PRIME SUBMODULES IN EXTENDED $BCK$-MODULE

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Abstract. In this paper, by considering the notion of $BCK$-module, we define the concept of extended $BCK$-module which is a generalization of $BCK$-module and we state and prove some related results. Specially, we define the notions of prime submodule and torsion free module and we investigate some important results. Finally, we define the concept of radical of any submodule in extended $BCK$-modules and we characterize the elements of it.

Keywords: $BCK$-algebra, $BCK$-module, extended $BCK$-module, prime submodule of $BCK$-module.

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1. Introduction

The notion of $BCK$-algebra was formulated first in 1966 by Imai and Iseki. This notion is originated from two different ways. One of the motivations is based on set theory. Another motivation is from classical and non-classical propositional calculus. The notion of $BCK$-module was introduced in 1994 [2] as an action of a $BCK$-algebra over a commutative group by M. Aslam, A.B. Thaheem and H.A.S. Abujaabal. The idea was further explored in 1994 by F. Kôpka and F. Chovanec [8]. The concept of $BCK$-module was extended by R. A. Borzooei, J. Shohane and M. Jafari in 2011 [4]. Now, we introduce a different extended $BCK$-module that we can obtain some interesting results by it. Since the notion of prime-submodule is fundamental notion in modules theory, in this paper we introduce and investigate it on $BCK$-modules and we obtain some results as mentioned in the abstract.
2. Preliminaries

Definition 2.1. [9] A BCK-algebra is a structure $X = (X, *, 0)$ of type $(2, 0)$ such that:

- $(BCK1)$ \[ ((x * y) * (x * z)) * (z * y) = 0, \]
- $(BCK2)$ \[ (x * (x * y)) * y = 0, \]
- $(BCK3)$ \[ x * x = 0, \]
- $(BCK4)$ \[ 0 * x = 0, \]
- $(BCK5)$ \[ x * y = y * x = 0 \text{ implies that } x = y, \text{ for all } x, y, z \in X. \]

The relation $x \leq y$ which is defined by $x * y = 0$ is a partial order with 0 as least element. In any BCK-algebra $X$, for all $x, y, z \in X$, we have

- $(BCK6)$ \[ x * y \leq x, \ (x * y) * z = (x * z) * y. \]

Definition 2.2. [9] Let $(X, *, 0)$ be a BCK-algebra. Then

- (i) $\emptyset \neq X_0 \subseteq X$ is called to be a subalgebra of $X$, if for any $x, y \in X_0$, $x * y \in X_0$,
- (ii) $\emptyset \neq I \subseteq X$ is called an ideal of $X$, if $0 \in I$ and for any $x, y \in X$, $x * y \in I$ and $y \in I$, implies that $x \in I$. Specially, generated ideal by $x$ is defined by $(x) = \{ y \in X : y * x = 0 \}$, for any $x \in X$,
- (iii) $X$ is called bounded, if there exists $1 \in X$ such that $x \leq 1$, for any $x \in X$. In this case, we set $N x = 1 * x$,
- (iv) $X$ is said to be commutative, if $y * (y * x) = x * (x * y)$, for all $x, y \in X$,
- (v) proper ideal $I$ of $X$, is called prime ideal if $X$ is commutative and $a \wedge b \in I$ implies that $a \in I$ or $b \in I$, for any $a, b \in X$,
- (iv) $X$ is said to be implicative if $x * (y * x) = x$, for all $x, y \in X$.

Note. In a BCK-algebra $X$, we let $x \wedge y = y * (y * x)$ and in a bounded BCK-algebra $X$, we let $x \vee y = N (Nx \wedge Ny)$, for all $x, y \in X$. Moreover, in bounded commutative BCK-algebra, $x \wedge y$ is the least upper bound and $x \vee y$ is the greatest lower bound of $x, y$, for any $x, y \in X$ and so $(L, \vee, \wedge)$ is a bounded lattice.

Lemma 2.3. [9] Let $X$ be a bounded implicative BCK-algebra. Then for all $x, y, z \in X$,

- (i) \[ x \wedge y = x * Ny, \]
- (ii) \[ x * (x \wedge y) = x * y, \]
- (iii) \[ x \wedge (y * z) = (x \wedge y) * (x \wedge z), \]
- (iv) \[ (x * y) + (y * x) = x + y, \text{ where } x + y = (x * y) \vee (y * x), \]
- (v) \[ (x + y) \wedge z = (x \wedge z) + (y \wedge z), \]
- (vi) \[ x + x = 0 \text{ and so } x = -x, \]
- (vii) \[ x + 0 = 0 + x = x. \]
Let $A$ be an ideal of $BCK$-algebra $X$. For any $x, y \in X$, we define $x \sim y$ if and only if $x \ast y \in A$ and $y \ast x \in A$. So $\sim$ is an equivalence relation on $X$. Denote the equivalence class containing $x$ by $C_x$ and $\frac{X}{A} = \{C_x : x \in X\}$. Then $(\frac{X}{A}, \ast, C_0)$ is a $BCK$-algebra (quotient $BCK$-algebra), where $C_x \ast C_y = C_{x \ast y}$, for all $x, y \in X$. Moreover, the relation $\leq$ which is defined by, $C_x \leq C_y$ if and only if $x \ast y \in A$, is a partial order relation. If $X$ is bounded and commutative, then $\frac{X}{A}$ is bounded and commutative, too. Let $(X, \ast, 0)$ and $(Y, \ast', 0)$ be two $BCK$-algebras. A mapping $f : X \rightarrow Y$ is called a homomorphism if $f(0) = 0$ and $f(x \ast y) = f(x) \ast' f(y)$, for any $x, y \in X$ (see [9]).

**Definition 2.4.** [1] Let $X$ be a $BCK$-algebra, $M$ be an abelian group under $"+"$ and $(x, m) \rightarrow x.m$ be a mapping of $X \times M \rightarrow M$ such that,

\begin{align*}
(XM1) & \quad (x \land y).m = x.(y.m), \\
(XM2) & \quad x.(m + n) = x.m + x.n, \\
(XM3) & \quad 0.m = 0, \text{ for all } x, y \in X \text{ and } m, n \in M.
\end{align*}

Then $M$ is called a $BCK$-module or briefly $X$-module. If $X$ is bounded and for any $m \in M$, $1.m = m$, then $M$ is called a unitary $X$-module.

**Definition 2.5.** [1] A map $f : M \rightarrow N$, where $M$ and $N$ are $X$-modules, is an $X$-homomorphism if the following hold:

\begin{enumerate}
\item[(i)] $f(m + n) = f(m) + f(n)$, for all $m, n \in M$,
\item[(ii)] $f(x.m) = x.f(m)$, for all $m \in M$ and $x \in X$.
\end{enumerate}

**Proposition 2.6.** [3] Let $M$ and $N$ be two $BCK$-modules over commutative $BCK$-algebra $X$ and $\text{Hom}(M, N) = \{f : f \text{ is a homomorphism from } M \text{ into } N\}$. Then $(\text{Hom}(M, N), +)$ forms an abelian group where $(f + g)(m) = f(m) + g(m)$, for any $f, g \in \text{Hom}(M, N)$ and $m \in M$. Moreover by operation $\bullet : X \times \text{Hom}(M, N) \rightarrow \text{Hom}(M, N), \text{Hom}(M, N)$ is an $X$-module, where $x \bullet f(m) = x.f(m)$.

**Theorem 2.7.** [4] Let $X$ be a bounded implicative $BCK$-algebra. Then $(X, +)$, is an abelian group and $X$ is an $X$-module, where $x + y = (x \ast y) \lor (y \ast x)$.

**Note.** From now on, in this paper $X$ is a $BCK$-algebra and $M$ is an abelian group.

3. Extended $BCK$-Modules

**Definition 3.8.** Let operation $\cdot : X \times M \rightarrow M$ satisfies the following axioms:

\begin{align*}
(XM1) & \quad (x \land y).m = x.(y.m), \\
(XM2) & \quad x.(m + n) = x.m + x.n, \\
(XM3) & \quad 0.m = 0, \\
(XM4) & \quad (x \ast y).m = x.m - y.m, \text{ where } x \ast y \neq 0, \text{ for } x \neq y,
\end{align*}
for all \(x, y \in X\) and \(m, n \in M\). Then \(M\) is called an extended BCK-module or briefly \(X^E\)-module. If \(X\) is bounded and \(1.m = m\), for any \(m \in M\), then \(M\) is called a unitary \(X^E\)-module.

**Example 3.9.** Let \(X\) be a bounded implicative BCK-algebra such that "\(\leq\)" is totally ordered and operations "\(+, \cdot\)"; \(X \times X \rightarrow X\) are defined by, \(x + y = (x * y) \lor (y * x)\), \(x.y = x \land y\), for all \(x, y \in X\). Then \(X\) is an \(X^E\)-module. By Theorem 2.7, it is enough to show that \((x + y).z = x.z - y.z\), for any \(x, y, z \in X\), where \(x \neq y\) for \(x \neq y\). If \(x = y\), then the proof is clear. Now, let \(x \neq y\), for \(x \neq y\). Since \(x \neq y\), \(x \neq y\) and so \(y \leq x\) and this means that \(y * x = 0\). Therefore,

\[
(x * y).z = (x * y) \land z,
\]

\[
= (x * y + 0) \land z, \text{ by Lemma 2.3(vii),}
\]

\[
= (x * y + y * x) \land z, \text{ since } y * x = 0,
\]

\[
= (x + y) \land z, \text{ by Lemma 2.3(iv),}
\]

\[
= (x \land z) + (y \land z), \text{ by Lemma 2.3(v),}
\]

\[
= x.z + y.z,
\]

\[
= x.z - y.z, \text{ by Lemma 2.3(vi).}
\]

**Example 3.10.** (i) Let \(X\) be a bounded commutative BCK-algebra such that \((X, \cdot)\) be an \(X^E\)-module and \(A\) be an ideal of \(X\). Then \(\frac{X}{A}, +', \cdot'\) is an abelian group, where \(C_x +' C_y = C_{x+y}\) and \(x + y = x + y \lor y', x \in X\). Moreover, if operation \(\bullet : X \times X \rightarrow X\) is defined by \(x \bullet C_y = C_{x.y}\), for any \(x, y \in X\), then \(\frac{X}{A}\) is an \(X^E\)-module.

(ii) Let \(X = \{0, x\}\) and operation "\(\ast\)" on \(X\) is defined by \(0 \ast x = 0 \ast 0 = 0 \ast x = 0\) and \(x \ast 0 = x\). Then \((X, \ast, 0)\) is a BCK-algebra. Now, let operation \(\cdot : X \times Z \rightarrow Z\) is defined by \(x.n = n\) and \(0.n = 0, \) for any \(n \in Z\). We claim that \(Z\) is an \(X^E\)-module. It is clear that \((x \land 0).n = 0 = 0.x, (0.x) = 0 = 0\) and so \((x \land 0).n = x.(0.n)\). Similarly \((x \land x).n = x.(x.n)\) and \((0 \land x).n = 0.(x.n)\). Then \((X M1)\) holds. The proof of \((X M2)\) and \((X M3)\) is clear. Moreover, since \((x \ast 0).n = n = x.n - 0.n\) and \((x \ast x).n = 0 = x.n - x.n\), \((X M4)\) holds.

(iii) It is easy to see that BCK-algebra \((X, \ast, 0)\) in (ii) is bounded with unit \(x\). Moreover, \((X, +)\) is an abelian group, where \(a + b = (a * b) \lor (b * a)\), for any \(a, b \in X\). Now, let operation \(\cdot : X \times X \rightarrow X\) is defined by \(a.b = a \land b\), for any \(a, b \in X\). Then \((X, +)\) is an \(X^E\)-module.

(iv) Let \(X = \{0, a, b, 1\}\) and operation "\(\ast\)" on \(X\) is defined by
Then \((X, *, 0)\) is a bounded \(BCK\)-algebra. Let \(M = \{0, a\} \subseteq X\). Then \((M, +)\) is an abelian group, where \(x + y = (x * y) \lor (y * x)\), for any \(x, y \in M\). We define the operation \(\cdot : X \times M \to M\) by
\[
x \cdot y = \begin{cases} a, & \text{if } x = b \text{ or } 1 \text{ and } y = a \\ 0, & \text{otherwise} \end{cases}
\]
Then \(M\) is an \(X^E\)-module.

(v) Let \((X, *, 0)\) be a bounded \(BCK\)-algebra with unit 1, \(1 \neq a \in X\) and \(1 * a = 1 \text{ or } a\). Now, if \(Y = \{0, a, 1\}\), then \(Y\) is a subalgebra of \(X\) and so it is a \(BCK\)-algebra. Moreover, let \(M = \{0, 1\} \subseteq X\). Then \((M, +)\) is an abelian group, where \(x + y = (x * y) \lor (y * x)\), for any \(x, y \in M\). Now, let the operation \(\cdot : Y \times M \to M\) is defined by \(y \cdot m = y \land m\) for any \(y \in Y\) and \(m \in M\). Then \(M\) is a \(Y^E\)-module.

(vi) Let \(X = \{0, 1, 2, 3, 4\}\) and the operation “*” is defined by
\[
\begin{array}{c|ccccc}
  * & 0 & 1 & 2 & 3 & 4 \\
\hline
  0 & 0 & 0 & 0 & 0 & 0 \\
  1 & 1 & 0 & 0 & 0 & 0 \\
  2 & 2 & 2 & 0 & 2 & 0 \\
  3 & 3 & 3 & 3 & 0 & 0 \\
  4 & 4 & 4 & 3 & 2 & 0 \\
\end{array}
\]
Then \((X, *, 0)\) is a bounded \(BCK\)-algebra. Let \(Y = \{0, 1, 4\}\) and \(M = \{0, 2, 3, 4\}\). It is clear that \(Y\) is a subalgebra of \(X\) and so is \(BCK\)-algebra. It is easy to show that \((M, +)\) is an abelian group, where \(x + y = (x * y) \lor (y * x)\), for any \(x, y \in M\). Now, we define the operation \(\cdot : Y \times M \to M\) by \(y \cdot m = y \land m\), for any \(y \in Y\) and \(m \in M\). Then \(M\) is a \(Y^E\)-module.

(vii) Let \(X = \{P, \{2\}, \{1, 2\}\}\) be a subset of \(BCK\)-algebra [7, Example 2.8]. Then it is easy to see that \((X, \odot, P)\) is a \(BCK\)-algebra. If operation \(\cdot : X \times \mathbb{Z} \to \mathbb{Z}\) is defined by \(\{2\}.n = n\) and \(\{1, 2\}.n = P.n = 0\), for any \(n \in \mathbb{Z}\), then \(\mathbb{Z}\) is an \(X^E\)-module.

**Theorem 3.11.** Every \(X^E\)-module is an \(X\)-module.

**Proof.** The proof is clear.

**Example 3.12.** Let \(X\) be a nonempty set. Then \((\mathcal{P}(X), -)\) is a bounded implicative \(BCK\)-algebra and \(\mathbb{Z}\) is a \(\mathcal{P}(X)\)-module with operation \(\cdot : \mathcal{P}(X) \times \mathbb{Z} \to \mathbb{Z}\) such that \(A.n = \mu(A)n\), for any \(A \subseteq X\), where for \(a \in X\),
\[
\mu(A) = \begin{cases} 0, \text{ if } a \notin A \\ 1, \text{ if } a \in A \end{cases}
\]
But $Z$ is not a $\mathcal{P}(X)^E$-module. Since for $A, B \in \mathcal{P}(X)$ such that $a \notin A, a \in B$, we have

$$(A - B).n = \mu(A - B)n = 0 \neq -n = 0 - n = A.n - B.n$$

and so $(XM4)$ is not true.

**Definition 3.13.** A map $f : M \to N$, where $M$ and $N$ are $X^E$-modules, is called an $X^E$-homomorphism, if the following hold:

(i) $f(m + n) = f(m) + f(n)$,

(ii) $f(x.m) = x.f(m)$, for all $m, n \in M$ and $x \in X$.

**Proposition 3.14.** Let $M, N$ be two $X^E$-modules and

$$\text{Hom}(M, N) = \{ f : f : M \to N \text{ is an } X^E - \text{homomorphism} \}.$$ 

Then $(\text{Hom}(M, N))$ is an $X^E$-module by the operation which is defined in Proposition 2.6.

**Proof.** By Proposition 2.6, it is enough to show that $(x \ast y) \bullet f(m) = x \bullet f(m) - y \bullet f(m)$, for any $x, y \in X$, where $x \ast y \neq 0$, and $x \neq y$. Now, since $N$ is an $X^E$-module, we have

$$(x \ast y) \bullet f(m) = (x \ast y).f(m) = x.f(m) - y.f(m) = x \bullet f(m) - y \bullet f(m).$$

**Theorem 3.15.** Let $X$ be a bounded implicative $BCK$-algebra. Then, by the assumption of Example 3.9, \(\left( \sum_{i \in I} X, +' \right)\) is an abelian group, where \(\{x_i\}_{i \in I}'\) \(\{y_i\}_{i \in I} \subseteq \sum_{i \in I} X\). Moreover, if the operation

\[ x.\left( \sum_{i \in I} X \right) \to \sum_{i \in I} X \]

is defined by $x.\{x_i\} = \{x \land x_i\}$, for any $x, x_i \in X$, $i \in \mathbb{N}$, then $\sum_{i \in I} X$ is an $X^E$-module.

**Proof.** Since by Theorem 2.7, $(X, +)$ is an abelian group, then it is clear that $\left( \sum_{i \in I} X, +' \right)$ is an abelian group.

Now, for any $x, y, x_i, y_i \in X$ and $i \in \mathbb{N}$, we have:

$(XM1)$: $x \land y).\{x_i\} = \{x \land y \land x_i\} = \{x \land (y \land x_i)\} = x.\{y \land x_i\} = x.(y.\{x_i\})$.

$(XM2)$: By Lemma 2.3(v),

\[
\begin{align*}
  x.\{x_i\} +' \{y_i\} &= x.\{x_i + y_i\} = x \land (x_i + y_i) = x \land x_i + x \land y_i \\
  &= \{x \land x_i\} +' \{x \land y_i\} = x.\{x_i\} +' x.\{y_i\}.
\end{align*}
\]
(XM3): $0.\{x_i\} = \{0 \land x_i\} = \{0\}$.

(XM4): Let $x \ast y \neq 0$ for $x \neq y$. Then by Lemma 2.3(v) and (vi),

\[
\begin{align*}
x \cdot \{x_i\} -' y \cdot \{x_i\} &= \{x \land x_i\} -' \{y \land x_i\} \\
&= \{x \land x_i\} +' \{y \land x_i\} \\
&= \{(x + y) \land x_i\} \\
&= (x + y).\{x_i\} \\
&= (x \ast y + y \ast x).\{x_i\} \\
&= (x \ast y + 0).\{x_i\} = (x \ast y).\{x_i\}.
\end{align*}
\]

\[\text{--- End of previous proof.} \]

**Theorem 3.16.** Let $X$ be BCK-algebra in Theorem 3.15, and $A$ be an ideal in $X$. Then $\left(\sum_{i \in I} \frac{X}{A}, \cdot'\right)$ is an abelian group, where $\{C_{x_i}\} = \{C_{x_i+y_i}\}$ and $x_i + y_i = x_i \ast y_i \lor y_i \ast x_i$, for any $x_i, y_i \in X$ and $i \in I$. Moreover, if we define $\cdot : X \times \sum_{i \in I} \frac{X}{A} \rightarrow \sum_{i \in I} \frac{X}{A}$ by $x \cdot \{C_{x_i}\} = \{C_{x\land x_i}\}$, then $\left(\sum_{i \in I} \frac{X}{A}\right)$ is an $X^E$-module.

**Proof.** It is easy to show that $\left(\sum_{i \in I} \frac{X}{A}, \cdot'\right)$ is an abelian group and $\sum_{i \in I} \frac{X}{A}$ is an $X^E$-module.

\[\text{--- End of previous proof.} \]

**Theorem 3.17.** Let $(X, \ast)$ and $(Y, \ast)$ be two BCK-algebras, $M$ be a $Y^E$-module and $\Phi : X \rightarrow Y$ be a BCK-homomorphism such that $x \neq 0$ implies that $\phi(x) \neq 0$, for any $x \in X$. If operation $\cdot : X \times M \rightarrow M$ is defined by $x \cdot m = \phi(x).m$, for any $x \in X$ and $m \in M$, then $M$ is an $X^E$-module.

**Proof.** Let $M$ be a $Y^E$-module and $\Phi : X \rightarrow Y$ be a BCK-homomorphism such that $x \neq 0$ implies that $\phi(x) \neq 0$, for any $x \in X$. Then for any $x, y \in X$ and $m, n \in M$, we have:

(XM1)$X$: By (XM1)$_Y$, we have

\[
(x \land y) \cdot m = \phi(x \land y).m = \phi(y \ast (y \ast x)).m = (\phi(y) \ast (\phi(y) \ast \phi(x))).m,
\]

\[
= (\phi(x) \land \phi(y)).m = \phi(x).(\phi(y)m) = x \cdot (y \cdot m).
\]

(XM2)$X$: By (XM2)$_Y$, we have

\[
x \cdot (m + n) = \phi(x).(m + n) = \phi(x)m + \phi(x)n = x \cdot m + x \cdot n
\]

(XM3)$_X$: $0 \cdot m = \phi(0).m = 0.m = 0$

(XM4)$_X$: By (XM4)$_Y$, where $x \ast y \neq 0$, for $x \neq y$ we have

\[
(x \ast y) \cdot m = \phi(x \ast y).m = (\phi(x) \ast \phi(y)).m = \phi(x).m - \phi(y).m = x \cdot m - y \cdot m.
\]
Lemma 3.3. Let $\mathfrak{m}$ be a submodule of $M$. Then $\frac{\mathfrak{m}}{N}$ is an $X^E$-module.

Proof. Let $X$ be a bounded commutative $BCK$-algebra, $(X, +)$ be an $X^E$-module and $A$ be an ideal of $X$. It is easy to show that $(\frac{X}{A}, +^{'})$ is an abelian group, where $C_x +^{' y} = C_{x+y}$ and $x + y = (x * y) \lor (y * x)$, for any $x, y \in X$. Let operation $\bullet: X \times \frac{X}{A} \rightarrow \frac{X}{A}$ is defined by $x \bullet C_y = C_{x,y}$, for any $x, y \in X$. Then we show that $\frac{X}{A}$ is an $X^E$-module. For $x, x', y \in X$,

\[(XM1) \frac{x \bullet (x' \bullet C_y)}{A} = x \bullet (x' \bullet C_y) \]

\[(XM2) \frac{x \bullet C_x +^{' y}}{A} = x \bullet C_{x \lor y'} = C_{x \lor y'} \]

\[(XM3) \frac{0 \bullet C_x}{A} = C_0 \]

\[(XM4) \frac{(x * y) \bullet C_{y'}}{A} = C_{x \lor y'} \]

3. Prime submodules in $X^E$-modules

Definition 3.1. A subgroup $N$ of $X^E$-module $M$ is a submodule of $M$ if for any $x \in X$ and any $n \in N$, $x \cdot n \in N$.

Example 3.2. (i) By considering the Example 3.10 (ii), $2\mathbb{Z}$ is a submodule of $\mathbb{Z}$.

(ii) Let $X$ be a bounded implicative $BCK$-algebra with the assumption of Example 3.9, and $M_r = \{x \in X: x \leq r\}$, where $r \in X$. Then $M_r$ is a submodule of $X$. First we show that $M$ is a subgroup of $X$. Let $m, n \in M_r$. By assumption, $m * n = 0$ or $n * m = 0$. W.L.G, $n * m = 0$. Hence, by Lemma 2.3(vi), $m - n = m + n = (m * n) \lor (n * m) = (m * n) \lor 0 = m * n$. On the other hand by $(BCK6)$, $m * n \leq m$ and $m \leq r$. Hence $m \leq n \leq r$ and so $m - n \in M_r$. It means that $M_r$ is a subgroup of $X$. Now, we will show that $x \cdot m \in M_r$, for any $x \in X$ and $m \in M_r$. By $(BCK4)$ and $(BCK6)$, we have

\[(x \cdot m) \cdot r = (x \land m) \cdot r = (m \cdot (m \land x)) \cdot r = (m \cdot r) \cdot (m \cdot x) = 0 \cdot (m \land r) = 0 \]

Hence, $x \cdot m \leq r$ and so $x \cdot m \in M_r$. Therefore, $M_r$ is a submodule of $X$.

Lemma 3.3. Let $M$ be an $X^E$-module and $N$ be a submodule of $M$. Then $\frac{M}{N}$ is an $X^E$-module.
Proof. Let $N$ be a submodule of $M$ and operation $\bullet : X \times \frac{M}{N} \to \frac{M}{N}$ is defined by $x \bullet (m+N) = x.m+N$, for any $x \in X$ and $m \in M$. Let $x = y$ and $m+N = m'+N$. Then $m - m' \in N$. Since $N$ is a submodule of $M$, $x.(m - m') = x.m - x.m' \in N$ and so $x \bullet m + N = x \bullet m' + N$. It means that "$\bullet$" is well defined. For any $x, y \in X$ and $m, m' \in M$,

\[(XM1)\text{ by (XM1),} \]
\[(x \land y) \bullet (m+N) = (x \land y).m+N = x \bullet (y.m+N) = x \bullet (y \bullet (m+N)) \]

\[(XM2)\text{ by (XM2),} \]
\[x \bullet (m + N + m' + N) = x.(m + m') + N = x.m + x.m' + N \]
\[= x.m + N + x.m' + N = x \bullet (m + N) + x \bullet (m' + N) \]

\[(XM3)\text{ by (XM3),} \]
\[0 \bullet (m+N) = 0.m+N = N \]

\[(XM4)\text{ by (XM4),} \]
\[x \ast y \neq 0, \text{ for } x \neq y. \text{ By (XM4),} \]
\[(x \ast y) \bullet (m+N) = (x \ast y).m+N = (x.m - y.m) + N = x.m + N - y.m + N \]
\[= x \bullet (m+N) - y \bullet (m+N). \]

Theorem 3.4. Let $M, M'$ be $X^E$-modules, $\phi : M \to M'$ be an $X^E$-homomorphism and $N$ be a submodule of $M$ such that $\phi(N) = 0$. Then there exists an $X^E$-homomorphism from $M/N$ to $M'$.

Proof. We define $\tilde{\phi} : \frac{M}{N} \to M'$ by $\tilde{\phi}(m+N) = \phi(m)$. It is easy to show that $\tilde{\phi}$ is well defined and it is an $X^E$-homomorphism.

Theorem 3.5. Let $M, M'$ be $X^E$-modules and $\phi : M \to M'$ be an $X^E$-homomorphism. Then

(i) $\text{Ker } \phi$ and $\text{Img } \phi$ are submodules of $M$ and $M'$, respectively,

(ii) $\frac{M}{\text{Ker } \phi} \cong \text{Img } \phi$.

Proof. (i) The proof is clear.

(ii) We know that, $\phi : M \to \text{Img } \phi$ is an epimorphism. Now, in Theorem 3.4, it is enough to consider $N = \text{Ker } \phi$.

Theorem 3.6. Let $M$ be an $X^E$-module and $N, K$ are submodules of $M$. Then

(i) $N + K = \{n + k : n \in N, k \in K\}$ and $N \cap K$ are $X^E$-modules,

(ii) $\frac{K}{N \cap K} \cong \frac{N + K}{N}$.
(iii) $\frac{K}{N}$ is a submodule of $\frac{M}{N}$ and $\frac{M}{N} \simeq \frac{M}{K}$, where $N \subseteq K$.

**Proof.** (i) $N + K$ is an $X$-module (see [3]). Now, let $x \neq 0$, where $x \neq y$, for any $x, y \in X$ and $n + k \in N + K$. Then,

$$(x \cdot y)(n + k) = (x \cdot y)n + (x \cdot y)k = x.n - y.n + x.k - y.k = x.(n + k) - y.(n + k)$$

and so we have $(X4)_{N+K}$. Therefore, $N + K$ is an $X^E$-module. Moreover, $N \cap K$ is an $X$-module (see [3]) and it is not difficult to verify the condition $(X4)_{N \cap K}$. Hence $N \cap K$ is an $X^E$-module, too.

(ii), (iii) The proofs are easy. ●

**Theorem 3.7.** Let $X$ be a bounded commutative $BCK$-algebra such that $x \neq y$ for $x, y \neq 0$, $M$ be an $X^E$-module and $K$ be a proper submodule of $M$. Then $(K : M) = \{x \in X : x.M \subseteq K\}$ is a prime ideal of $X$.

**Proof.** First, we show that $(K : M)$ is an ideal of $X$. If $(K : M) = X$, then $1.M \subseteq K$ and so $M \subseteq K$, which is a contradiction. Since $K$ is a subgroup of $M$, $0.m = 0 \in K$, for any $m \in M$ and so $0 \in (K : M)$. Now, for any $x, y \in X$, let $x \cdot y \in (K : M) = \{x \in X : x.M \subseteq K\}$ and $y \in (K : M)$. Then $(x \cdot y).m \in K$ and $y.m \in K$, for any $m \in M$. If $x = y$ or $x = 0$, then it is clear that $x.m \in M$. So let $x \neq y$. If $x \neq y = 0$, then $x \cdot (x \cdot y) = x$ and so $x \cdot y = x$. Since $y.m \in K$, $(x \cdot y).m \in K$, for any $m \in M$ and $K$ is a submodule of $M$ and so $x.m = (x \cdot y).m \in K$. If $x \neq y$, then by $(X4)$, $x.m - y.m = (x \cdot y)m \in K$. Since $(K, +)$ is a subgroup of $M$ and $y.m \in K$, we have $x.m \in K$. Therefore, $(K : M)$ is an ideal of $X$.

Now, we prove that $(K : M)$ is prime. Let $x \cdot y \in (K : M)$, for $x, y \in X$. Then for any $m \in M$, $(x \cdot y).m \in K$ and so $(y \cdot (y \cdot x)).m \in K$. Now if $x = y$, then $x.m = (x \cdot 0).m = (x \cdot (x \cdot x)).m = (x \cdot y).m \in K$. If $x = 0$ or $y = 0$, then it is clear that $x \in (K : M)$ or $y \in (K : M)$. If $x \neq y$, $x, y \neq 0$ and $x \cdot y = 0$, then $x.m = (x \cdot 0).m = (x \cdot (x \cdot y)).m = (y \cdot x).m = (x \cdot y).m \in K$, for any $m \in M$. If $x \neq y$, $x, y \neq 0$, $x \cdot y \neq 0$, $x \cdot x \cdot y$ and $x \cdot (x \cdot y) \neq 0$. Then by $(X4)$, $y.m = x.m - (x.m - y.m) = x.m - (x \cdot y).m = x \cdot (x \cdot y).m = (y \cdot x).m = (x \cdot y).m \in K$, for any $m \in M$. Finally, if $x \neq y$, $x, y \neq 0$, $x \cdot y \neq 0$, $x \neq x \cdot y$ and $x \cdot (x \cdot y) = 0$, then by $(BCK6)$, we have $(x \cdot y) \cdot x = 0$ and so $x = x \cdot y$, which is a contradiction. Therefore, $(K : M)$ is a prime ideal of $X$. ●

**Proposition 3.8.** Let $M$ be an $X^E$-module. If for any $x, y \in X$, $x \neq y$ implies that $x \cdot y \neq 0$, then $Ann_X(M) = \{x \in X : x.m = 0, \forall m \in M\}$ is an ideal of $X$.

**Proof.** It is clear that $0 \in Ann_X(M)$. Now, let $x \cdot y, y \in Ann_X(M)$ and $x \neq y$, for any $x, y \in X$. If $x = 0$, then it is clear that $x \in Ann_X(M)$. Now, let $x \neq 0$. Then by $(X4)$, $x.m = x.m - 0 = x.m - y.m = (x \cdot y).m = 0$, for any $m \in M$, and so $x \in Ann_X(M)$. Therefore, $Ann_X(M)$ is an ideal of $X$. ●

**Theorem 3.9.** Let $M$ be an $X^E$-module and $I$ be an ideal of $X$ such that $I \subseteq Ann_X(M)$. If the operation $\bullet : X/I \times M \longrightarrow M$ is defined by $c_x \bullet m = x.m$, for any $x, y \in X$ and $m \in M$, then $M$ is an $(X/I)^E$-module.
Proof. Let $\bullet : X/I \times M \longrightarrow M$ is defined by $c_x \bullet m = x.m$, for any $x \in X$ and $m \in M$. First we prove that $\bullet$ is well defined. Let $c_x = c_y$, $m = n$ and $x \neq y$, for all $x, y \in X$ and $m, n \in M$. Then $x * y \in I$ and $y \star x \in I$. If $x * y = y * x = 0$, then by $(BC'K5)$, $x = y$, which is a contradiction. If $x * y \neq 0$ or $y \star x \neq 0$, since $I \subseteq \text{Ann}_X(M)$, by $(XM4)$, $0 = (x \star y).m = x.m - y.m$ or $0 = (y \star x).m = y.m - x.m$ and so $x.m = y.m$. Now, since $m = n$, $x.m = y.n$. Hence, $\bullet$ is well defined. Now, we will show that $M$ is an $(X/I)^E$-module.

$(XM1)_{X/I}$: We have $c_x \wedge c_y = c_y \ast (c_y \ast c_x) = c_y \ast (x \ast x)$, then by $(XM1)_X$,
\[
(c_x \wedge c_y) \bullet m = (y \ast (y \ast x)).m = (x \wedge y).m = x.(y.m) = c_x \bullet (c_y \bullet m)
\]

$(XM2)_{X/I}$: By $(XM2)_X$,
\[
c_x \bullet (m + n) = x.(m + n) = x.m + x.n = c_x \bullet m + c_x \bullet n.
\]

$(XM3)_{X/I}$: $c_0 \bullet m = 0$. $= 0$. 

$(XM4)_{X/I}$: Let $c_x \ast c_y \neq c_0$, for $c_x \neq c_y$. Hence $c_x \ast y \neq c_y$. Since $0 \ast (x \ast y) = 0 \in I$, $(x \ast y) \ast 0 = x \ast y \neq I$ and so $x \ast y \neq 0$. Therefore, by $(XM4)_X$,
\[
(c_x \ast c_y) \bullet m = c_x \ast y \bullet m = (x \ast y).m = x.m - y.m = c_x \bullet m - c_y \bullet m. \quad \blacksquare
\]

Notation. For $X^E$-module $M$, $Y \subseteq X$ and submodule $N$ of $M$, we consider

\[
Y.M = YM = \{x.m : x \in Y, m \in M\}.
\]

Lemma 3.10. Let $X$ be a commutative $BCK$-algebra, $M$ be an $X^E$-module, $N$ be a submodule of $M$ and $I$ be an ideal of $X$. Then

\[
I.M + N = \left\{ \sum_{i=1}^{n} t_i.m_i + n : t_i \in I, \ m_i \in M, \ n \in N \right\}
\]
is a submodule of $M$.

Proof. Let $N$ be a submodule of $M$ and $I$ be an ideal of $X$. It is clear that “$+$” is an associative operation in $I.M + N$ and $0 \in I.M + N$. Moreover, by $(XM4)$,
\[
\sum_{i=1}^{n} t_i.m_i + n - \left( \sum_{i=1}^{n} t_i.m_i + n \right) = \sum_{i=1}^{n} (t_i \ast t_i).m_i = 0,
\]
for any $\sum_{i=1}^{n} t_i.m_i + n \in I.M + N$. Hence, every element in $I.M + N$ has an inverse element and so $I.M + N$ is a subgroup of $M$. Now, by $(XM1)$ and $(XM2)$,
\[
x. \left( \sum_{i=1}^{n} t_i.m_i + n \right) = \sum_{i=1}^{n} x.(t_i.m_i) + x.n = \sum_{i=1}^{n} (x \wedge x).m_i + x.n = \sum_{i=1}^{n} (t_i \wedge x).m_i + x.n = \sum_{i=1}^{n} t_i.(x.m_i) + x.n \in I.M + N,
\]
for any \( \sum_{i=1}^{n} t_i m_i + n \in I.M + N \) and \( x \in X \). Therefore, \( I.M + N \) is a submodule of \( M \).

**Theorem 3.11.** Let \( X \) be a bounded BCK-algebra, \( I \) be a proper ideal of \( X \) and \( M \) be an \( X^{E} \)-module. Then \( M/IM \) is an \( (X/I)^{E} \)-module.

**Proof.** Let \( I \) be a proper ideal of \( X \) and \( M \) be an \( X^{E} \)-module. By Lemma 3.10, \( IM \) is a submodule of \( M \). Now, we define \( \bullet : X/I \times M/IM \rightarrow M/IM \) by \( c_x \bullet m + IM = x.m + IM \), for any \( x \in X \) and \( m \in M \). Since \( I \bullet (M/IM) = \{ x \bullet (m + IM) : x \in I, m \in M \} = \{ x.m + IM : x \in I, m \in M \} = IM \), then \( I \subseteq \text{ann}_X(M/IM) \). By Lemma 3.9, "\( \bullet \)" is well defined. Now, we show that \( M/IM \) is an \( (X/I)^{E} \)-module, for any \( x, y \in X \) and \( m, n \in M \).

\((XM1)_{X/I} : \) Since \( (c_x \land c_y) = c_y \ast (c_y \ast c_x) = c_{y \ast (y \ast x)} \), by \((XM1)_{X} \),

\[ (c_x \land c_y) \bullet (m + IM) = c_{y \ast (y \ast x)} \bullet (m + IM) = (y \ast (y \ast x)).m + IM, \]

\[ = (x \land y).m + IM = x.(y.m) + IM, \]

\[ = c_x \bullet (y.m + IM) = c_x \bullet (c_y \bullet (m + IM)) \]

\((XM2)_{X/I} : \) By \((XM2)_{X} \),

\[ c_x \bullet ((m + IM) + (n + IM)) = c_x \bullet (m + n + IM) = x.(m + n) + IM \]

\[ = (x.m + x.n) + IM = x.m + IM + x.n + IM \]

\[ = c_x \bullet (m + IM) + c_x \bullet (n + IM) \]

\((XM3)_{X/I} : \) \( c_0 \bullet (m + IM) = 0.m + IM = 0 + IM = IM = 0_{M/IM} \)

\((XM4)_{X/I} : \) If \( c_x = c_y \), then by \((XM4)_{X} \),

\[ (c_x \ast c_y) \bullet (m + IM) = c_{x \ast y} \bullet (m + IM) = c_0 \bullet (m + IM) \]

\[ = 0.m + IM = (x \ast x).m + IM, \]

\[ = x.m + IM - x.m + IM, \]

\[ = c_x \bullet (m + IM) - c_x \bullet (m + IM) \]

Now, let \( c_x \ast c_y \neq 0 \) where \( c_x \neq c_y \). Then \( c_{x \ast y} \neq c_0 \) i.e., \( (x \ast y) \ast 0 = x \ast y \notin I \) and so \( x \ast y \neq 0 \). Hence, by \((XM4)_{X} \),

\[ (c_x \ast c_y) \bullet (m + IM) = c_{x \ast y} \bullet (m + IM) = (x \ast y).m + IM = (x.m - y.m) + IM, \]

\[ = x.m + IM - y.m + IM = c_x \bullet (m + IM) - c_y \bullet (m + IM) \]

Therefore, \( M/IM \) is an \( (X/I)^{E} \)-module.

**Definition 3.12.** Let \( M \) be an \( X^{E} \)-module and \( N \) be a submodule of \( M \). Then \( N \) is called a **prime** submodule of \( M \), if \( N \neq M \) and for any \( x \in X \), \( x.m \in N \) implies that \( m \in N \) or \( x \in (N : M) \).
Example 3.13. By considering the Example 3.10 (ii), $2\mathbb{Z}$ is a prime submodule of $\mathbb{Z}$. It is clear that $2\mathbb{Z}$ is a subgroup of $\mathbb{Z}$. Now, let $x.n \in 2\mathbb{Z}$. If $x \neq 0$, $x.n = n$, then $n \in 2\mathbb{Z}$. If $x = 0$, then $x.n = 0.n = 0$ and so $0 \in (2\mathbb{Z} : \mathbb{Z})$. Hence, $2\mathbb{Z}$ is a prime submodule of $\mathbb{Z}$.

Theorem 3.14. Let $X$ be a commutative $BCK$-algebra, $M$ be an $X^E$-module and $N \neq M$ be a submodule of $M$. Then $N$ is a prime submodule of $M$ if and only if for any ideal $I$ in $X$ and for any submodule $D$ of $M$, $ID \subseteq N$ implies that $I \subseteq (N : M)$ or $D \subseteq N$.

Proof. $(\Rightarrow)$ Let $N$ be a prime submodule of $M$, $I$ be an ideal in $X$ and $D$ be a submodule of $M$ such that $ID \subseteq N$. We show that $I \subseteq (N : M)$ or $D \subseteq N$. Let $I \nsubseteq (N : M)$ and $D \nsubseteq N$. Then there exist $x \in X$ and $d \in D$ such that $x.M \nsubseteq N$ and $d \notin N$. On the other hand, $ID \subseteq N$ implies that $x.d \in N$. Since $N$ is a prime submodule of $M$, $x.M \subseteq N$, which is a contradiction.

$(\Leftarrow)$ Let $x \in X$ and $m \in M$ such that $x.m \in N$ and $m \notin N$. Let $I = \{x \in X : y * x = 0\}$ and $D = \langle x \rangle$. Then $ID = \{y'(m) : y' \in X\} \subseteq N$ and so $I \subseteq (N : M)$ or $D \subseteq N$. Since $m \notin N$, $I \subseteq (N : M)$ and this implies that $x.M \subseteq N$. Therefore, $N$ is a prime submodule of $M$.

Proposition 3.15. Let $M$ be an $X^E$-module and $N$ be a submodule of $M$. Then $P$ is a prime submodule of $M$ if and only if $\frac{P}{N}$ is a prime submodule of $\frac{M}{N}$, where $N \subseteq P$.

Proof. By Lemma 3.3, the proof is easy.

Definition 3.16. Let $M$ be an $X^E$-module. $M$ is called torsion free if $x.m = 0$ implies that $m = 0$ or $x = 0$, for any $x \in X$ and $m \in M$.

Example 3.17. (i) In Example 3.10(ii), $\mathbb{Z}$ is a torsion free.

(ii) In Example 3.10(iv), $M$ is not a torsion free. Because, $a.a = 0$ but $a \neq 0$.

Theorem 3.18. Let $X$ be bounded, $M$ be a unitary $X^E$-module and $K$ be a submodule of $M$. Then $K$ is a prime submodule of $M$ if and only if $P = (K : M)$ is a prime ideal of $X$ and $\frac{M}{K}$ is a torsion free $\left(\frac{X}{P}\right)^E$-module, where $\left(\frac{X}{P}, \ast, P\right)$ is a quotient $BCK$-algebra.

Proof. $(\Rightarrow)$ Let $K$ be a prime submodule of $M$. By Theorem 3.7, $P = (K : M)$ is an ideal of $X$. If $X = (K : M)$, then $1 \in P$ and so $M = K$, which is a contradiction. Now, let $x \land y \in P$, for any $x, y \in X$. Then for any $m \in M$, $(x \land y).m \in K$ and so by (XM1), $x.(y.m) \in K$. Since $K$ is a prime submodule of $M$, we have $y.m \in K$ or $x \in (K : M)$. It means that $y \in (K : M)$ or
Let $c.m \in \text{submodule of } M$. Hence, $(K : M)$ is a prime ideal. Now, we show that $\frac{M}{K}$ is a torsion free $\left( \frac{X}{P} \right)^E$-module. Let the operation $\bullet : \frac{X}{P} \times \frac{M}{K} \longrightarrow \frac{M}{K}$ be defined by $c_x \bullet (m + K) = x.m + K$, for any $x \in X$, $m \in M$. Similar to the proof of Theorem 3.9, $\bullet$ is well defined. Finally, for any $c_x \in \frac{X}{P}$ and $m + K \in \frac{M}{K}$, we will show that, $c_x \bullet (m + K) = K$ implies that $c_x = c_0$ or $m + K = K$. Let $c_x \bullet (m + K) = K$, for any $x \in X$ and $m \in M$. Then $x.m + K = K$ and so $x.m \in K$. Since $K$ is a prime submodule of $M$, $m \in K$ or $x \in (K : M)$. If $m \in K$, then $m + K = K$. Now, if $x \in (K : M) = P$, then $c_x = c_0 = P$ (because $x \ast 0 = x \in P$ and $0 \ast x = 0 \in P$).

Therefore, $\frac{M}{K}$ is a torsion free.

$(\Leftarrow)$ Let $P$ be a prime ideal in $X$ and $\frac{M}{K}$ is a torsion free $\left( \frac{X}{P} \right)^E$-module.

First we show that $K \nsubseteq \not{\exists} M$. Since, if $K = M$, $P = (K : M) = (M : M) = X$, which is a contradiction. Now, let $x.m \in K$, for any $x \in X$, $m \in M$. Hence $x.m + K = K$ and so $c_x \bullet (m + K) = K$. Since $\frac{M}{K}$ is torsion free, $c_x = c_0 = P$ or $m + K = K$. This means that $x \in P$ or $m \in K$. Therefore, $K$ is a prime submodule of $M$.

**Theorem 3.19.** Let $X$ be a bounded commutative BCK-algebra, $M$ be a unitary $X^E$-module, $N$ be a submodule of $M$ and $P$ be a prime ideal of $X$. Then $K(N, P) = \{m \in M : c.m \in P.M + N, \forall c \in X - P\}$ is a submodule of $M$ and $P.M + N \subseteq K(N, P)$.

**Proof.** First, we show that $K(N, P)$ is a subgroup of $M$. Let $m, n \in K(N, P)$. Then there exists $c, c' \in X - P$ such that $c.m, c'.n \in P.M + N$. Let $t = c \land c'$. Then

$$
t.(m - n) = (c \land c').(m - n),$$

$$= c.(c'.(m - n)) \text{ by (XM1)},$$

$$= c.(c'.m - c'.n) \text{ by (XM2)},$$

$$= c.(c'.m) - c.(c'.n) \text{ by (XM2)},$$

$$= (c \land c').m - c.(c'.n) \text{ by (XM1)},$$

$$= (c \land c).m - c.(c'.n),$$

$$= c'.(c.m) - c.(c'.n) \in P.M + N, \text{ by Lemma 3.10.}$$

and so $m - n \in K(N, P)$, which means that $K(N, P)$ is a subgroup. Now, let $x \in X$ and $m \in K(N, P)$. Since $m \in K(N, P)$, there exists $c \in X - P$ such that $c.m \in P.M + N$. Now, by Lemma 3.10,

$$c.(x.m) = (c \land x).m = (x \land c).m = x.(c.m) \in P.M + N$$

Hence, $x.m \in K(N, P)$ and so $K(N, P)$ is a submodule of $M$. Finally, for any $m \in P.M + N$, if we let $c = 1$ then $c.m = 1.m = m \in P.M + N$, then $m \in K(N, P)$.
Let $p.m \in K(N, P)$, for any $x \in X$, $m \in M$. Then there exists $c \in X - P$ such that $c.(x.m) \in P,M + N$. We will show that $m \in K(N, P)$ or $x \in (K(N, P) : M)$. If $x \in P$, then $x.M \subseteq P.M + N \subseteq K(N, P)$ and so $x.M \subseteq K(N, P)$. Hence $x \in (K(N, P) : M)$. If $x \notin P$, then $x \in X - P$. Since $P$ is a prime ideal of $X$, $c \wedge x \in X - P$. because, if $c \wedge x \in P$, then $c \in P$ or $x \in P$, which is a contradiction. Theorem 3.20. Let $X$ be a bounded commutative $BCK$-algebra, $M$ be a unitary $X^E$-module, and $N$ be a submodule of $M$ and $P$ be a prime ideal of $X$. Then $K(N, P) = M$ or $K(N, P)$ is a prime submodule of $M$ such that

$$P = (K(N, P) : M).$$

**Proof.** Let $K(N, P) \neq M$. We will show that $K(N, P)$ is a prime submodule of $M$ and $P = (K(N, P) : M)$. By Theorem 3.19, $K(N, P)$ is a submodule of $M$. Let $x.m \in K(N, P)$, for any $x \in X$, $m \in M$. Then there exists $c \in X - P$ such that $c.(x.m) \in P,M + N$. We will show that $m \in K(N, P)$ or $x \in (K(N, P) : M)$. If $x \in P$, then $x.M \subseteq P.M + N \subseteq K(N, P)$ and so $x.M \subseteq K(N, P)$. Hence $x \in (K(N, P) : M)$. If $x \notin P$, then $x \in X - P$. Since $P$ is a prime ideal of $X$, $c \wedge x \in X - P$. Because, if $c \wedge x \in P$, then $c \in P$ or $x \in P$, which is a contradiction. Therefore, $K(N, P)$ is a prime submodule of $M$. Now, we will prove that $P = (K(N, P) : M)$. Let $p \in P$. Then for any $m \in M, p.m \in P,M + N$. Let $c = 1$. Then $c.(p.m) \in P,M + N$ and so $p.m \in K(N, P)$, which implies that $P,M \subseteq K(N, P)$. Hence, $P \subseteq (K(N, P) : M)$. Now, let $q \in (K(N, P) : M)$ such that $q \notin P$. Since $q,M \subseteq K(N, P), q.m \in K(N, P)$, for any $m \in M$. Hence there exists $c \in X - P$ such that $c.(q.t) \in P,M + N$ and so $(c \wedge q).t \in P,M + N$. Now, since $P$ is prime, $c \wedge q \notin P$ i.e., $c \wedge q \notin X - P$ and so $t \in K(N, P)$. Hence, $M = K(N, P)$, which is a contradiction. Then $q \in P$ and so $(K(N, P) : M) \subseteq P$. Therefore, $P = (K(N, P) : M)$.

**Definition 3.21.** Let $M$ be an $X^E$-module and $N$ be a submodule of $M$. The intersection of all prime submodules of $M$, including $N$, is called radical of $N$ and it is shown by $rad_M(N)$. If there exists no prime submodule of $M$ consisting of $N$, then we let $rad_M(N) = M$.

**Theorem 3.22.** Let $X$ be a bounded commutative $BCK$-algebra and $M$ be an $X^E$-module. Then for any submodule $N$ of $M$,

$$rad_M(N) = \bigcap\{K(N, P) : P is a prime ideal of X\}.$$

**Proof.** Let $T = \bigcap\{K(N, P) : P is a prime ideal of X\}$ and $m \in T$. Let $L$ be a prime submodule of $M$ including of $N$. Hence, by Theorem 3.7, $Q = (L : M)$ is a prime ideal of $Q$. Since for any prime ideal $P$ of $X, m \in K(N, P), m \in K(N, Q)$ and so there exists $c \in X - Q$ such that $c.m \in Q.M + N = (L : M).M + N \subseteq L + L \subseteq L$. Since $L$ is a prime submodule of $M$ and $c \notin Q = (L : M), m \in L$. Hence $T \subseteq rad_M(N)$. Now, let $m \in rad_M(N)$. Hence, $m \in L$, where $L$ is any prime submodule of $M$ consisting of $N$ and $P$ be a prime ideal of $X$. If $K(N, P) = M$, then the proof is complete. Let $K(N, P) \neq M$. By Theorem 3.20, $K(N, P)$ is a prime submodule of $M$ and $P = (K(N, P) : M)$. Now, we show that $N \subseteq K(N, P)$. By Theorem 3.19, we have $P,M + N \subseteq K(N, P)$ and so $N \subseteq K(N, P)$ . Since $m \in rad_M(N)$, then $m \in K(N, P)$. Hence $m \in T$ and so $rad_M(N) \subseteq T$. Therefore, $rad_M(N) = T$. 


References


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