FUZZY PARAMETERIZED FUZZY SOFT RINGS AND APPLICATIONS

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Abstract. The concepts of $FP$-fuzzy soft rings, $FP$-equivalent fuzzy soft rings and $FP$-increasing(decreasing) fuzzy soft rings are introduced. Then some properties of them are given. Finally, aggregate fuzzy subrings are proposed by aggregate fuzzy sets of $FP$-fuzzy soft rings.

Keywords: $FP$-fuzzy soft rings; $FP$-equivalent fuzzy soft rings; $FP$-increasing (decreasing) fuzzy soft rings; $FP$-fuzzy soft homomorphism; aggregate fuzzy subrings.

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1. Introduction

In dealing with the complicated problems in economics, engineering and environmental sciences, we are usually unable to apply the classical methods because there are various uncertainties in these problems. Some kinds of theories were developed like theory of fuzzy sets, soft sets, i.e., which can be used as the fundamental tools for dealing with uncertainties.

The concept of fuzzy sets and fuzzy set operations, introduced by L.A. Zadeh [18], have been extensively applied to many scientific fields. In 1971, A. Rosenfeld [17] applied the concept to the theory of groupoids and groups. In 1982, W. Liu [8] defined and studied fuzzy subrings as well as fuzzy ideals. Since then many papers concerning various fuzzy algebraic structures have appeared in the literature.

The concept of soft sets was introduced by D. Molodtsov in 1999 [15], which was another mathematical tool for dealing with uncertainties. At present, the algebraic structure of set theories dealing with uncertainties has been studied by

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many authors. H. Aktaş et al. [1] applied the notion of soft sets to the theory of groups. Y.B. Jun [6] introduced the notions of soft BCK/BCI-algebras, and then investigated their basic properties [7]. We also noticed that F. Feng et al. [5] have already investigated the definition of soft rings and established three isomorphism theorems. Furthermore, we gave three fuzzy isomorphism theorems of soft rings in [10].

In 2001, P.K. Maji et al. [13] presented the definition of fuzzy soft set, and Roy et al. presented some applications of this notion to decision-making problems in [14]. We notice that E. İnan et al. [4] have already introduced the definition of fuzzy soft rings and studied some of their basic properties.

Furthermore, N. Çağman introduced fuzzy parameterized soft sets [2] and fuzzy parameterized fuzzy soft sets [3], in short written $FP$-soft sets and $FP$-fuzzy soft sets, respectively, whose parameters sets are fuzzy sets and have improved several results. In [3], the authors also defined their operation and soft aggregation operator to form $FP$-fuzzy soft decision making method that allows constructing more efficient decision processes. $FP$-soft sets and $FP$-fuzzy soft sets have already been studied by some authors. We have studied $FP$-soft rings on $FP$-soft set theory in [11].

In this paper, we study $FP$-fuzzy soft rings on $FP$-fuzzy soft set theory. We first introduce $FP$-fuzzy soft rings generated by $FP$-fuzzy soft sets and some properties of $FP$-fuzzy soft rings will be given. Then $FP$-equivalent soft rings and $FP$-accelerating(decelerating) fuzzy soft rings will be studied. Moreover, the notions of $FP$-fuzzy homomorphisms of $FP$-fuzzy soft rings are proposed and some examples are given. Finally, aggregate fuzzy subrings will be proposed by aggregate operator and an example will be given to show that the methods can be successfully applied to many problems that contain uncertainties.

2. Preliminaries

**Definition 2.1** [16]

(i) A fuzzy set $\mu$ in a ring $R$ is said to be a fuzzy subring of $R$ if the following conditions hold for all $x, y \in R$:

1. $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$, and
2. $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$.

(ii) A fuzzy set $\mu$ in a ring $R$ is said to be a fuzzy left (right) ideal of $R$ if the following conditions hold for all $x, y \in R$: (1) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$, and (3) $\mu(xy) \geq \mu(y)$ ($\mu(xy) \geq \mu(x)$).

(iii) A fuzzy set $\mu$ is said to be a fuzzy ideal of $R$ if it is both a fuzzy left ideal of $R$ and a fuzzy right ideal of $R$.

**Definition 2.2** [12] Let $f : X \rightarrow Y$ be a mapping of sets, $\mu$ a fuzzy set of $X$ and $\nu$ a fuzzy set of $Y$. Then the image $f(\mu)$ of $\mu$ and preimage $f^{-1}(\nu)$ of $\nu$ are both fuzzy sets defined respectively as follows:
f(u)(y) = \begin{cases} 
\sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \\
0 & \text{otherwise.}
\end{cases}

f^{-1}(\nu)(x) = \nu(f(x)), x \in X.

**Definition 2.3** [3] Let $U$ be an initial universe, $E$ be the set of all parameters and $X$ be a fuzzy set over $E$ with the membership function $\mu_X : E \rightarrow [0, 1]$ and $\gamma_X(x)$ be a fuzzy set over $U$ for all $x \in E$, $F(U)$ be the set of all fuzzy set of $U$. Then an fuzzy parameterized fuzzy soft set $\Gamma_X$ on $U$ is defined by a function $\gamma_X$ representing a mapping $\gamma_X : E \rightarrow F(U)$ such that $\gamma_X(x) = \emptyset$ if $\mu_X(x) = 0$. Here, $\gamma_X$ is called the fuzzy approximate function of the fuzzy parameterized fuzzy soft set $\Gamma_X$, and the value $\gamma_X(x)$ is a fuzzy set called $x$-element of the fuzzy parameterized fuzzy soft set for all $x \in E$. Thus a fuzzy parameterized fuzzy soft set $\Gamma_X$ over $U$ can be represented by the set of ordered pairs

$$\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) : x \in E, \gamma_X(x) \in F(U), \mu_X(x) \in [0, 1]\}.$$ 

A fuzzy parameterized fuzzy soft set is briefly said to be an $FP$-fuzzy soft set. The set of all $FP$-fuzzy soft sets is denoted by $FPFS(U)$.

**Definition 2.4** [3] Let $\Gamma_X \in FPFS(U)$.

(i) If $\gamma_X(x) = \emptyset$ for all $x \in E$, then $\Gamma_X$ is called an $X$-empty $FP$-fuzzy soft set, denoted by $\Gamma_{\emptyset_X}$.

(ii) If $X = \emptyset$, then the $\Gamma_X$ is called an empty $FP$-fuzzy soft set, denoted by $\Gamma_{\emptyset}$.

(iii) If $\mu_X(x) = 1$ and $\gamma_X(x) = U$ for all $x \in E$, then $\Gamma_X$ is called an $X$-universal $FP$-fuzzy soft set, denoted by $(\Gamma_U)_X$.

(iv) If $X = E$, then the $X$-universal $FP$-fuzzy soft set is called an universal $FP$-fuzzy set, denoted by $\Gamma_E$.

**Definition 2.5** [3] Let $\Gamma_X, \Gamma_Y \in FPFS(U)$. Then

(i) $\Gamma_X$ is an $FP$-fuzzy soft subset of $\Gamma_Y$, denoted by $\Gamma_X \subseteq \Gamma_Y$, if $\mu_X(x) \leq \mu_Y(x)$ and $\gamma_X(x) \subseteq \gamma_Y(x)$ for all $x \in E$.

(ii) $\Gamma_X$ and $\Gamma_Y$ are $FP$-equal, denoted by $\Gamma_X = \Gamma_Y$, if $\mu_X(x) = \mu_Y(x)$ and $\gamma_X(x) = \gamma_Y(x)$ for all $x \in E$. 
Definition 2.6  [3] Let $\Gamma_X \in FPFS(U)$. Then the complement of $\Gamma_X$, denoted by $\Gamma_X^c$, is an $FP$-fuzzy soft set defined by
$$\mu_X^c(x) = 1 - \mu(x) \text{ and } \gamma_X^c(x) = U \setminus \gamma_X(x).$$

Definition 2.7  [3] Let $\Gamma_X, \Gamma_Y \in FPFS(U)$.

(i) The union of $\Gamma_X$ and $\Gamma_Y$, denoted by $\Gamma_X \cup \Gamma_Y$, is defined by
$$\mu_{X \cup Y}(x) = \max\{\mu_X(x), \mu_Y(x)\} \text{ and } \gamma_{X \cup Y}(x) = \gamma_X(x) \cup \gamma_Y(x) \text{ for all } x \in E.$$  

(ii) The intersection of $\Gamma_X$ and $\Gamma_Y$, denoted by $\Gamma_X \cap \Gamma_Y$, is defined by
$$\mu_{X \cap Y}(x) = \min\{\mu_X(x), \mu_Y(x)\} \text{ and } \gamma_{X \cap Y}(x) = \gamma_X(x) \cap \gamma_Y(x) \text{ for all } x \in E.$$  

3. Fuzzy parameterized fuzzy soft rings

Definition 3.1 Let $R$ be a ring, $E$ be a set of parameters and $X$ be a fuzzy set over $E$, $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) : x \in E, \gamma_X(x) \in F(R), \mu_X(x) \in [0, 1]\} \in FPFS(R)$. Then $\Gamma_X$ is said to be an $FP$-fuzzy soft ring over $R$ if, for any $x \in E$, $\gamma_X(x)$ is a fuzzy subring of $R$.

Example 3.2 Let $R = Z_4 = \{0, 1, 2, 3\}$ be a ring and $E = \{a, b\}$ be a set of parameters. If $X = \{0.2/a, 0.4/b\}$, $\gamma_X(a) = \{0/0.5, 1/0.3, 2/0.4, 3/0.3\}$, $\gamma_X(b) = \{0/0.4, 1/0.2, 2/0.3, 3/0.2\}$, then $\Gamma_X$ is an $FP$-fuzzy soft ring over $R$.

Theorem 3.3 Let $R$ be a ring, $E$ be a set of parameters. Then

(1) $\Gamma_{\emptyset_X}$ and $\Gamma_{\emptyset}$ are $FP$-fuzzy soft rings.

(2) $(\Gamma_R)_X$ and $(\Gamma_R)_E$ are $FP$-fuzzy soft rings.

(3) $\Gamma_X^c$ and $\Gamma_E^c$ are $FP$-fuzzy soft rings.

Proof. By Definitions 2.4 and 3.1, the proofs of (1) and (2) are straightforward. Since $\Gamma_{\emptyset} = (\Gamma_R)_E$ and $\Gamma_{\emptyset}^c = \Gamma_{\emptyset}$, then (3) is hold.  

Theorem 3.4 Let $R$ be a ring, $E$ be a set of parameters and $\Gamma_X$ and $\Gamma_Y$ be $FP$-fuzzy soft rings over $R$. Then their intersection $\Gamma_X \cap \Gamma_Y$ is still an $FP$-fuzzy soft ring over $R$.

Proof. We can write $\Gamma_X \cap \Gamma_Y = \Gamma_{X \cap Y}$. For all $x \in E$, $\mu_{X \cap Y}(x) = \min\{\mu_X(x), \mu_Y(x)\}$, $\gamma_X(x)$ and $\gamma_Y(x)$ are fuzzy subrings of $R$, then $\gamma_{X \cap Y}(x) = \gamma_X(x) \cap \gamma_Y(x)$ is a fuzzy subring of $R$. Therefore, $\Gamma_X \cap \Gamma_Y$ is an $FP$-fuzzy soft ring over $R$.  

Theorem 3.5 Let $R$ be a ring, $E$ be a set of parameters and $\Gamma_X$ and $\Gamma_Y$ be $FP$-fuzzy soft rings over $R$ with $X \cap Y = \emptyset$. Then their union $\Gamma_X \cup \Gamma_Y$ is still an $FP$-fuzzy soft ring over $R$.  

Proof. We can write $\Gamma_\bar{X} \cap \Gamma_Y = \Gamma_{X \cap Y}$. For all $x \in E$, $\mu_{\bar{X} \cap Y}(x) = \max\{\mu_X(x), \mu_Y(x)\}$, then $\mu_{\bar{X} \cap Y}(x) = \mu_X(x) \text{ or } \mu_{\bar{X} \cap Y}(x) = \mu_Y(x)$ since $X \cap Y = \emptyset$. Therefore, $\gamma_{\bar{X} \cap Y}(x) = \gamma_X(x)$ or $\gamma_{\bar{X} \cap Y}(x) = \gamma_Y(x)$, so $\gamma_{\bar{X} \cap Y}(x)$ is a fuzzy subring of $R$, then $\Gamma_{X \cup Y}$ is an $FP$-fuzzy soft ring over $R$.

**Definition 3.6** Let $\Gamma_X, \Gamma_Y$ be $FP$-fuzzy soft rings over $R$. Then $\Gamma_X$ is said to be an $FP$-fuzzy soft subring of $\Gamma_Y$, if $\mu_X(x) \leq \mu_Y(x)$ and $\gamma_X(x)$ is a fuzzy subset of $\gamma_Y(x)$ for all $x \in E$.

**Example 3.7** Let $R = Z_4 = \{0, 1, 2, 3\}$ be a ring and $E = \{a, b\}$ be a set of parameters. If $X = \{0/2, a, 0/b\}$, $\gamma_X(a) = \{\emptyset, 0/0.5, 1/0.3, 2/0.4, 3/0.3\}$, $\gamma_X(b) = \emptyset$, and $Y = \{0.4/a, 0.3/b\}$, $\gamma_Y(a) = \{0/0.6, 1/0.4, 2/0.5, 3/0.4\}$, $\gamma_Y(b) = \{0/0.4, 1/0.2, 2/0.3, 3/0.2\}$, then $\Gamma_X$ and $\Gamma_Y$ are $FP$-fuzzy soft rings over $R$, and $\Gamma_X$ is an $FP$-fuzzy soft subring of $\Gamma_Y$.

**Theorem 3.8** Let $R$ be a ring, $E$ be a set of parameters, $\Gamma_X$ and $\Gamma_Y$ are $FP$-fuzzy soft subrings of $\Gamma_Z$.

1. $\Gamma_X \cap \Gamma_Y$ is an $FP$-fuzzy soft subring of $\Gamma_Z$.

2. If $X \cap Y = \emptyset$, then $\Gamma_X \cup \Gamma_Y$ is an $FP$-fuzzy soft subring of $\Gamma_Z$.

**Proof.** The proofs are similar to the proofs of Theorems 3.4 and 3.5.

**Definition 3.9** Let $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) : x \in A, \gamma_X(x) \in F(R), \mu_X(x) \in [0, 1]\}$ and $\Gamma_Y = \{(\mu_Y(y)/y, \gamma_Y(y)) : y \in B, f_Y(y) \in F(K), \mu_Y(y) \in [0, 1]\}$ be $FP$-fuzzy soft rings over rings $R$ and $K$, respectively. If $f : R \to K$ and $g : A \to B$ are two functions, then $(f, g)$ is called an $FP$-fuzzy soft homomorphism such that $(f, g)$ is an $FP$-fuzzy soft homomorphism from $\Gamma_X$ to $\Gamma_Y$. The latter is written by $\Gamma_X \sim \Gamma_Y$ if the following conditions are satisfied:

1. $f$ is an epimorphism from $R$ to $K$,

2. $g$ is a surjective mapping, and

3. $f(\gamma_X(x)) = \gamma_Y(g(x))$ and $\mu_X(x) = \mu_Y(g(x))$ for all $x \in A$.

In the above definition, if $f$ is an isomorphism from $R$ to $K$ and $g$ is a bijective mapping, then $(f, g)$ is called an $FP$-fuzzy soft isomorphism so that $(f, g)$ is an $FP$-fuzzy soft isomorphism from $\Gamma_X$ to $\Gamma_Y$, denoted by $\Gamma_X \simeq \Gamma_Y$.

**Example 3.10** Let $R = (Z, +, \times)$ and $K = (4Z, +, \times)$, $A = \{1, 3\}$ and $B = \{2, 6\}$. Define a homomorphism $f$ from $R$ onto $K$ by $f(r) = 4r$ for $r \in R$, and a mapping $g$ from $A$ onto $B$ by $g(x) = 2x$, for $x \in A$.

Let $X$ be a fuzzy set over $A$ defined by $\mu_X = \{1/0.5, 3/0.8\}$.

Let $Y$ be a fuzzy set over $B$ defined by $\mu_Y = \{2/0.5, 6/0.8\}$.
Let $\gamma_X : A \to F(R)$ defined by

$$(\gamma_X(1))(r) = \begin{cases} 0.1, & r = 2k + 1, k \in \mathbb{Z}, \\ 0.3, & r = 2k, k \in \mathbb{Z}. \end{cases}$$

$$(\gamma_X(3))(r) = \begin{cases} 0.2, & r = 2k + 1, k \in \mathbb{Z}, \\ 0.4, & r = 2k, k \in \mathbb{Z}. \end{cases}$$

Let $\gamma_Y : B \to F(K)$ defined by

$$(\gamma_Y(2))(r) = \begin{cases} 0.1, & r = 8k + 4, k \in \mathbb{Z}, \\ 0.3, & r = 8k, k \in \mathbb{Z}. \end{cases}$$

$$(\gamma_Y(6))(r) = \begin{cases} 0.2, & r = 8k + 4, k \in \mathbb{Z}, \\ 0.4, & r = 8k, k \in \mathbb{Z}. \end{cases}$$

It is clear that $\Gamma_X$ and $\Gamma_Y$ are FP-fuzzy soft rings over $R$ and $K$, respectively. We can immediately see that $f$ is an isomorphism from $R$ to $K$ and $g$ is a bijective mapping, $\mu_X(x) = \mu_Y(g(x))$ and we can deduce that $f(\gamma_X(x)) = \gamma_Y(g(x))$ for all $x \in A$. Hence $(f, g)$ is an FP-fuzzy soft isomorphism from $\Gamma_X$ to $\Gamma_Y$.

The following lemma is similar to fuzzy subgroups in [12], and we omit the proof.

Lemma 3.11 If $f : R \to K$ is an epimorphism of rings and $\mu$ a fuzzy subring ideal of $R$, then $f(\mu)$ is a fuzzy subring ideal of $K$.

Theorem 3.12 Let $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) : x \in A, \gamma_X(x) \in F(R), \mu_X(x) \in [0, 1]\}$ be an FP-fuzzy soft ring over $R$ and $\Gamma_Y = \{(\mu_Y(y)/y, \gamma_Y(y)) : y \in B, \gamma_Y(y) \in F(K), \mu_Y(y) \in [0, 1]\}$ be an FP-fuzzy soft set over ring $K$. If $\Gamma_X$ is FP-fuzzy soft homomorphic to $\Gamma_Y$, then $\Gamma_Y$ is an FP-fuzzy soft ring over $K$.

Proof. Let $(f, g)$ be an FP-fuzzy soft homomorphism from $\Gamma_X$ to $\Gamma_Y$. Since $\Gamma_X$ is an FP-fuzzy soft ring over $R$, $f(R) = K$ and $\gamma_X(x)$ is a fuzzy subring of $R$ for all $x \in A$. Now, for all $y \in B$, there exists $x \in A$ such that $g(x) = y$. Hence, $\gamma_Y(y) = \gamma_Y(g(x)) = f(\gamma_X(x))$ is a fuzzy subring of the ring $K$ and $\mu_Y(y) = \mu_Y(g(x)) = \mu_X(x)$, so $\Gamma_Y$ must be an FP-fuzzy soft ring over $K$ as well.

4. FP-equivalent fuzzy soft rings

Definition 4.1 Let $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) : x \in E, \gamma_X(x) \in F(R), \mu_X(x) \in [0, 1]\}$ be an FP-fuzzy soft ring over $R$. Then $\Gamma_X$ is said to be FP-equivalent fuzzy soft ring over $R$ if, for any $x, y \in E$, $\mu_X(x) = \mu_X(y)$, we have $\gamma_X(x) = \gamma_X(y)$.

Example 4.2 Let $R = \mathbb{Z}_4$, $E = \{x_1, x_2, x_3, x_4\}$ be and $X$ be a fuzzy set over $E$ defined by $X = \{0.1/x_1, 0.5/x_2, 0.5/x_3, 0.3/x_4\}$, $\gamma_X(x_1) = \{0/0.1, 1/0.4, 2/0.5, 3/0.4\}$, $\gamma_X(x_2) = \{0/0.8, 1/0.5, 2/0.6, 3/0.5\}$, $\gamma_X(x_3) = \{0/0.8, 1/0.5, 2/0.6, 3/0.5\}$, $\gamma_X(x_4) = \{0/0.7, 1/0.3, 2/0.5, 3/0.3\}$, It is clearly that $\Gamma_X$ is an FP-equivalent fuzzy soft ring over $R$. 
**Theorem 4.3** Let \( R \) be a ring, \( E \) be a set of parameters. Then

1. \( \Gamma_0 \) and \( \Gamma_0 \) are \( FP \)-equivalent fuzzy soft rings.
2. \( \Gamma_R \) and \( \Gamma_R \) are \( FP \)-equivalent fuzzy soft rings.
3. \( \Gamma_\delta \) and \( \Gamma_\delta \) are \( FP \)-equivalent fuzzy soft rings.

**Proof.** By Definitions 2.4 and 4.1, the proofs of (1) and (2) are straightforward. Since \( \Gamma_\delta = \Gamma_R \) and \( \Gamma_\delta = \Gamma_0 \), then (3) is hold. 

**Notation 4.4** If \( \Gamma_X = \{ \mu_X(x)/x, \gamma_X(x) : x \in E, \gamma_X(x) \in F(R), \mu_X(x) \in [0,1] \} \) and \( \Gamma_Y = \{ (\mu_Y(x)/x, \gamma_Y(x) : x \in E, \gamma_Y(x) \in F(R), \mu_Y(x) \in [0,1] \} \) are \( FP \)-equivalent fuzzy soft sets over ring \( R \), \( \Gamma_X \cap \Gamma_Y \) is not always an \( FP \)-equivalent fuzzy soft ring over \( R \).

**Example 4.5** Let \( R = Z_4 \), \( E = \{ x_1, x_2, x_3, x_4 \} \). Let \( \Gamma_X \) be an \( FP \)-fuzzy soft set over \( R \) defined by

\[
X = \{ 0.1/x_1, 0.5/x_2, 0.5/x_3, 0.3/x_4 \},
\gamma_X(x_1) = \{ 0/0.1, 1/0.4, 2/0.5, 3/0.4 \},
\gamma_X(x_2) = \{ 0/0.8, 1/0.5, 2/0.6, 3/0.5 \},
\gamma_X(x_3) = \{ 0/0.8, 1/0.5, 2/0.6, 3/0.5 \},
\gamma_X(x_4) = \{ 0/0.7, 1/0.3, 2/0.5, 3/0.3 \}.
\]

And let \( \Gamma_Y \) be an \( FP \)-fuzzy soft set over \( R \) defined by

\[
Y = \{ 0.3/x_1, 0.3/x_2, 0.6/x_3, 0.1/x_4 \},
\gamma_Y(x_1) = \{ 0/0.5, 1/0.3, 2/0.4, 3/0.3 \},
\gamma_Y(x_2) = \{ 0/0.5, 1/0.3, 2/0.4, 3/0.3 \},
\gamma_Y(x_3) = \{ 0/0.6, 1/0.4, 2/0.5, 3/0.4 \},
\gamma_Y(x_4) = \{ 0/0.8, 1/0.2, 2/0.7, 3/0.2 \}.
\]

It is clearly that \( \Gamma_X \) and \( \Gamma_Y \) are \( FP \)-equivalent fuzzy soft rings over \( R \).

We can see that

\[
\gamma_X \cap \gamma_Y(x_1) = \{ 0/0.5, 1/0.3, 2/0.4, 3/0.3 \},
\gamma_X \cap \gamma_Y(x_4) = \{ 0/0.7, 1/0.2, 2/0.5, 3/0.2 \}.
\]

Then \( \Gamma_X \cap \Gamma_Y \) is not an \( FP \)-equivalent fuzzy soft ring over \( R \).

**Notation 4.6** If \( \Gamma_X = \{ \mu_X(x)/x, \gamma_X(x) : x \in E, \gamma_X(x) \in F(R), \mu_X(x) \in [0,1] \} \) and \( \Gamma_Y = \{ (\mu_Y(x)/x, \gamma_Y(x) : x \in E, \gamma_Y(x) \in F(R), \mu_Y(x) \in [0,1] \} \) are \( FP \)-equivalent fuzzy soft rings over ring \( R \) with \( X \cap Y = \emptyset \), \( \Gamma_X \cup \Gamma_Y \) is not always an \( FP \)-equivalent fuzzy soft ring over \( R \).

**Example 4.7** Let \( R = Z_4 \), \( E = \{ x_1, x_2, x_3, x_4 \} \). Let \( \Gamma_X \) be an \( FP \)-fuzzy soft set over \( R \) defined by

\[
X = \{ 0/x_1, 0.5/x_2, 0.5/x_3, 0/x_4 \},
\gamma_X(x_1) = \emptyset, \gamma_X(x_2) = \{ 0/0.6, 1/0.4, 2/0.5, 3/0.4 \},
\gamma_X(x_3) = \{ 0/0.6, 1/0.4, 2/0.5, 3/0.4 \}, \gamma_X(x_4) = \emptyset.
\]

Let \( \Gamma_Y \) be an \( FP \)-fuzzy soft set over \( R \) defined by

\[
Y = \{ 0.5/x_1, 0/x_2, 0/x_3, 0.5/x_4 \},
\gamma_Y(x_1) = \{ 0/0.4, 1/0.2, 2/0.3, 3/0.2 \}, \gamma_Y(x_2) = \emptyset,
\gamma_Y(x_3) = \emptyset, \gamma_Y(x_4) = \{ 0/0.4, 1/0.2, 2/0.3, 3/0.2 \}.
\]

It is clear that \( \Gamma_X \) and \( \Gamma_Y \) are \( FP \)-equivalent fuzzy soft rings over \( R \) and \( X \cap Y = \emptyset \).
We can see that
\[ \mu_{(X \cup Y)}(x_1) = \mu_{(X \cup Y)}(x_2), \]
but
\[ \gamma_{(X \cup Y)}(x_1) = \{0/0.4, 0.1/0.2, 0/0.3, 0/0.2\} \neq \gamma_{(X \cup Y)}(x_2) = \{0/0.6, 0.1/0.4, 0/0.5, 0/0.4\}. \]

Then \( \Gamma_X \tilde\cap \Gamma_Y \) is not an FP-equivalent fuzzy soft ring over \( R \).

**Theorem 4.8** Let \( \Gamma_X = \{(x, \gamma_X(x)) : x \in A, \gamma_X(x) \in F(R), \mu_X(x) \in [0, 1]\} \) be an FP-equivalent fuzzy soft ring over \( R \) and \( \Gamma_Y = \{(x, \gamma_Y(x)) : x \in B, \gamma_Y(x) \in F(K), \mu_Y(x) \in [0, 1]\} \) be an FP-fuzzy soft set over ring \( K \). If \( \Gamma_X \) is FP-fuzzy soft homomorphic to \( \Gamma_Y \), then \( \Gamma_Y \) is an FP-equivalent fuzzy soft ring over \( K \).

**Proof.** Let \( (f, g) \) be an FP-fuzzy soft homomorphism from \( \Gamma_X \) to \( \Gamma_Y \). Since \( \Gamma_X \) is an FP-equivalent fuzzy soft ring over \( R \), \( \gamma_X(x_1) = \gamma_X(x_2) \) for all \( x_1, x_2 \in A \), \( \mu_X(x_1) = \mu_X(x_2) \). Now, for all \( y_1, y_2 \in B \) and \( \mu_Y(y_1) = \mu_Y(y_2) \), there exist \( x_1, x_2 \in A \) such that \( g(x_1) = y_1, g(x_2) = y_2 \). Since \( \mu_Y(y_1) = \mu_Y(g(x_1)) = \mu_X(x_1) \) and \( \mu_Y(y_2) = \mu_Y(g(x_2)) = \mu_X(x_2) \), then \( \gamma_X(x_1) = \gamma_X(x_2) \). Hence, \( \gamma_Y(y_1) = \gamma_Y(g(x_1)) = f(\gamma_X(x_1)) = f(\gamma_X(x_2)) = \gamma_Y(g(x_2)) = \gamma_Y(y_2) \) and \( \Gamma_Y \) must be an FP-soft fuzzy ring over \( K \) as well.

5. **FP-increasing(decreasing) fuzzy soft rings**

**Definition 5.1** Let \( \Gamma_X = \{(x, \gamma_X(x)) : x \in E, \gamma_X(x) \in F(R), \mu_X(x) \in [0, 1]\} \) be an FP-fuzzy soft ring over \( R \). Then \( \Gamma_X \) is said to be an FP-increasing fuzzy soft ring over \( R \) if, for any \( x, y \in E, \mu_X(x) \leq \mu_X(y) \), we have \( \gamma_X(x) \subseteq \gamma_X(y) \), and \( \Gamma_X \) is said to be FP-decreasing fuzzy soft ring over \( R \) if, for any \( x, y \in E, \mu_X(x) \leq \mu_X(y) \), we have \( \gamma_X(x) \supseteq \gamma_X(y) \).

**Example 5.2** Let \( R = Z_4 \), \( E = \{x_1, x_2, x_3, x_4\} \) and \( X \) be a fuzzy set over \( E \) defined by \( X = \{0.6/x_1, 0.5/x_2, 0.3/x_3, 0.2/x_4\} \), \( \gamma_X(x_1) = \{0/1, 0.1/0.6, 0/0.7, 0/0.3\} \), \( \gamma_X(x_2) = \{0/0.8, 0.1/0.5, 0/0.6, 0/0.5\} \), \( \gamma_X(x_3) = \{0/0.8, 0.1/0.5, 0/0.6, 0/0.5\} \), \( \gamma_X(x_4) = \{0/0.7, 0.1/0.3, 0/0.5, 0/0.3\} \). It is clearly that \( F_X \) is an FP-increasing fuzzy soft ring over \( R \).

**Notation 5.3** If \( \Gamma_X = \{(x, \gamma_X(x)) : x \in E, \gamma_X(x) \in F(R), \mu_X(x) \in [0, 1]\} \) and \( \Gamma_Y = \{(x, \gamma_Y(x)) : x \in E, \gamma_Y(x) \in F(R), \mu_Y(x) \in [0, 1]\} \) are FP-increasing fuzzy soft rings over ring \( R \), \( \Gamma_X \tilde\cap \Gamma_Y \) is not always an FP-increasing fuzzy soft ring over \( R \).

**Example 5.4** Let \( R = Z_4 \), \( E = \{x_1, x_2\} \). Let \( X \) be a fuzzy set over \( E \) defined by \( X = \{0.5/x_1, 0.4/x_2\} \), \( \gamma_X(x_1) = \{0/1, 0.1/0.6, 0/0.7, 0/0.6\} \), \( \gamma_X(x_2) = \{0/0.7, 0.1/0.2, 0/0.3, 0/0.2\} \). Let \( Y \) be a fuzzy set over \( E \) defined by \( Y = \{0.1/x_1, 0.9/x_2\} \).
\[ \gamma_Y(x_1) = \{ \overline{0}/0.6, \overline{1}/0.4, \overline{2}/0.5, \overline{3}/0.4 \}, \quad \gamma_Y(x_2) = \{ \overline{0}/0.8, \overline{1}/0.5, \overline{2}/0.6, \overline{3}/0.5 \}. \]

It is clear that \( \Gamma_X \) and \( \Gamma_Y \) are FP-increasing fuzzy soft rings over \( R \). We can see that

\[ X \cap Y = \{0.1/x_1, 0.4/x_2\}, \quad \gamma_{(X \cap Y)}(x_1) = \{ \overline{0}/0.6, \overline{1}/0.4, \overline{2}/0.5, \overline{3}/0.4 \}, \quad \gamma_{(X \cap Y)}(x_2) = \{ \overline{0}/0.7, \overline{1}/0.2, \overline{2}/0.3, \overline{3}/0.2 \}. \]

Then \( \Gamma_X \cap \Gamma_Y \) is not an FP-increasing fuzzy soft ring over \( R \).

**Notation 5.5** If \( \Gamma_X = \{ \mu_X(x)/x, \gamma_X(x) : x \in E, \gamma_X(x) \in F(R), \mu_X(x) \in [0, 1] \} \) and \( \Gamma_Y = \{ (\mu_Y(x)/x, \gamma_Y(x)) : x \in E, \gamma_Y(x) \in F(R), \mu_Y(x) \in [0, 1] \} \) are FP-increasing fuzzy soft rings over ring \( R \) with \( X \cap Y = \emptyset \), \( \Gamma_X \cup \Gamma_Y \) is not always an FP-increasing fuzzy soft ring over \( R \).

**Example 5.6** Let \( R = Z_1, E = \{ x_1, x_2 \} \). Let \( X \) be a fuzzy set over \( E \) defined by

\[ X = \{0/x_1, 0.5/x_2\}, \quad \gamma_X(x_1) = \emptyset, \quad \gamma_X(x_2) = \{ \overline{0}/0.6, \overline{1}/0.4, \overline{2}/0.5, \overline{3}/0.4 \}. \]

Let \( Y \) be a fuzzy set over \( E \) defined by

\[ Y = \{0.9/x_1, 0/x_2\}, \quad \gamma_Y(x_1) = \{ \overline{0}/0.5, \overline{1}/0.2, \overline{2}/0.3, \overline{3}/0.2 \}, \quad \gamma_Y(x_2) = \emptyset. \]

It is clear that \( F_X \) and \( F_Y \) are FP-increasing fuzzy soft rings over \( R \) and \( X \cap Y = \emptyset \). We can see that

\[ (X \cup Y) = \{0.9/x_1, 0.5/x_2\}, \quad \gamma_{(X \cup Y)}(x_1) = \{ \overline{0}/0.5, \overline{1}/0.2, \overline{2}/0.3, \overline{3}/0.2 \}, \quad \gamma_{(X \cup Y)}(x_2) = \{ \overline{0}/0.6, \overline{1}/0.4, \overline{2}/0.5, \overline{3}/0.4 \}. \]

Then \( \Gamma_X \cup \Gamma_Y \) is not an FP-increasing fuzzy soft ring over \( R \).

**Theorem 5.7** Let \( \Gamma_X = \{ (\mu_X(x)/x, \gamma_X(x)) : x \in A, \gamma_X(x) \in F(R), \mu_X(x) \in [0, 1] \} \) be an FP-increasing fuzzy soft ring over \( R \) and \( \Gamma_Y = \{ (\mu_Y(x)/x, \gamma_Y(x)) : x \in B, \gamma_Y(x) \in F(K), \mu_Y(x) \in [0, 1] \} \) be an FP-fuzzy soft set over ring \( K \). If \( \Gamma_X \) is FP-fuzzy soft homomorphic to \( \Gamma_Y \), then \( \Gamma_Y \) is an FP-increasing fuzzy soft ring over \( K \).

**Proof.** Let \( (f, g) \) be an FP-fuzzy soft homomorphism from \( \Gamma_X \) to \( \Gamma_Y \). Since \( \Gamma_X \) is an FP-increasing fuzzy soft ring over \( R \), for all \( x_1, x_2 \in A, \mu_X(x_1) \leq \mu_X(x_2) \), \( \gamma_X(x_1) \subseteq \gamma_X(x_2) \). Now, for all \( y_1, y_2 \in B \) and \( \mu_Y(y_1) \leq \mu_Y(y_2) \), then there exist \( x_1, x_2 \in A \) such that \( g(x_1) = y_1, g(x_2) = y_2 \). Since \( \mu_Y(y_1) = \mu_Y(g(x_1)) = \mu_X(x_1) \) and \( \mu_Y(y_2) = \mu_Y(g(x_2)) = \mu_X(x_2) \), then \( \mu_X(x_1) \leq \mu_X(x_2) \). Hence, \( \gamma_Y(y_1) = \gamma_Y(g(x_1)) = f(\gamma_X(x_1)) \subseteq f(\gamma_X(x_2)) = \gamma_Y(g(x_2)) = \gamma_Y(y_2) \) and \( \Gamma_Y \) must be an FP-increasing fuzzy soft ring over \( K \) as well.

**Corollary 5.8** If \( \Gamma_X = \{ (\mu_X(x)/x, \gamma_X(x)) : x \in E, \gamma_X(x) \in F(R), \mu_X(x) \in [0, 1] \} \) is both a FP-equivalent fuzzy soft ring and FP-increasing fuzzy soft ring over ring \( R \), then \( \Gamma_X = \{ (\mu_X(\tilde{x})/\tilde{x}, \gamma_X(\tilde{x})) : \tilde{x} \in \tilde{E}, \gamma_X(\tilde{x}) \in F(R), \mu_X(\tilde{x}) \in [0, 1] \} \) is an FP-increasing fuzzy soft ring over ring \( R \).
6. Aggregate fuzzy subrings

In [3], N. Çağman et al. defined an aggregate fuzzy set of an $FP$-fuzzy soft set. They also defined $FPFS$-aggregation operator that produced an aggregate fuzzy set from an $FP$-fuzzy soft set and its fuzzy parameter set.

**Definition 6.1** [3] Let $\Gamma_X \in FPFS(U)$. Then $FPFS$-aggregation operator, denoted by $FPFS_{agg}$ is defined by

$$FPFS_{agg} : F(E) \times FPFS(U) \rightarrow F(U),$$

$$FPFS_{agg}(X, \Gamma_X) = \Gamma_X^*$$

where

$$\Gamma_X^* = \{ \mu_{\Gamma_X^*}(u) / u : u \in U \}$$

which is a fuzzy set over $U$. The value $\Gamma_X^*$ is called aggregate fuzzy set of the $\Gamma_X$.

Here the membership degree $\mu_{\Gamma_X^*}(u)$ of $u$ is defined as follows

$$\mu_{\Gamma_X^*}(u) = \frac{1}{|E|} \sum_{x \in E} \mu_X(x) \mu_{\gamma_X(x)}(u)$$

where $|E|$ is the cardinality of $E$.

**Theorem 6.2** Let $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) : x \in E, \gamma_X(x) \in F(R), \mu_X(x) \in [0,1]\}$ be an $FP$-fuzzy soft ring over $R$. Then the aggregate fuzzy set $\Gamma_X^*$ of $\Gamma_X$ is a fuzzy subring of $R$.

**Proof.** For any $x \in E$, $\gamma_X(x)$ is a fuzzy subring of $R$. Then for all $r, s \in R$, $\mu_{\gamma_X(x)}(r-s) \geq \min\{\mu_{\gamma_X(x)}(r), \mu_{\gamma_X(x)}(s)\}$ and $\mu_{\gamma_X(x)}(rs) \geq \min\{\mu_{\gamma_X(x)}(r), \mu_{\gamma_X(x)}(s)\}$. Then

$$\mu_{\Gamma_X^*}(r-s) = \frac{1}{|E|} \sum_{x \in E} \mu_X(x) \mu_{\gamma_X(x)}(r-s)$$

$$\geq \min \left\{ \frac{1}{|E|} \sum_{x \in E} \mu_X(x) \mu_{\gamma_X(x)}(r), \frac{1}{|E|} \sum_{x \in E} \mu_X(x) \mu_{\gamma_X(x)}(s) \right\}$$

$$= \min\{\mu_{\Gamma_X^*}(r), \mu_{\Gamma_X^*}(s)\}.$$ 

In the same way, we can obtain $\mu_{\Gamma_X^*}(rs) \geq \min\{\mu_{\Gamma_X^*}(r), \mu_{\Gamma_X^*}(s)\}$. Which is to say that $\Gamma_X^*$ is a fuzzy subring of $R$.

**Notation 6.3** Above $\Gamma_X^*$ is called an aggregate fuzzy subring of $FP$-fuzzy soft ring $\Gamma_X$. 


Example 6.4 Let $R$ be a full matrix ring, written by $M_n$, let $A$ be an upper triangular matrix ring and $B$ a symmetrical matrix ring. And let $E = \{a, b\}$, the parameters $a, b$ stand for “upper triangular” and “symmetrical”, respectively. And $X$ be a fuzzy set over $E$ defined by

$$
\mu_X(x) = \begin{cases}
1, & x = a, \\
0.8, & x = b.
\end{cases}
$$

Let $\gamma_X$ be defined by

$$
\mu_{\gamma_X(a)}(r) = \begin{cases}
0, & r \text{ is not an upper triangular matrix,} \\
1, & r \text{ is an upper triangular matrix.}
\end{cases}
$$

$$
\mu_{\gamma_X(b)}(r) = \begin{cases}
0, & r \text{ is not symmetrical,} \\
1, & r \text{ is symmetrical.}
\end{cases}
$$

It is clear that $\Gamma_X$ is an $FP$-fuzzy soft ring over $M_n$. The aggregate fuzzy set can be found as

$$
\Gamma_X^*(m) = \begin{cases}
0.9, & \text{if } m \in A \cap B, \\
0.5, & \text{if } m \in A - B, \\
0.4, & \text{if } m \in B - A, \\
0, & \text{otherwise.}
\end{cases}
$$

We can verify that $\Gamma_X^*$ is a fuzzy ring of $M_n$.

Notation 6.5 Let $R$ be a subring of $M_n$, if $\Gamma_X$ is defined as in Example 6.4, then $R$ is a diagonal matrix ring if and only if the aggregate fuzzy subring of $\Gamma_X$ is $\Gamma_X^* = 0.9$.

Notation 6.6 Let $R$ be a subring of $M_n$, if $\Gamma_X$ is defined as in Example 6.4, then the $\Gamma_X^*$ is called a fuzzy diagonal subring of $R$ related to the $FP$-fuzzy soft ring $\Gamma_X$.

Remark 6.7 We can define another fuzzy diagonal subring of $R$ related to another $FP$-fuzzy soft ring.

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References


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