., V_n$ are nonempty disjoint $n$-subsets of $V(G_I(P))$.

Let $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in V_j$ for some $j$. Then we have $x_j, y_j \notin I_j$ which implies $(x_j, y_j) \nsubseteq I_j$, so $(x, y)^l \nsubseteq I$. Thus no two vertices of $V_j$ are adjacent and hence $G_I(P)$ is an $n$-partite graph.

For moreover case, take

$$X_1 = (P_1 \backslash I_1) \times I_2 \times ... \times I_n,$$
$$X_2 = I_1 \times (P_2 \backslash I_2) \times ... \times I_n,$$
$$X_3 = I_1 \times I_2 \times (P_3 \backslash I_3) \times ... \times I_n, ...,$$
$$X_n = I_1 \times I_2 \times ... \times (P_n \backslash I_n).$$

Then $X_i's$ are subset of $V_i's$ and forms $K_{|X_1|,|X_2|,...,|X_n|}$ and $\{x_1, x_2, ..., x_n\}$ forms $K_n$ for $x_i \in X_i$. ∎

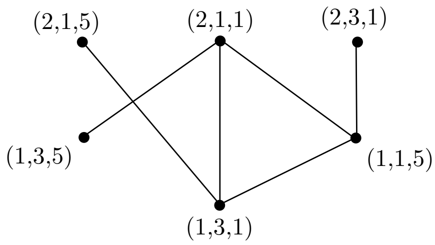
**Corollary 2.5** *Let $I_1, I_2, ..., I_n$ be prime semi-ideals of posets $P_1, P_2, ..., P_n$, respectively. Then the clique of $G_I(P)$ is $n$.*

**Theorem 2.6** *Let $I_1$ and $I_2$ be prime semi-ideals of posets $P_1$ and $P_2$, respectively. Then $G_I(P)$ is complete bipartite graph.*

**Proof.** By Theorem 2.3 and Theorem 2.4, $G_I(P)$ is bipartite graph with vertex set $V_1 = (P_1 \backslash I_1) \times I_2$ and $V_2 = I_1 \times (P_2 \backslash I_2)$ which forms $K_{|V_1|,|V_2|}$. ∎

The following example shows that $G_I(P)$ need not be a complete n-partite graph if $n > 2$.

**Example 2.7** Let $P_1 = \{1, 2\}$, $P_2 = \{1, 3\}$ and $P_3 = \{1, 5\}$ be posets with respect to division and $I_1 = \{1\}$, $I_2 = \{1\}$ and $I_3 = \{1\}$ be prime semi-ideals of posets $P_1$, $P_2$ and $P_3$, respectively. Then $G_I(P)$ with respect to $I = (1, 1, 1)$ is:




Here $G_I(P)$ is 3-partite graph, but not complete 3-partite graph. ∎

**Lemma 2.8** *Let $I_1$ be semi-ideal of poset $P_1$ with $diam(G_{I_1}(P_1)) = 1$. Then $(x, y)^l \subseteq I_1$ for all $x,\ y \in V(G_{I_1}(P_1))$, also if $P_1 = V(G_{I_1}(P_1)) \cup I_1$, then $(x, y)^l \subseteq I_1$ for all $x,\ y \in P_1$.*

**Proof.** It is trivial. ∎

**Theorem 2.9** *Let $I_1, I_2, ..., I_n$ be semi-ideals of posets $P_1, P_2, ..., P_n$, respectively. Then $G_I(P)$ is connected and $diam(G_I(P)) \leq 3$.*

**Proof.** Let $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in V(G_I(P))$. Then there exist $a = (a_1, a_2, ..., a_n)$, $b = (b_1, b_2, ..., b_n) \in V(G_I(P))$ such that $(x, a)^l \subseteq I$ and $(y, b)^l \subseteq I$. If $(x, y)^l \subseteq I$, then $diam(G_I(P)) = 1$. Suppose that $(x, y)^l \nsubseteq I$. If $(a, b)^l \subseteq I$, then we have a path $x - a - b - y$ of length 3. Suppose that $(a, b)^l \nsubseteq I$. Then $(a_j, b_j)^l \nsubseteq I_j$ for some $j$, so we can find $t_j \in (a_j, b_j)^l$ with $t_j \notin I_j$ for some $a_j, b_j \in P_j \backslash I_j$. Now for $t = (i_1, i_2 ..., i_{j-1}, t_j, i_{j+1}, ..., i_n) \notin I$, we have $(x, t)^l \subseteq I$ and $(y, t)^l \subseteq I$, which imply $x - t - y$ is a path of length 2. Hence $G_I(P)$ is connected and $diam(G_I(P)) \leq 3$. ∎

**Lemma 2.10** *Let $I_1, I_2, ..., I_n$ be semi-ideals of posets $P_1, P_2, ..., P_n$, respectively. If $diam(G_{I_1}(P_1)) = diam(G_{I_2}(P_2)) = .... = diam(G_{I_n}(P_n)) = 1$, then the following hold:*

(i) *$diam(G_I(P)) \neq 1$*

(ii) *$diam(G_I(P)) = 2$ if and only if $P_j = V(G_{I_j}(P_j)) \cup I_j$ for all $j \in \{1, 2, ..., n\}$.*

(iii) *$diam(G_I(P)) = 3$ if and only if $P_j \neq V(G_{I_j}(P_j)) \cup I_j$ for some $j \in \{1, 2, ..., n\}$.*

**Proof.** (i) Let $a = (a_1, a_2, ...., a_n)$ and $b = (a_1, i_2, ..., i_n) \in V(G_I(P))$ with $a \neq b$, where $a_1 \in V(G_{I_1}(P_1))$. Then $(a, b)^l \nsubseteq I$ and hence $diam(G_I(P)) \neq 1$.

(ii) Assume that $P_j = V(G_{I_j}(P_j)) \cup I_j$ for all $j \in \{1, 2, ..., n\}$. If there exist distinct vertices $c = (c_1, c_2, ..., c_n)$, $d = (d_1, d_2, ..., d_n)$ in $V(G_I(P))$ with $a - c - d - b$ is a path of length 3, then $(a_j, d_j)^l \nsubseteq I_j$ for some $a_j, d_j \in P_j \backslash I_j$, a contradiction to $diam(G_{I_j}(P_j)) = 1$. Thus we have a path $a - t - b$ of length 2 for all $t \in V(G_I(P))$. So $diam(G_I(P)) = 2$. Conversely, assume that $diam(G_I(P)) = 2$. Suppose that $P_j \neq V(G_{I_j}(P_j)) \cup I_j$ for some $j \in \{1, 2, ..., n\}$. Then there exists $x_j \in P_j \backslash (V(G_{I_j}(P_j)) \cup I_j)$ for some $j \in \{1, 2, ..., n\}$. Since for each $z_k \in V(G_{I_k}(P_k))$, there exists $z'_k \in V(G_{I_k}(P_k))$ such that $(z_k, z'_k)^l \subseteq I_k$ for all $k$. So, if $a = (z_1, x_2, ..., x_n)$ and $b = (x_1, z_2, x_3, ..., x_n)$, then $(a, (z'_1, i_2, ..., i_n))^l \subseteq I$ and $(b, (i_1, z'_2, ..., i_n))^l \subseteq I$ which imply $a$, $b \in V(G_I(P))$. Since $(a, b)^l \nsubseteq I$ and by assumption, there exists $c = (c_1, c_2, ..., c_n) \in V(G_I(P))$ such that $(a, c)^l \subseteq I$ and $(b, c)^l \subseteq I$ which imply $c_j \in I_j$, a contradiction. Thus $P_j = V(G_{I_j}(P_j)) \cup I_j$ for all $j \in \{1, 2, ..., n\}$.

(iii) This follows from (i) and (ii). ∎

**Theorem 2.11** *Let $I_1, I_2, ..., I_n$ be semi-ideals of posets $P_1, P_2, ..., P_n$, respectively. If $diam(G_{I_1}(P_1)) = diam(G_{I_2}(P_2)) = ... = diam(G_{I_n}(P_n)) = 2$, then the following hold:*

(i) *$diam(G_I(P)) \neq 1$.*

(ii) *$diam(G_I(P)) = 2$ if and only if $P_j = V(G_{I_j}(P_j)) \cup I_j$, for all $j \in \{1, 2, ..., n\}$.*

(iii) *$diam(G_I(P)) = 3$ if and only if $P_j \neq V(G_{I_j}(P_j)) \cup I_j$, for some $j \in \{1, 2, ..., n\}$.*

**Proof.** (i) It is clear.

(ii) Let $P_j = V(G_{I_j}(P_j)) \cup I_j$ for all $j \in \{1, 2, ..., n\}$. By (i), there are elements $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in V(G_I(P))$ such that $x \neq y$ and $(x, y)^l \nsubseteq I$ which imply $(x_j, y_j)^l \nsubseteq I_j$ for some $j$. Since $x_j$, $y_j \in P_j$ and by assumption, we have $(x_j, z_j)^l \subseteq I_j$ and $(y_j, z_j)^l \subseteq I_j$ for some $z_j \in V(G_{I_j}(P_j))$. Now let $z = (i_1, ...i_{j-1}, z_j, i_{j+1}, ..., i_n)$. Then $z \notin I$ with $(x, z)^l \subseteq I$ and $(y, z)^l \subseteq I$ which imply $x - z - y$ is a path of length 2. So $diam(G_I(P)) = 2$.

Conversely, assume that $diam(G_I(P)) = 2$ and let $P_j \neq V(G_{I_j}(P_j)) \cup I_j$ for some $j \in \{1, 2, ..., n\}$. Then we can find some $m_j \in P_j \backslash (V(G_{I_j}(P_j)) \cup I_j)$. Since for each $i$, $e_i \in V(G_{I_i}(P_i))$, there is an element $e'_i$ of $V(G_{I_i}(P_i))$ such that $(e_i, e'_i)^l \subseteq I_i$. If $a = (e_1, m_2, ..., m_n)$ and $b = (m_1, e_2, m_3, ..., m_n)$, then $(a, (e'_1, i_2, ..., i_n))^l \subseteq I$ and $(b, (i_1, e'_2, i_3, ..., i_n))^l \subseteq I$. So $a$, $b \in V(G_I(P))$ and $(a, b)^l \nsubseteq I$. Since $diam(G_I(P)) = 2$, there exists $c = (c_1, ..., c_n) \in V(G_I(P))$ such that $(a, c)^l \subseteq I$ and $(b, c)^l \subseteq I$. Thus $c_j \in I_j$, a contradiction. Thus $P_i = V(G_{I_i}(P_i)) \cup I_i$ for all $i \in \{1, 2..., n\}$.

(iii) It follows from (i) and (ii). ∎

**Theorem 2.12** *Let $I_1, I_2, ..., I_n$ be semi-ideals of posets $P_1, P_2, ..., P_n$, respectively. If $diam(G_{I_1}(P_1)) = diam(G_{I_2}(P_2)) = ... = diam(G_{I_n}(P_n)) = 3$, then*

$$diam(G_I(P)) = 3.$$

**Proof.** Assume that for each $j \in \{1, 2, ..., n\}$, $diam(G_{I_j}(P_j)) = 3$, there exist $x_j,\ y_j \in V(G_{I_j}(P_j))$ with $x_j \neq y_j$, $(x_j, y_j)^l \nsubseteq I_j$ such that there is no $z_j \in V(G_{I_j}(P_j))$ with $(x_j, z_j)^l \subseteq I_j$ and $(y_j, z_j)^l \subseteq I_j$. Consider $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$. For each $j \in \{1, 2, ..., n\}$, there are elements $x'_j, y'_j \in V(G_{I_j}(P_j))$ such that $(x_j, x'_j)^l \subseteq I_j$ and $(y_j, y'_j)^l \subseteq I_j$. So $x,\ y \in V(G_I(P))$ and $(x, y)^l \nsubseteq I$. If $diam(G_I(P)) = 2$, then there exists $a = (a_1, ..., a_n) \in V(G_I(P))$ with $(x, a)^l \subseteq I$ and $(y, a)^l \subseteq I$ which imply $(x_j, a_j)^l \subseteq I_j$ and $(y_j, a_j)^l \subseteq I_j$, a contradiction. So $diam(G_I(P)) = 3$. ∎

**Theorem 2.13** *Let $I_1, I_2, ..., I_n$ be semi-ideals of posets $P_1, P_2, ..., P_n$, respectively. If $diam(G_{I_j}(P_j)) = 1$, $diam(G_{I_k}(P_k)) = 2$ for some $j,\ k \in \{1, 2, ..., n\}$ and there is no $m \in \{1, 2, ..., n\}$ with $diam(G_{I_m}(P_m)) = 3$, then the following hold:*

(i) *$diam(G_I(P)) \neq 1$.*

(ii) *$diam(G_I(P)) = 2$ if and only if $P_j = V(G_{I_j}(P_j)) \cup I_j$, for all $j \in \{1, 2, ..., n\}$.*

(iii) *$diam(G_I(P)) = 3$ if and only if $P_j \neq V(G_{I_j}(P_j)) \cup I_j$, for some $j \in \{1, 2, ..., n\}$.*

**Proof.** (i) It is clear.

(ii) Let $P_j = V(G_{I_j}(P_j)) \cup I_j$ for all $j \in \{1, 2, ..., n\}$. By Lemma 2.8, $(x_j, y_j)^l \subseteq I_j$ for all $x_j,\ y_j \in V(G_{I_j}(P_j)) \cup I_j$. By (i), there are distinct vertices $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ in $V(G_I(P))$ such that $(x, y)^l \nsubseteq I$.

We divided the proof into two cases.

**Case (a):** $diam(G_{I_j}(P_j)) = 1$. If $z_j \in V(G_{I_j}(P_j))$, then

$$(x, (i_1, ..., i_{j-1}, z_j, i_{j+1}, ..., i_n))^l \subseteq I \text{ and } (y, (i_1, ...i_{j-1}, z_j, i_{j+1}, ..., i_n))^l \subseteq I.$$

Thus we have a path $x - z - y$ of length 2. So $diam(G_I(P)) = 2$.

**Case (b):** $diam(G_{I_j}(P_j)) = 2$. Then, for some $x_j,\ y_j \in V(G_{I_j}(P_j))$, there exists $z_j \in V(G_{I_j}(P_j))$ such that $(x_j, z_j)^l \subseteq I_j$ and $(y_j, z_j)^l \subseteq I_j$. Set $z = (i_1, ...i_{j-1}, z_j, i_{j+1}, ..., i_n)$. Then $(x, z)^l \subseteq I$ and $(y, z)^l \subseteq I$. So we must have a path $x - z - y$ of length 2 and hence $diam(G_I(P)) = 2$.

Conversely, assume that $diam(G_I(P)) = 2$. Suppose that $P_j \neq V(G_{I_j}(P_j)) \cup I_j$ for some $j \in \{1, 2, ..., n\}$. Then we can find some elements $m_j \in P_j \setminus (V(G_{I_j}(P_j)) \cup I_j)$. Since for each $i$, $x_i \in V(G_{I_i}(P_i))$, there is an element $x'_i$ of $V(G_{I_i}(P_i))$ such that $(x_i, x'_i)^l \subseteq I_i$ for all $i$. If $a = (x_1, m_2, ..., m_n)$ and $b = (m_1, x_2, m_3, ..., m_n)$, then

$(a,(x'_1,i_2,...,i_n))^l \subseteq I$ and $(b,(i_1,x'_2,i_3,...,i_n))^l \subseteq I$. So $a,\ b \in V(G_I(P))$ and $(a,b)^l \nsubseteq I$. Since $diam(G_I(P)) = 2$, there exists $c = (c_1,...,c_n) \in V(G_I(P))$ such that $(a,c)^l \subseteq I$ and $(b,c)^l \subseteq I$ which imply $c_j \in I_j$, a contradiction. Thus $P_j = V(G_{I_j}(P_j)) \cup I_j$ for all $j \in \{1,2...,n\}$.

(iii) It follows from (i) and (ii). ∎

**Theorem 2.14** *Let $I_1,I_2,...,I_n$ be semi-ideals of posets $P_1,P_2,...,P_n$, respectively. If $diam(G_{I_j}(P_j)) = 1$, $diam(G_{I_k}(P_k)) = 3$ for some $j,\ k \in \{1,2,...,n\}$ and there is no $m \in \{1,2,...,n\}$ with $diam(G_{I_m}(P_m)) = 2$, then the following hold:*

(i) *$diam(G_I(P)) \neq 1$.*

(ii) *$diam(G_I(P)) = 2$ if and only if $diam(G_{I_j}(P_j)) = 1$ and $P_j = V(G_{I_j}(P_j)) \cup I_j$ for all $j \in \{1,2,...,n\}$.*

(iii) *$diam(G_I(P)) = 3$ if and only if there is no $j \in \{1,2,...,n\}$ with $diam(G_{I_j}(P_j)) = 1$ and $P_j = V(G_{I_j}(P_j)) \cup I_j$.*

**Proof.** (i) It is clear.

(ii) Assume that $diam(G_{I_j}(P_j)) = 1$ and $P_j = V(G_{I_j}(P_j)) \cup I_j$ for all $j \in \{1,2,...,n\}$. Then by Lemma 2.8, $(x_j,y_j)^l \subseteq I_j$ for all $x_j,\ y_j \in V(G_{I_j}(P_j)) \cup I_j$. By (i), there are distinct vertices $x = (x_1,x_2,...,x_n)$ and $y = (y_1,y_2,...,y_n)$ in $V(G_I(P))$ such that $(x,y)^l \nsubseteq I$. Let $a_j \in V(G_{I_j}(P_j))$ and set $a = (i_1,...i_{j-1},a_j,i_{j+1},...,i_n)$. Then $(x,a)^l \subseteq I$ and $(y,a)^l \subseteq I$ which imply $a \in V(G_I(P))$ and $x - a - y$ is a path of length 2. So $diam(G_I(P)) = 2$.

Conversely, assume that $diam(G_I(P))$=2. We now show that $diam(G_{I_i}(P_i))$=1 and $P_i = V(G_{I_i}(P_i)) \cup I_i$ for all $i \in \{1,2,...,n\}$. Suppose not. Let $i_1,i_2,...,i_k$ be such that $diam(G_{I_{i_r}}(P_{i_r})) = 1$ $(1 \leq r \leq k)$, and let $j_1,j_2,...,j_t$ be such that $diam(G_{I_{j_s}}(P_{j_s})) = 3$ $(1 \leq s \leq t)$. Then for each $s$, there exist distinct vertices $x_{j_s},y_{j_s} \in V(G_{I_{j_s}}(P_{j_s}))$ such that $(x_{j_s},y_{j_s})^l \nsubseteq I_{j_s}$ and there is no $z_{j_s} \in V(G_{I_j}(P_{j_s}))$ with $(x_{j_s},z_{j_s})^l \subseteq I_{j_s}$ and $(z_{j_s},y_{j_s})^l \subseteq I_{j_s}$. Moreover, for each $s$ $(1 \leq s \leq t)$, there must exist $x'_{j_s},y'_{j_s} \in V(G_{I_{j_s}}(P_{j_s}))$ with $(x_{j_s},x'_{j_s})^l \subseteq I_{j_s}$ and $(y_{j_s},y'_{j_s})^l \subseteq I_{j_s}$. Now, for each $r$ $(1 \leq r \leq k)$, let $m_{i_r} \in P_{i_r} \backslash (V(G_{I_{i_r}}(P_{i_r})) \cup I_{i_r})$. Set $c = (m_{i_1},...,x_{j_1},...,x_{j_t},..)$ and $d = (m_{i_1},...,y_{j_1},...,y_{j_t},..)$. Then $(c,(i_1,...,x'_{j_1},i_{j+1},...,i_n)) \subseteq I$ and $(d,(i,...,y'_{j_1},i,...,i)) \subseteq I$, which imply $c,d \in V(G_I(P))$ and $(c,d)^l \nsubseteq I$. Since $diam(G_I(P)) = 2$, there exists $e = (e_1,...,e_n) \in V(G_I(P))$ such that $(c,e)^l \subseteq I$ and $(d,e)^l \subseteq I$. Thus $e_i \in I_i$, a contradiction. Thus $diam(G_{I_i}(P_i)) = 1$ and $P_i = V(G_{I_i}(P_i)) \cup I_i$ for all $i \in \{1,2...,n\}$.

(iii) If $diam(G_I(P)) = 2$, then by (ii), we have $diam(G_{I_i}(P_i)) = 1$ and $P_i$=$V(G_{I_i}(P_i))$∪$I_i$ for all $i \in \{1,2...,n\}$, a contradiction. Thus $diam(G_I(P)) = 3$. ∎

**Theorem 2.15** *Let $I_1,I_2,...,I_n$ be semi-ideals of posets $P_1,P_2,...,P_n$, respectively. If $diam(G_{I_j}(P_j)) = 2$, $diam(G_{I_k}(P_k)) = 3$ for some $j,\ k \in \{1,2,...,n\}$ and there is no $m \in \{1,2,...,n\}$ with $diam(G_{I_m}(P_m)) = 1$, then the following hold:*

(i) $diam(G_I(P)) \neq 1$.

(ii) $diam(G_I(P)) = 2$ *if and only if* $diam(G_{I_j}(P_j)) = 2$ *and* $P_j = V(G_{I_j}(P_j)) \cup I_j$ *for all* $j \in \{1, 2, ..., n\}$.

(iii) $diam(G_I(P)) = 3$ *if and only if there is no* $j \in \{1, 2, ..., n\}$ *with* $diam(G_{I_j}(P_j)) = 2$ *and* $P_j = V(G_{I_j}(P_j)) \cup I_j$.

**Proof.** (i) It is clear.

(ii) Assume that $diam(G_{I_j}(P_j)) = 2$ and $P_j = V(G_{I_j}(P_j)) \cup I_j$ for all $j \in \{1, 2, ..., n\}$. By (i), there are elements $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in V(G_I(P))$ such that $(x, y)^l \nsubseteq I$ which implies $(x_j, y_j)^l \nsubseteq I_j$ for some $j$. Then there exists $a_j \in V(G_{I_j}(P_j))$ such that $(a_j, x_j)^l \subseteq I_j$ and $(a_j, y_j)^l \subseteq I_j$. Set $a = (i_1, ...i_{j-1}, a_j, i_{j+1}, ..., i_n)$. Then $(x, a)^l \subseteq I$ and $(y, a)^l \subseteq I$ which imply $a \in V(G_I(P))$ and $x - a - y$ is a path of length 2, so $diam(G_I(P)) = 2$.

Conversely, assume that $diam(G_I(P))$=2. We now show that $diam(G_{I_i}(P_i))$=2 and $P_i = V(G_{I_i}(P_i)) \cup I_i$ for all $i \in \{1, 2, ..., n\}$. Suppose that for some $i$ $(1 \leq i \leq n)$, if $diam(G_{I_i}(P_i)) = 2$, then $P_i = V(G_{I_i}(P_i)) \cup I_i$. Let $i_1, i_2, ..., i_k$ be such that $diam(G_{I_{i_r}}(P_{i_r})) = 2$ $(1 \leq i \leq k)$, and let $j_1, j_2, ..., j_t$ be such that $diam(G_{I_{j_s}}(P_{j_s})) = 3$ $(1 \leq s \leq t)$. Now for each $r (1 \leq r \leq k)$, $P_{i_r} \neq V(G_{I_{i_r}}(P_{i_r})) \cup I_{i_r}$. For each $r$ $(1 \leq r \leq k)$, let $m_{i_r} \in P_{i_r} \backslash (V(G_{I_{i_r}}(P_{i_r})) \cup I_{i_r})$. Since for each $s$ $(1 \leq s \leq t)$, $diam(G_{I_{j_s}}(P_{j_s})) = 3$, there exist $x_{j_s}, y_{j_s} \in V(G_{I_{j_s}}(P_{j_s}))$ with $x_{j_s} \neq y_{j_s}, (x_{j_s}, y_{j_s})^l \nsubseteq I_{j_s}$ such that there is no $z_{j_s} \in V(G_{I_{j_s}}(P_{j_s}))$ with $(x_{j_s}, z_{j_s})^l \subseteq I_{j_s}$ and $(z_{j_s}, y_{j_s})^l \subseteq I_{j_s}$. Moreover, for each $s (1 \leq s \leq t)$, there must exist $x'_{j_s}, y'_{j_s} \in V(G_{I_{j_s}}(P_{j_s}))$ with $(x_{j_s}, x'_{j_s})^l \subseteq I_{j_s}$ and $(y_{j_s}, y'_{j_s})^l \subseteq I_{j_s}$. Set $c = (m_{i_1}, ..., x_{j_1}, ..., x_{j_t}, ..)$ and $d$=$(m_{i_1}, ..., y_{j_1}, ..., y_{j_t}, ..)$. Then $(c, (i, ..., x'_{j_1}, i, ..., i)) \subseteq I$ and $(d, (i, ..., y'_{j_1}, i, ..., i)) \subseteq I$, and so $c, d \in V(G_I(P))$. Since $(c, d)^l \nsubseteq I$ and $diam(G_I(P)) = 2$, there must be some $e = (e_1, ..., e_n)$ such that $(c, e)^l \subseteq I$ and $(d, e)^l \subseteq I$. Thus $e_i \in I_i$, a contradiction. Thus $diam(G_{I_i}(P_i)) = 2$ and $P_i = V(G_{I_i}(P_i)) \cup I_i$ for all $i \in \{1, 2..., n\}$.

(iii) It follows from (i) and (ii). ∎

**Theorem 2.16** *Let* $I_1, I_2, ..., I_n$ *be semi-ideals of posets* $P_1, P_2, ..., P_n$*, respectively. If* $diam(G_{I_j}(P_j)) = 1$, $diam(G_{I_k}(P_k)) = 2$ *and* $diam(G_{I_m}(P_m)) = 3$ *for some* $j, k, m \in \{1, 2, ..., n\}$*, then the following hold:*

(i) $diam(G_I(P)) \neq 1$.

(ii) $diam(G_I(P)) = 2$ *if and only if* $diam(G_{I_j}(P_j)) \leq 2$ *and* $P_j = V(G_{I_j}(P_j)) \cup I_j$ *for some* $j \in \{1, 2, ..., n\}$.

(iii) $diam(G_I(P)) = 3$ *if and only if there is no* $j \in \{1, 2, ..., n\}$ *with* $diam(G_{I_j}(P_j)) \leq 2$ *and* $P_j = V(G_{I_j}(P_j)) \cup I_j$.

**Proof.** (i) It is clear.

(ii) Let $diam(G_{I_i}(P_i)) \leq 2$ and $P_i = V(G_{I_i}(P_i)) \cup I_i$ for all $i \in \{1, 2, ..., n\}$. We divide the proof into two cases.

**Case (a):** $diam(G_{I_i}(P_i)) = 1$ and $P_i = V(G_{I_i}(P_i)) \cup I_i$ for all $i \in \{1, 2, ..., n\}$. By a similar argument as in Theorem 2.14 (ii), we get

$$diam(G_I(P)) = 2.$$

**Case (b):** $diam(G_{I_i}(P_i)) = 2$ and $P_i = V(G_{I_i}(P_i)) \cup I_i$ for all $i \in \{1, 2, ..., n\}$. By a similar argument as in Theorem 2.15 (ii), we get

$$diam(G_I(P)) = 2.$$

Conversely, suppose that $diam(G_I(P)) = 2$. It is easy to see from Theorem 2.15(ii) that

$$diam(G_{I_i}(P_i)) \leq 2 \text{ and } P_i = V(G_{I_i}(P_i)) \cup I_i \text{ for all } i.$$

(iii) It follows from (i) and (ii). ■

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