OD-CHARACTERIZATION OF ALTERNATING GROUP OF DEGREE $p + 3$

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Abstract. Let A_{p+3} be the alternating group of degree $p+3$, where p is a prime, $p+4$ is a composite number, $p+6$ is a prime and $7 \neq p \in \pi(1000!)$. In the present paper, we prove that A_{p+3} is OD-characterizable by using the classification theorem of finite simple groups and Magma soft of computational group theory. This new method is introduced in order to deal with the subtle changes of the prime graph of a group in the discussion of its OD-characterization, which might occur. As a consequence of this theorem not only generalizes the result in [1] (Hoseini, A.A. and Moghaddamfar, A.R., *Frontiers of Mathematics in China*, 5 (3), 2010) but also gives a positive answer to a conjecture in [2] (Shi, W.J., Contemporary Math., 82, 1989).

Keywords: prime graph, degree pattern, degree of a vertex, finite simple group, alternating group.

2000 AMS Subject Classification: 20D05.

1. Introduction

Throughout this paper, groups under consideration are finite, and by a simple group, we always mean a nonabelian simple. For any group G, we denote by $\pi_e(G)$ the set of orders of its elements and by $\pi(G)$ the set of prime divisors of $|G|$. We associate to $\pi(G)$ a graph of G, denoted by $\Gamma(G)$ (cf. [3]). The vertex set of this graph is $\pi(G)$, and two distinct vertices p, q are adjacent by an edge if

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and only if $pq \in \pi_e(G)$, in this case, we write $p \sim q$. We also denote by $\pi(l)$ the set of all primes dividing l, where l is a positive integer.

In this article, we also use the following symbols. Let G be a finite group, then the socle of G is defined as the subgroup generated by minimal normal subgroups of G, denoted as $Soc(G)$. $Syl_p(G)$ denotes the set of all Sylow p-subgroups of G, where $p \in \pi(G)$, and P_r denotes a Sylow r-subgroup of G for $r \in \pi(G)$. Moreover, we use A_n to denote alternating group of degree n. Let q be a prime, we denote by $Exp(n, q)$ the exponent of the largest power of a prime q in the factorization of a positive integer $n(> 1)$. All further unexplained symbols and notations are standard and can be found in [4], for instance.

Definition 1.1 ([5]) Let G be a finite group and $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_i s$ are primes and $\alpha_i s$ are integers. For $p \in \pi(G)$, let $deg(p) := |\{q \in \pi(G) | p \sim q\}|$, which we call the *degree* of p. We also define $D(G) := (deg(p_1), deg(p_2), ..., deg(p_k))$, where $p_1 < p_2 < \cdots < p_k$. We call $D(G)$ the *degree pattern* of G.

Definition 1.2 ([5]) A group M is called *k-fold OD-characterizable* if there exist exactly k non-isomorphic groups G such that

(1) $|G| = |M|$,

(2) $D(G) = D(M)$.

Moreover, a 1-fold OD-characterizable group is simply called an *OD-characterizable group.*

It is an interesting and difficult topic to determine the structure of finite groups by their orders and degree patterns. This topic is related to following open problem:

Open problem. ([5]) *Let G and M be finite groups satisfying the conditions*

(1) $|G| = |M|$,

(2) $D(G) = D(M)$.

Then

(i) *How far do these conditions effect the structure of G?*

(ii) *Is the number of non-isomorphic groups satisfying* (1) *and* (2) *finite?*

At present, we mention that the problem is still unsolved completely and till now we may not be able to provide a suitable answer for the above questions. This topic was studied in several articles. For example, in a series of articles (Ref. to [1],[6]-[17]), it was shown that many finite almost simple groups are m-fold OD-characterizable, where m is a positive integer and $m \geq 1$. For convenience, we summarize some results of these articles which will be used later in the following propositions:

Proposition 1.3 ([1], [6]-[8]) *A finite group G is OD-characterizable if G is isomorphic to one of the following groups:*

(1) *The alternating groups A_p, A_{p+1} and A_{p+2}, where p is a prime;*
(2) *The alternating groups A_{p+3}, where p is a prime and $7 \neq p \in \pi(100!)$;*
(3) *All finite simple K_4-groups except A_{10};*
(4) *All finite simple groups whose orders are less than 10^8 except for A_{10} and $U_4(2)$.*

Proposition 1.4 ([9]) *A finite group G is 2-fold OD-characterizable if and only if $|G| = |A_{10}|$ and (2) $D(G) = D(A_{10})$.*

2. Main results

According to Proposition 1.3, the alternating groups A_p, A_{p+1} and A_{p+2} are OD-characterizable, and A_{p+3} with $7 \neq p \in \pi(100!)$ is OD-characterizable. Proposition 1.4 says that the alternating group A_{10} is 2-fold OD-characterizable. On the other hand, by [10], we see that all A_n with $10 \neq n \leq 100$ is OD-characterizable. Now, omitting all the above alternating groups except A_{10}, there remains the following groups:

$$A_{10}, A_{106}, A_{112}, A_{116}, ..., A_{126}, A_{134}, A_{135}, A_{136}, A_{142}, ... \tag{2.1}$$

By [3], we know that all the alternating groups in (2.1) have connected prime graph. By these facts above, we see that it is very difficult to investigate how many-fold OD-characterization of alternating groups. In this paper, we continue to investigate this topic and get the following theorem:

Main Theorem. *All alternating groups A_{p+3}, where $p+2$ is a composite number and $p+4$ is a prime and $7 \neq p \in \pi(1000!)$, are OD-characterizable.*

Proposition 1.4 says A_{10} is 2-fold OD-characterizable. It is worth mentioning that A_{10} is the first alternating group which has not been considered for OD-characterizable. Up to now, we do not know whether there exists an alternating group A_n $(n \neq 10)$ which isOD-characterizable. Hence, we put forward the following question:

Question. *Are the alternating groups A_n $(n \neq 10)$ OD-characterizable?*

In fact, Main Theorem and Proposition 1.3 imply the following corollary.

Corollary 2.1 *Let A_n be an alternating group of degree n. Assume one of the following conditions is fulfilled:*

(1) *$n = p$, $p+1$ or $p+2$, where p is a prime;*
(2) *$n = p+3$, where $p+2$ is a composite number and $p+4$ is a prime and $7 \neq p \in \pi(1000!)$.*

Then A_n is OD-characterizable.

3. Preliminaries

In this section, we give some results which will be applied for our further investigations. We shall utilize the following Lemma 3.1 concerning the set of elements of the alternating and symmetric groups ([18]).

Lemma 3.1 ([18]) *The group S_n (or A_n) has an element of order $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, where $p_1, p_2, ..., p_s$ are distinct primes and $\alpha_1, \alpha_2, ..., \alpha_s$ are natural numbers, if and only if $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq n$ (or $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq n$ for m odd, and $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq n-2$ for m even).*

As an immediately corollary of Lemma 3.1, we have

Lemma 3.2 *Let A_n (or S_n) be an alternating group (or a symmetric group) of degree n. Then, the following assertions hold:*

(1) *Let $p, q \in \pi(A_n)$ be odd primes. Then $p \sim q$ if and only if $p + q \leq n$.*
(2) *Let $p \in \pi(A_n)$ be an odd prime. Then $2 \sim p$ if and only if $p + 4 \leq n$.*
(3) *Let $p, q \in \pi(S_n)$. Then $p \sim q$ if and only if $p + q \leq n$.*

Lemma 3.3 ([19]) *Let G be a finite solvable group all of whose elements are of prime power order. Then $|\pi(G)| \leq 2$.*

Lemma 3.4 *Let A_{p+3} be an alternating group of degree $p+3$, where p is a prime and $p+2$ is a composite number. Suppose that $|\pi(A_{p+3})| = d$. Then the following assertions hold.*

(i) *$deg(2) = d - 2$. Particularly, $2 \sim r$ for each $r \in \pi(A_{p+3})\backslash\{p\}$.*

(ii) *$deg(3) = d - 1$, i.e., $3 \sim r$ for each $r \in \pi(A_{p+3})$.*

(iii) *$deg(p) = 1$. In other words, $p \sim r$, where $r \in \pi(A_{p+3})$, if and only if $r = 3$.*

(iv) *$Exp(|A_{p+3}|, 2) = \sum_{i=1}^{\propto} \left[\frac{p+3}{2^i}\right] - 1$. In particular, $Exp(|A_{p+3}|, 2) < p + 3$.*

(v) *$Exp(|A_{p+3}|, r) = \sum_{i=1}^{\propto} \left[\frac{p+3}{r^i}\right]$ for each $r \in \pi(A_{p+3})\backslash\{2\}$.*

Furthermore, $Exp(|A_{p+3}|, r) < \frac{p-1}{2}$, where $5 \leq r \in \pi(A_{p+3})$.

Particularly, if $r > \left[\frac{p+3}{2}\right]$, then $Exp(|A_{p+3}|, r) = 1$.

Proof. By Lemma 3.2, one has that $2 \not\sim p$. Obviously, $r + 4 \leq p + 3$ for each $r \in \pi(A_{p+3})\backslash\{p\}$, it follows that $2 \sim r$ and so $deg(2) = d - 2$. By the same reason, we have $deg(3) = d - 1$. For $r \in \pi(A_{p+3})\backslash\{2, p\}$, by Lemma 3.2, it is easy to see that $p \sim r$ if and only if $p + r \leq p + 3$. Hence $r = 3$ and $deg(p) = 1$. Therefore, (i), (ii) and (iii) hold.

By the definition of Gauss's integer function, we can get that

$$\begin{aligned} Exp(|A_{p+3}|, 2) &= \sum_{i=1}^{\infty} \left[\frac{p+3}{2^i}\right] - 1 \\ &= \left(\left[\frac{p+3}{2}\right] + \left[\frac{p+3}{2^2}\right] + \left[\frac{p+3}{2^3}\right] + \cdots\right) - 1 \\ &\leq \left(\frac{p+3}{2} + \frac{p+3}{2^2} + \frac{p+3}{2^3} + \cdots\right) - 1 \\ &= (p+3)\left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right) - 1 = p+2. \end{aligned}$$

Hence

$$Exp(|A_{p+3}|, 2) < p+3.$$

Thus (iv) follows.

By the same reason as above, we can prove

$$Exp(|A_{p+3}|, r) < \frac{p-1}{2},$$

where $5 \leq r \in \pi(A_{p+3})$.

Clearly, if $r > \left[\frac{p+3}{2}\right]$, then we have

$$Exp(|A_{p+3}|, r) = 1.$$

Hence (v) follows. This completes the proof of Lemma 3.4. ∎

Lemma 3.5 ([20]) *Let a be an arbitrary integer and m be a positive integer. If $(a, m) = 1$, then the equation $a^x \equiv 1\,(mod\ m)$ has solutions. Moreover, if the order of a modulo m is $h(a)$, then $h(a)|\varphi(m)$, where $\varphi(m)$ is Euler's function of m.*

Lemma 3.6 *Let A_{p+3} be an alternating group of degree $p+3$, where $p+2$ is a composite number and $p+4$ is a prime and $97 < p \in \pi(1000!)$. Set $P \in Syl_p(A_{p+3})$ and $Q \in Syl_q(A_{p+3})$, where $5 \leq q < p$. Then, the following assertions hold.*

(i) $q^{s(q)} \nmid |N_G(P)|$, *where* $s(q) = Exp(|A_{p+3}|, q)$.

(ii) *If* $p \in \{103, 109, 163, 193, 223, 229, 277, 349, 439, 463, 499, 613, 643, 739, 769, 823, 853, 877, 907, 967\}$, *then* $p \nmid |N_G(Q)|$.

(iii) *If* $p \in \{127, 307, 313, 379, 397, 457, 487, 673, 757, 859, 883, 937\}$, *then there exists at least a prime number, say r, such that the order of r modulo p is less than $p-1$, where $5 \leq r < p$ and $r \in \pi(A_{p+3})$.*

Proof. It is easy to see that the equation $q^x \equiv 1(\,mod\,p)$ has solutions by Lemma 3.5. Suppose that the order of q modulo p is $h(q)$. If $h(q) = p-1$, then q is a primitive root of modulo p. By hypothesis, using Magma soft of computational group theory, we know that there are only 32 such groups satisfying the conditions that $p+2$ is a composite number and $p+4$ is a prime number and $97 < p \in \pi(1000!)$. Again, using Magma of mathematics soft, we can obtain $h(q)$. For convenience, we have computed the values of p and $h(q)$ listed in Table 1 of this article.

Table 1 (p **and** $h(q)$)

p	$h(q)$	Condition	p	$h(q)$	Condition	p	$h(q)$	Condition
103	$2{\cdot}3{\cdot}17$	none	109	$2^2{\cdot}3^3$	none	163	$2{\cdot}3^4$	none
127	$2{\cdot}3^2{\cdot}7$	$q \neq 19$	127	3	$q = 19$	193	$2^6{\cdot}3$	none
223	$2{\cdot}3{\cdot}37$	none	229	$2^2{\cdot}3{\cdot}19$	none	277	$2^2{\cdot}3{\cdot}23$	none
307	$2{\cdot}3^2{\cdot}17$	$q \neq 17$	307	3	$q = 17$	349	$2^2{\cdot}3{\cdot}29$	none
313	$2^3{\cdot}3{\cdot}13$	$q \neq 5$	313	8	$q = 5$	439	$2{\cdot}3{\cdot}73$	none
379	$2{\cdot}3^3{\cdot}7$	$q \neq 5$	379	21	$q = 5$	397	$2^2{\cdot}3^2{\cdot}11$	$q \neq 31$
397	11	$q = 31$	457	$2^3{\cdot}3{\cdot}19$	$q \neq 109$	457	4	$q = 109$
463	$2{\cdot}3{\cdot}7{\cdot}11$	none	487	$2{\cdot}3^5$	$q \neq 5, 41$	487	54	$q = 5$
487	9	$q = 41$	499	$2{\cdot}3{\cdot}83$	none	613	$2^2{\cdot}3^2{\cdot}17$	none
643	$2{\cdot}3{\cdot}107$	none	673	$2^5{\cdot}3{\cdot}7$	$q \neq 23$	673	14	$q = 23$
739	$2{\cdot}3^2{\cdot}41$	none	757	$2^2{\cdot}3^3{\cdot}7$	$q \neq 59$	757	7	$q = 59$
769	$2^8{\cdot}3$	none	823	$2{\cdot}3{\cdot}137$	none	853	$2^2{\cdot}3{\cdot}71$	none
859	$2{\cdot}3{\cdot}11{\cdot}13$	$q \neq 13$	859	11	$q = 13$	877	$2{\cdot}3^2{\cdot}73$	none
883	$2{\cdot}3^2{\cdot}7^2$	$q \neq 71$	883	7	$q = 71$	907	$2{\cdot}3{\cdot}151$	none
937	$2^3{\cdot}3^2{\cdot}13$	$q \neq 13, 23$	937	18	$q = 13$	937	24	$q = 23$
967	$2{\cdot}3{\cdot}7{\cdot}23$	none						

By N-C Theorem, the factor group $N_G(P)/C_G(P)$ is isomorphic to a subgroup of $Aut(P) \cong \mathbb{Z}_{p-1}$. Thus, $|N_G(P)/C_G(P)|\big|(p-1)$. By Table 1, if there exists a prime q, where $5 \leq q < p$ and $q \in \pi(A_{p+3})$, such that $q^{s(q)}\big||N_G(P)|$, then $q\big||C_G(P)|$. Hence $deg(p) \geq 2$, a contradiction, and so (i) is proved.

We next assume that

$$p \in \{103, 109, 163, 193, 223, 229, 277, 349, 439, 463, 499,\\ 613, 643, 739, 769, 823, 853, 877, 907, 967\}.$$

If $p\big||N_G(Q)|$, by Table 1, and $Exp(|A_{p+3}|, q) < p$, then $p\big||C_G(Q)|$, which leads to a similar contradiction. Thus (ii) holds. The remaining parts of (iii) follows at once from Table 1. This completes the proof of Lemma 3.6.

Lemma 3.7 ([4], [21]) *Let M be a finite nonabelian simple group with order having prime divisors at most* 997. *Then M is isomorphic to one of the simple groups listed in Table* 1-3 *in* [21]. *Particularly, if* $|\ \pi(Out(M))\ | \neq 1$, *then* $\pi(Out(M)) \subseteq \{2, 3, 5, 7\}$.

Lemma 3.8 ([22]) *Let $S = P_1 \times P_2 \times \cdots \times P_r$, where P_is are isomorphic non-abelian simple groups. Then $Aut(S) = (Aut(P_1) \times Aut(P_2) \times \cdots \times Aut(P_r)) \rtimes \mathbb{S}_r$.*

4. OD-Characterization of Alternating Group A_{p+3}

We again recall that all the alternating groups A_p, A_{p+1} and A_{p+2}, where p is a prime number, are OD-characterizable (see Proposition 1.3 (1)). On the other hand, it has been shown that all the alternating groups A_{p+3} with $7 \neq p \in \pi(100!)$ are OD-characterizable (see Proposition 1.3 (2)). It is worth to mention that alternating group A_{10} is 2-fold OD-characterizable (see Proposition 1.4). Moreover, So far we have not found an alternating group which is not OD-characterizable. Hence, Professor Moghaddamfar, A. R. in [11] put forward the following conjecture 1 of this article:

Conjecture 1 ([11]) *All alternating groups A_{p+3} with $p \neq 7$ are OD-characterizable.*

In this section, we are going to give an affirmative answer to this conjecture for all the alternating groups A_{p+3}, where $p+2$ is a composite number and $p+4$ is a prime and $7 \neq p \in \pi(1000!)$. In other words, we will prove the following Main Theorem. We need to mention that this result not only generalizes the results in [1] but also gives an affirmative answer to the **Question** of this article for the alternating group A_{p+3}.

Main Theorem *All alternating groups A_{p+3}, where $p+2$ is a composite and $p+4$ is a prime and $7 \neq p \in \pi(1000!)$, are OD-characterizable.*

Proof. Let G be a finite group satisfying the conditions: (1) $|G| = |S_{p+3}|$ and (2) $D(G) = D(S_{p+3})$, where $p+2$ is a composite number and $p+4$ is a prime and $7 \neq p \in \pi(1000!)$. By [10], we only discuss the alternating groups A_{p+3}, where $p+2$ is a composite and $p+4$ is a prime and $97 < p \in \pi(1000!)$. By these hypotheses, we deduce that $\{r\} \cup \{rs | r+s \leq p+3\} \subseteq \pi_e(G)$ and $\{rs | r+s > p+3\} \cap \pi_e(G) = \varnothing$, where r, $s \in \pi(G)$. By Lemma 3.4, the prime graph of G is connected since $deg(3) = |\pi(G)| - 1$. Moreover, by the structure of $D(G)$, it is easy to check that $\Gamma(G) = \Gamma(A_{p+3})$. In the following, we write the proof in a number of separate cases. ∎

Case 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3\}$-group. Particularly, G is nonsolvable.

Proof. We first assert that K is a p'-group. If not, let p divides the order of K. Set $P \in Syl_p(G)$. By Lemma 3.6 (i), one has that $q^{s(q)} \nmid |N_G(P)|$ for each prime $q \in \pi(G)$ and $5 \leq q < p$. If $q \big| |N_G(P)|$, then either $q \big| |C_G(P)|$ or $q \in \pi(K)$. For the former, by Lemma 3.4 (iii), this leads to an obvious contradiction since $q \sim p$. In the latter case, i.e., $q \in \pi(K)$. In this case, by Table 1, it is easy to check that there necessarily exists such a prime r such that $r \not\sim q$, where $5 \leq r < p$

and $r \in \pi(K)$. In fact, by Lemma 3.2 (1), it is sufficient to find such a prime r such that $r+q>p$, then $r \not\sim q$. Since K is solvable, it possesses a Hall $\{p,q,r\}$-subgroup T. It follows that T is solvable. Since there exists no edge between p, q and r in $\Gamma(G)$, all elements in T are of prime power order. Hence $|\pi(T)| \leq 2$ by Lemma 3.3, a contradiction. Thus K is a p'-group.

Next, we show that K is a q'-group for each $q \in \pi(G) \backslash \{2,3,p\}$. Set $Q \in Syl_q(K)$, where $q \in \pi(K)$. Suppose that the order of q modulo p is $h(q)$. By the Frattini argument, $G = KN_G(Q)$, hence p divides the order of $N_G(Q)$. By Lemma 3.6 (ii) and (iii), it is easy to see that p is one of the possible primes: 127, 307, 313, 379, 397, 457, 487, 673, 757, 859, 883 and 937. In this case, there necessarily exists at least a prime, say q, such that $h(q) < p-1$. We prove the lemma up to choice of p one by one. The proof is divided into 3 cases.

Case 1.1. $p = 127$.

By Table 1, If there exists a prime q such that $p \big| |N_G(Q)|$, where $Q \in Syl_q(G)$, then $q = 19$. Now, by N-C Theorem, $N_G(Q)/C_G(Q) \lesssim Aut(Q)$. By Lemma 3.4 (v), we have $Exp(|G|, 19) = 6$, thus $|N_G(Q)/C_G(Q)| \big| \prod_{i=1}^{6} 19^{15} \cdot (19^i - 1)$. It is to check that $113 \nmid \prod_{i=1}^{6} 19^{15} \cdot (19^i - 1)$. If $113 \big| |N_G(Q)|$, then $113 \in \pi(C_G(Q))$. Thus $19 \sim 113$, a contradiction. Therefore, $113 \in \pi(K)$. Since K is a solvable group, it possesses a Hall $\{19, 113\}$-subgroup H of order $19^6 \cdot 113$. Clearly, H is nilpotent, so $19 \sim 113$, which leads to a contradiction as above.

Case 1.2. To prove the lemma follows if $p = 307$.

It is easy to see that there exists a prime, say q, such that $p \big| |N_G(Q)|$, where $Q \in Syl_q(G)$. Then $q = 17$ by Table 1. Since $N_G(Q)/C_G(Q) \lesssim Aut(Q)$ by N-C Theorem and $Exp(|G|, 17) = 19$ by Lemma 3.4, we have $|N_G(Q)/C_G(Q)| \big| \prod_{i=1}^{19} 17^{171} \cdot (17^i - 1)$. Using Magma soft, one has that $31 \nmid \prod_{i=1}^{19} 17^{171} \cdot (17^i - 1)$. If $31 \big| |N_G(Q)|$, then $31 \in \pi(C_G(Q))$. Set $N = N_G(Q)$, $C = C_G(Q)$ and $K_{31} \in Syl_{31}(C_G(Q))$. By Lemma 3.4, we have $Exp(|G|, 31) = 9$. By the Frattini argument, we can obtain that $N = CN_N(K_{31})$, and hence $p \nmid |N_N(K_{31})|$. Thus $p \big| |C_G(Q)|$, and so $deg(p) \geq 3$, which is a contradiction. Therefore, $31 \nmid |N_G(Q)|$ and $31 \in \pi(K)$. Set $P_{31} \in Syl_{31}(K)$. Then $P_{31} \in Syl_{31}(G)$. Since $G = KN_G(P_{31})$, then $p \big| |N_G(P_{31})|$. By Table 1, we see that this is impossible.

Case 1.3. Till now we have proved that K is a q'-group while $p = 127$ or 307. Assume that p is equal to one of the remaining primes. Now, we have to discuss ten cases up choice of p one by one. If K is a q-group for each $q \in \pi(G) \backslash \{2,3,p\}$, we can prove that this is impossible by checking each choice of p. Since the methods used below is completely the same as in Case 2, hence, we omitted the detailed processes. Therefore K is a $\{2,3\}$-group. Since $K \neq G$, it follows at once that G is a nonsolvable group, which completes the proof of Case 1.3 and also completes the proof of Case 1.

Case 2. G/K is an almost simple group. In other words, there exists a nonabelian simple group S such that $S \lesssim G/K \lesssim Aut(S)$.

Proof. Let $\overline{G} := G/K$ and $S := Soc(\overline{G})$. Then $S = P_1 \times P_2 \times \cdots \times P_s$, where P_i $(i = 1, 2, ..., s)$ are nonabelian simple groups and $S \lesssim \overline{G} \lesssim Aut(S)$. We will show that $s = 1$, and hence $S = P_1$.

Suppose that $s \geq 2$. We assert that p does not divide the order of S. Otherwise, there exists a prime r such that $r \sim p$, where $5 \leq r < p$ and $r \in \pi(G)$, an obvious contradiction to Lemma 3.4 (iii). Hence, for every i we have $P_i \in \mathcal{F}_p$ (cf.[21]). By Lemma 3.7, we observe that $p \in \pi(\overline{G}) \subseteq \pi(Aut(S))$. Thus $p\big||Out(S)|$. But $Out(S) = Out(S_1) \times Out(S_2) \times \cdots \times Out(S_r)$, where the groups S_j are direct products of all isomorphic $P_i^{'}s$ such that $S = S_1 \times S_2 \times \cdots \times S_r$. Therefore, for some j, p divides the order of an outer automorphism group of a direct product S_j of t isomorphic simple groups P_i for some $1 \leq i \leq s$. Since $P_i \in \mathcal{F}_p$, it follows that $|Out(P_i)|$ is not divided by p by Lemma 3.7. Now, by Lemma 3.8, we obtain that $|Aut(S_j)| = |Aut(P_i)|^t \cdot t!$. Therefore, $t \geq p$ and so 2^{2p} must divide the order of G. However, $Exp(|S_{p+3}|, 2) \leq p+3 < 2p$ by Lemma 3.4 (iv), which is a contradiction. Thus $s = 1$ and $S = P_1$. This completes the proof of Case 2.

Case 3 $G \cong A_{p+3}$. In other words, A_{p+3} is OD-characterizable.

Proof. By Lemma 3.7 and Case 1, we may assume that $|S| = \dfrac{|G|}{2^{k_1} \cdot 3^{k_2}} \cdot 2^{\beta_1} \cdot 3^{\beta_2}$, where $2 \leq \beta_1 \leq Exp(|A_{p+3}|, 2) = k_1$, $1 \leq \beta_2 \leq Exp\,(|A_{p+3}|, 3) = k_2$. Let $p_1, p_2, p_3, ..., p_s$ be distinct consecutive primes and $2 = p_1 < p_2 < p_3 < \cdots < p = p_s$, then $|G|_{p_j} = Exp\,(|A_{p+3}|, p_j)$ for any $j \geq 3$. Using Table 1-3 in [21], we deduce that S can only be isomorphic to one of the simple groups: A_p, A_{p+1}, A_{p+2} and A_{p+3}.

If $S \cong A_p$, then K is a 2-group. In this case, it is easy to see that $3p \in \pi_e(\overline{G}) \backslash \pi_e(S_p)$, a contradiction.

By the similar reason as above, $S \ncong A_{p+1}$ or A_{p+2}. Therefore, $S \cong A_{p+3}$. According to the consequence of Case 2, one has that

$$A_{p+3} \lesssim G/K \lesssim Aut(A_{p+3}) \cong S_{p+3}.$$

If $G/K \cong S_{p+3}$, then $2^{Exp(|S_{p+3}|,2)}\big||G|$, a contradiction.

If $G/K \cong A_{p+3}$, then $|K| = 1$. Therefore $G \cong A_{p+3}$. This completes the proof of Case 3 and also the proof of Main Theorem. ∎

In 1989, in [2] Professor Shi, W. J. put forward the following conjecture.

Conjecture 1 [2] *Let G be a group and M a finite simple group. Then $G \cong M$ if and only if*

(1) $|G| = |M|$, *and*

(2) $\pi_e(G) = \pi_e(M)$.

The above Conjecture 1 was proved by joint works of many mathematicians, the last part of the proof was given by V.D. Mozurov et al. in [23]. That is, the following theorem holds.

Theorem 4.1 [23] *Let G be a group and M a finite simple group. Then $G \cong M$ if and only if*

(1) $|G| = |M|$, *and*

(2) $\pi_e(G) = \pi_e(M)$.

In fact, Theorem 4.1 is valid for alternating group A_{p+3} since $\pi_e(G) = \pi_e(A_{p+3})$ implies G and A_{p+3} have the same prime graph. Thus they have the same degree pattern. Therefore, we can get the following corollary.

Corollary 4.2 *If G is a finite group such that*

(1) $|G| = |A_{p+3}|$,*and*

(2) $\pi_e(G) = \pi_e(A_{p+3})$, *where* $7 \neq p \in \pi(1000!)$,

then $G \cong A_{p+3}$.

Acknowledgements. This paper was supported by the Natural Science Foundation of China (Grant Nos. 11171364; 11271301; 11471266); by "the Fundamental Research Funds for the Central Universities" (Grant Nos. XDJK2014C163; XDJK2014C162); by the Natural Science Foundation Project of CQ CSTC (Grant No. cstc2014jcyjA00010); by Chongqing Postdoctoral Science Foundation) (Grant No. Xm2014029) and China Postdoctoral Science Foundation) (Grant No. 2014M1453).

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Accepted: 20.09.2014