PROPERTIES OF HYPERIDEALS IN ORDERED SEMIHYPERGROUPS

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Abstract. An ordered semihypergroup is a semihypergroup $(S, \circ)$ together with a partial order $\leq$ on S such that the monotone condition holds, i.e., for all $x, y, a \in S$, if $x \leq y$, then for all $u \in x \circ a$ there exists $v \in y \circ a$ such that $u \leq v$, and similarly, for all $u' \in a \circ x$ there exists $v' \in a \circ y$ such that $u' \leq v'$. Indeed, the concept of ordered semihypergroups is a generalization of the concept of ordered semigroups. In this paper, we study some aspects of hyperideals of ordered semihypergroups. We give a necessary and sufficient condition of a subset of Cartesian product of two ordered semihypergroups to be a prime hyperideal. Also, we study right simple element ordered semihypergroups containing right simple elements.

Keywords: algebraic hyperstructure, ordered semigrouip, ordered semihypergroup, hyperideal, prime hyperideal, simple element.
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1. Introduction and basic definitions

The concept of algebraic hyperstructures was introduced in 1934 by Marty [11] and has been studied in the following decades and nowadays by many mathematicians. Let S be a nonempty set. A mapping $\circ : S \times S \to \mathcal{P}^*(S)$, where $\mathcal{P}^*(S)$ denotes the family of all nonempty subsets of S, is called a *hyperoperation* on S. The couple $(S, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two nonempty subsets of S and $x \in S$, then we denote

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \; x \circ A = \{x\} \circ A \;\text{ and }\; A \circ x = A \circ \{x\}.$$

A hypergroupoid $(S, \circ)$ is called a *semihypergroup* if for every $x, y, z \in S$, $x \circ (y \circ z) = (x \circ y) \circ z$, that is

$$\bigcup_{u \in y \circ z} x \circ u = \bigcup_{v \in x \circ y} v \circ z.$$

A nonempty subset A of S is called a *subsemihypergroup* if $x \circ y \subseteq A$ for all x, y in A. Semihypergroups are studied by many authors, for example, Bonansinga and Corsini [2], Davvaz [4], [5], De Salvo et al. [6], Freni [7], Hila et al. [9], Leoreanu [14], and many others. The concept of ordering hypergroups investigated by Chvalina [3] as a special class of hypergroups and studied by him and many others. In [8], Heidari and Davvaz studied a semihypergroup $(S, \circ)$ besides a binary relation $\leq$, where $\leq$ is a partial order relation such that satisfies the monotone condition. Indeed, an *ordered semihypergroup* $(S, \circ, \leq)$ is a semihypergroup $(S, \circ)$ together with a partial order $\leq$ that is *compatible* with the hyperoperation, meaning that for any x, y, z in S,

$$x \leq y \Rightarrow z \circ x \leq z \circ y \text{ and } x \circ z \leq y \circ z.$$

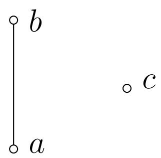
Here, $z \circ x \leq z \circ y$ means for any $a \in z \circ x$ there exists $b \in z \circ y$ such that $a \leq b$. The case $x \circ z \leq y \circ z$ is defined similarly.

Example 1 We have $(S, \circ, \leq)$ is an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c
a	a	$\{a, b\}$	$\{a, c\}$
b	a	$\{a, b\}$	$\{a, c\}$
c	a	$\{a, b\}$	c

$$\leq \, = \{(a, a), (b, b), (c, c), (a, b)\}.$$

The covering relation and the figure of S are given by: $\prec = \{(a, b)\}$

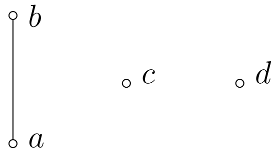


Example 2 We have $(S, \circ, \leq)$ is an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c	d
a	a	$\{a, b\}$	$\{a, c\}$	$\{a, d\}$
b	a	$\{a, b\}$	$\{a, c\}$	$\{a, d\}$
c	a	b	c	d
d	a	b	c	d

$$\leq = \{(a, a), (b, b), (c, c), (d, d), (a, b)\}.$$

The covering relation and the figure of S are given by: $\prec = \{(a, b)\}$



Note that the concept of ordered semihypergroups is a generalization of the concept of ordered semigroups [1], [10], [13]. Indeed, every ordered semigroup is an ordered semihypergroup.

For a nonempty subset A of an ordered semihypergroup $(S, \circ, \leq)$, we write

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

The following is easy to see for nonempty subsets A, B of an ordered semihypergroup $(S, \circ, \leq)$:

(1) $A \subseteq (A]$;

(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $(A] \circ (B] \subseteq (A \circ B]$;

(4) $((A] \circ (B]] = (A \circ B]$;

(5) $(A] \cup (B] = (A \cup B]$.

Let $(S, \circ, \leq)$ be an ordered semihypergroup. A subset A of S is called a *hyperideal* of S if it satisfies the following conditions:

(1) $x \circ y \subseteq A$ and $y \circ x \subseteq A$ for all x in A, y in S;

(2) for $x \in A, y \in S$, $y \leq x$ implies $y \in A$.

Let $(S, \circ, \leq_S)$ and $(T, \diamond, \leq_T)$ be two ordered semihypergroups. Under the coordinatewise multiplication, i.e.,

$$(s_1, t_1) \star (s_2, t_2) = s_1 \circ s_2 \times t_1 \diamond t_2$$

where $(s_1, t_1), (s_2, t_2) \in S \times T$, the Cartesian product $S \times T$ of S and T forms a semihypergroup. Define a partial order $\leq$ on $S \times T$ by

$$(s_1, t_1) \leq (s_2, t_2) \text{ if and only if } s_1 \leq_S s_2 \text{ and } t_1 \leq_T t_2$$

where $(s_1, t_1), (s_2, t_2) \in S \times T$. Then, $(S \times T, \star, \leq)$ is an ordered semihypergroup.

2. Prime ideals of the Cartesian product of two ordered semihypergroups

A hyperideal P of an ordered semihypergroup $(S, \circ, \leq)$ is said to be *prime* if $S \setminus P$ is a subsemihypergroup of S. Note that if a hyperideal P of S is prime, then $P \neq S$. In this section we accept the empty set to be a prime hyperideal. Similar to the method of Petrich [15], we give a necessary and sufficient condition of a subset of Cartesian product of two ordered semihypergroups to be a prime hyperideal.

Theorem 2.1 *Let $(S, \circ, \leq_S)$ and $(T, \diamond, \leq_T)$ be ordered semihypergroups. Then, a subset L of $S \times T$ is a prime hyperideal of $S \times T$ if and only if there exist a prime hyperideal I of S and a prime hyperideal J of T such that $L = (I \times T) \cup (S \times J)$.*

Proof. Assume that there exist a prime hyperideal I of S and a prime hyperideal J of T such that

$$L = (I \times T) \cup (S \times J).$$

If $I = \emptyset$ and $J = \emptyset$, then $L = \emptyset$; hence L is a prime hyperideal of $S \times T$.

Suppose that $I \neq \emptyset$ or $J \neq \emptyset$. Then, $L \neq \emptyset$. We will show that L is a prime hyperideal of $S \times T$. Let $(x, u) \in L$ and $(y, v) \in S \times T$. If $x \in I$, then $x \circ y \subseteq I$ and $y \circ x \subseteq I$; hence

$$(x, u) \star (y, v) = x \circ y \times u \diamond v \subseteq I \times T$$

and

$$(y, v) \star (x, u) = y \circ x \times v \diamond u \subseteq I \times T.$$

Similarly, if $u \in J$, then $(x, u) \star (y, v) \subseteq S \times J$ and $(y, v) \star (x, u) \subseteq S \times J$. Let $(x, u) \in L$ and $(y, v) \in S \times T$ be such that $(y, v) \leq (x, u)$, i.e., $y \leq_S x$, $v \leq_T u$. If $x \in I$, then $y \in I$; hence $(y, v) \in I \times T$. Thus, $(y, v) \in L$. Similarly, if $u \in J$, then $(y, v) \in L$. Therefore, L is a hyperideal of $S \times T$. Next, we assert that $(S \times T) \setminus L$ is a subsemihypergroup of $S \times T$. Since $S \setminus I \neq \emptyset$ and $T \setminus J \neq \emptyset$, it follows that $S \setminus I$ and $T \setminus J$ are semihypergroups of S and of T, respectively. We have

$$(S \times T) \setminus L = (S \setminus I) \times (T \setminus J) \neq \emptyset.$$

Then, $(S \setminus I) \times (T \setminus J)$ is a subsemihypergroup of $S \times T$. Hence, L is a prime hyperideal of $S \times T$.

Conversely, assume that L is a prime hyperideal of $S \times T$. If $L = \emptyset$, then $L = (\emptyset \times T) \cup (S \times \emptyset)$. Assume that $(x, u) \in L$. We assert that $\{x\} \times T \subseteq L$ or $S \times \{u\} \subseteq L$. Suppose that $\{x\} \times T \nsubseteq L$ and $S \times \{u\} \nsubseteq L$. Then, there exist $v \in T$ and $y \in S$ such that $(x, v) \notin L$ and such that $(y, u) \notin L$. We have

$$(x, v) \star (y, u) \star (x, v) \star (y, u) \quad = \quad x \circ y \circ x \circ y \times v \diamond u \diamond v \diamond u$$

and

$$(x \circ y, v) \star (x, u) \star (y, v \diamond u) \quad = \quad x \circ y \circ x \circ y \times v \diamond u \diamond v \diamond u.$$

Since $(x, v) \star (y, u) \star (x, v) \star (y, u) \subseteq (S \times T) \setminus L$, we have

$$(x \circ y, v) \star (x, u) \star (y, v \diamond u) \subseteq (S \times T) \setminus L.$$

But, since $(x, u) \in L$, we have $(x \circ y, v) \star (x, u) \star (y, v \diamond u) \subseteq L$. This is a contradiction. Hence,, $\{x\} \times T \subseteq L$ or $S \times \{u\} \subseteq L$. Let

$$A = \{x \in S \mid \{x\} \times T \subseteq L\} \text{ and } B = \{u \in T \mid S \times \{u\} \subseteq L\},$$

and let

$$I = (A] \text{ and } J = (B].$$

Let $(x,u) \in L$. Then, $\{x\} \times T \subseteq L$ or $S \times \{u\} \subseteq L$. Thus, $x \in I$ or $u \in J$. Hence, $(x,u) \in (I \times T) \cup (S \times J)$. Thus, $L \subseteq (I \times T) \cup (S \times J)$. The reverse inclusion is clear. Hence,

$$L = (I \times T) \cup (S \times J).$$

We will show that I is a prime hyperideal of S. That J is a prime hyperideal of T can be proved similarly. If $I = \emptyset$, then I is a prime hyperideal of S. Assume that $I \neq \emptyset$. If $I = S$, then $L = S \times T$. This is a contradiction since L is a prime hyperideal of $S \times T$. Hence, $S \setminus I \neq \emptyset$. Similarly, $T \setminus J \neq \emptyset$. Let $x, y \in S \setminus I$ and $u \in T \setminus J$. Then,

$$(x,u), (y,u) \in (S \setminus I) \times (T \setminus J).$$

Since L is prime, we have $(S \setminus I) \times (T \setminus J)$ is a subsemihypergroup of $S \times T$. Since

$$x \circ y \times u \diamond u = (x,u) \star (y,u) \subseteq (S \setminus I) \times (T \setminus J)$$

we get $x \circ y \subseteq S \setminus I$. Thus, $S \setminus I$ is a subsemihypergroup of S. Let $x \in I, y \in S$ and $u \in T \setminus J$. Since $x \in I$, $(x,u) \in L$. Since L is a hyperideal of $S \times T$, we have

$$(x,u) \star (y,u) = x \circ y \times u \diamond u \subseteq L$$

and

$$(y,u) \star (x,u) = y \circ x \times u \diamond u \subseteq L.$$

Since $T \setminus J$ is a subsemihypergroup, so $u \diamond u \subseteq T \setminus J$. Since

$$x \circ y \times u \diamond u, y \circ x \times u \diamond u \subseteq L$$

we obtain

$$x \circ y \times u \diamond u, y \circ x \times u \diamond u \subseteq I \times T$$

and hence $x \circ y, y \circ x \subseteq I$. It is clear that if $x \in I$ and $y \in S$ such that $y \leq x$, then $y \in I$. Therefore, I is a prime hyperideal of S. ■

Suppose that $(S, \circ)$ and $(T, \diamond)$ are semihypergroups. Then, the Cartesian product $S \times T$ is a semihypergroup under the coordinatewise multiplication. Define a partial order $\leq_S$ on S by

$$x \leq_S y \text{ if and only if } x = y \text{ for all } x, y \in S.$$

Then, S forms an ordered semihypergroup. Similarly, T forms an ordered semihypergroup with a partial order $\leq_T$ defined in a similar way. Using Theorem 2.1, we have the following result proved in [15].

Corollary 2.2 *Let $(S, \circ)$ and $(T, \diamond)$ be semihypergroups. Then, a subset L of $S \times T$ is a prime hyperideal of $S \times T$ if and only if $L = (I \times T) \cup (S \times J)$ for some prime hyperideals I and J of S and of T, respectively.*

3. Right simple ordered semihypergroups

Let $(S, \circ, \leq)$ be an ordered semihypergroup. An element a of S is said to be *right simple* if $S = (a \circ S]$. If S contains a right simple element then it is called a *right simple element ordered semihypergroup.* If every element of S is right simple, then S is called a *right simple ordered semihypergroup.*

Theorem 3.1 *If $(S, \circ, \leq_S)$ and $(T, \diamond, \leq_T)$ are two right simple element ordered semihypergroups, then $S \times T$ is a right simple element ordered semihypergroup, too. Moreover, if A and B are the sets of all right simple elements of S and of T, respectively, then $A \times B$ is the set of all right simple elements of $S \times T$.*

Proof. Assume that $(S, \circ, \leq_S)$ and $(T, \diamond, \leq_T)$ are right simple element ordered semihypergroups with the sets of all right simple elements A and B, respectively. If $(a, b) \in A \times B$, then

$$S \times T = (a \circ S] \times (b \diamond T].$$

If $(s, t) \in S \times T$, then $s \in a \circ s'$ for some s' in S, and $t \in b \diamond t'$ for some t' in T. Since $(s, t) \in a \circ s' \times b \diamond t'$, it follows that

$$(s, t) \in \left(\bigcup_{(s,t) \in S \times T} a \circ s \times b \diamond t \right] = ((a, b) \star (S \times T)].$$

Hence, (a, b) is a right simple element of $S \times T$. If (a, b) is a right simple element of $S \times T$, then

$$S \times T = ((a, b) \star (S \times T)] = \left(\bigcup_{(s,t) \in S \times T} a \circ s \times b \diamond t \right].$$

If $(s, t) \in S \times T$, then $(s, t) \leq (u, v)$ for some $(u, v) \in a \circ s' \times b \diamond t'$ where $s' \in S, t' \in T$. Since

$$s \leq u \in a \circ s' \subseteq (a \circ S],$$

we have $s \in (a \circ S]$, and so $S = (a \circ S]$.

Similarly, $T = (b \diamond T]$. Hence, $(a, b) \in A \times B$. ∎

It is well known the following result in semigroup theory. Let S be a right simple element semigroup and let R denote the set of all right simple elements of S. Then, the following conditions holds: (1) R is a subsemigroup of S; (2) If $S \setminus R$ is nonempty, then it is the maximal right ideal of S and is prime, too ([12]). In the following, we extend the above result based on ordered semihypergroups.

Theorem 3.2 *Let $(S, \circ, \leq)$ be right simple element ordered semihypergroup with the set of all right simple elements R. The following statements hold:*

(1) *R is a subsemihypergroup of S.*

(2) *If $S \setminus R$ is nonempty, then it is the maximal right hyperideal of S and is prime, too.*

Proof. If $(S, \circ, \leq)$ is right simple ordered semihypergroup, then it is clear that (1) and (2) hold. Then, we assume that $(S, \circ, \leq)$ is not a right simple ordered semihypergroup.

(1) Let $a, b \in R$. Since $S = (a \circ S]$ and $S = (b \circ S]$, we have

$$S = (a \circ S] = (a \circ (b \circ S]] \subseteq ((a] \circ (b \circ S]] = (a \circ b \circ S],$$

and so $a \circ b \subseteq R$.

(2) Assume that $S \setminus R \neq \emptyset$. Let $x \in S$ and $a \in S \setminus R$. If $a \circ x \subseteq R$, then $S = (a \circ x \circ S] \subseteq (a \circ S]$; hence $a \in R$. This is a contradiction. Thus, $a \circ x \subseteq S \setminus R$. Let $x \in S \setminus R$ and $y \in S$ be such that $y \leq x$. If $y \in R$, then $S = (y \circ S] \subseteq (x \circ S]$; hence $x \in R$. This is a contradiction. Thus, $S \setminus R$ is a right hyperideal of S. Let A be a right hyperideal of S such that $S \setminus R \subset A$. Then, there is an element a in $A \setminus (S \setminus R)$. Since $S = (a \circ S] \subseteq A$, so $A = S$. By (1), it follows directly that $S \setminus R$ is prime. ■

Theorem 3.3 *If an ordered semihypergroup $(S, \circ, \leq)$ has a unique maximal right hyperideal A such that $S \setminus A \neq (b]$ for all b in $S \setminus A$, then $S \setminus A$ is the set of all right simple elements of S.*

Proof. Let R denote the set of all right simple elements of S. Let $a \in R$. If $a \in A$, then $S = (a \circ S] \subseteq A$. This is a contradiction. Thus, $R \subseteq S \setminus A$. Let $b \in S \setminus A$. We have $(b \circ S]$ is a right hyperideal of S. If $(b \circ S] \subset S$, then by assumption we have $(b \circ S] \subseteq A$; hence $(A \cup \{b\}]$ is a right hyperideal of S. By $A \subset (A \cup \{b\}]$, $S = (A \cup \{b\}]$. Hence, $S \setminus A = (b]$. This is a contradiction. Hence, $(b \circ S] = S$. ■

Let $(S, \circ, \leq)$ be an ordered semihypergroup. An equivalence relation $\mathcal{R}$ is defined on S by

$$a\mathcal{R}b \ \text{ if and only if } \ (a \cup a \circ S] = (b \cup b \circ S]$$

for any a, b in S.

An element a of an ordered semihypersroup $(S, \circ, \leq)$ is said to be *right regular* if $a \in (a^2 \circ S]$.

Theorem 3.4 *Let $(S, \circ, \leq)$ be a right simple element ordered semihypergroup with set of all right simple elements R. Then,*

(1) *R is an $\mathcal{R}$-class of S;*

(2) *every element of R is right regular.*

Proof. (1) If $a, b \in R$, then $S = (a \circ S]$ and $S = (b \circ S]$; hence $(a \cup a \circ S] = (b \cup b \circ S]$. This shows that $a\mathcal{R}b$. Let $x \in S$ be such that $x\mathcal{R}a$ for some a in R. Then, $(x \cup x \circ S] = (a \cup a \circ S] = S$. If $S \setminus R = \emptyset$, then $x \in R$. If $S \setminus R \neq \emptyset$ and $x \in S \setminus R$, then $S = (x \cup x \circ S] \subseteq S \setminus R$; hence $S = S \setminus R$. This is a contradiction. Hence, $x \in R$.

(2) If $a \in R$, then $a \in (a \circ S] \subseteq (a \circ (a \circ S]] \subseteq (a^2 \circ S]$; hence a is right regular. ∎

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