QUOTIENT RINGS VIA FUZZY CONGRUENCE RELATIONS

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Abstract. This paper aims to introduce fuzzy congruence relations over rings and give constructions of quotient rings induced by fuzzy congruence relations. The Fuzzy First, Second and Third Isomorphism Theorems of rings are established. Finally, we investigate the relationships between fuzzy ideals and fuzzy congruence relations on rings.

Keywords: ring; fuzzy congruence relation; fuzzy ideal; quotient ring.

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1. Introduction

Fuzzy set theory, proposed by L.A. Zadeh [14], has been extensively applied to many scientific fields. In fact, the field grew enormously, and applications were found in areas by many authors (see [1], [13]) as diverse as washing machines to handwriting recognition and other applications.

Following the discovery of fuzzy sets, much attention has been paid to generalize the basic concepts of classical algebra in a fuzzy framework, and thus developing a theory of fuzzy algebras. In recent years, much interest is shown to generalize algebraic structures of groups, rings, modules, etc. The notion of fuzzy ideals of a ring R was put forward and the operations on fuzzy ideals was discussed by several researchers (see, e.g., [4], [5], [6], [7]). Fuzzy congruence relations and fuzzy normal subgroups on groups was shown by N. Kuroki [3]. Later on, L. Filep and I. Maurer [2] and by V. Murali [11] further studied fuzzy congruence relations

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on universal algebras. Fuzzy isomorphism theorems of soft rings were shown by X.P. Liu [8], [9]. General algebraic structure, such as group and ring of congruence relations and ideals to depict the algebraic structure has played a very important role. The various constructions of quotient groups and quotient rings by fuzzy ideals was introduced by Y.L. Liu [7]. Moreover, N. Kuroki has been shown that there exists a one-to-one mapping from all fuzzy normal subgroups and all fuzzy congruence relations of groups. Naturally, the study of the definition and properties about fuzzy congruence relations on rings is a meaningful work.

In this paper, we introduce the notion of fuzzy congruence relations on rings and introduce the notion of quotient rings by fuzzy congruence relations and give the Fuzzy First, Second and Third Isomorphism Theorems of rings based on fuzzy congruence relation. Moreover, we give some properties between fuzzy ideals and fuzzy congruence relations on rings.

2. Preliminaries

From the properties of fuzzy set theory, we know that a fuzzy set defined on a set as follows: let R be a non-empty set, then $\mu : R \to [0,1]$ is called a fuzzy set of R. In this paper, R is always a ring.

Definition 2.1 [10]

(1) A fuzzy set μ of R is called a fuzzy left (resp., right) ideal of R if it satisfies:

  (i) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$ for all x, y of R,

  (ii) $\mu(xy) \geq \mu(y)$ (resp., $\mu(xy) \geq \mu(x)$) for all x, y of R.

(2) A fuzzy set μ of R is called a fuzzy ideal of R if it is both a fuzzy left ideal and a fuzzy right ideal of R.

Clearly, let μ be a fuzzy set of R, if it satisfies $\mu(xy) \geq \mu(x) \vee \mu(y)$, then μ is a fuzzy ideal of R. We denote the set of all fuzzy ideals of R by $FI(R)$.

Definition 2.2 [10] Let μ and ν be two fuzzy sets of R. Then the product $\mu + \nu$ is defined by the following:

$$(\mu+\nu)(z) = \bigvee_{x+y=z} [\mu(x) \wedge \nu(y)],$$

and

$$(\mu+\nu)(z) = 0$$

if z cannot be expressed as $z = x + y$, for all x, y and z of R.

Definition 2.3 [11]

(1) A function α from $R \times R$ to the unit interval [0,1] is called a fuzzy relation on R. Let α and β be two fuzzy relations on R, then the product $\alpha \circ \beta$ is defined by the following:

$$(\alpha \circ \beta)(a, b) = \bigvee_{x \in R} [\alpha(a, x) \wedge \beta(x, b)]$$

for all a, b of R.

(2) Let α and β be two fuzzy relations on R, then the product $\alpha \cap \beta$ is defined by the following way:

$$(\alpha \cap \beta)(x, y) = \alpha(x, y) \wedge \beta(x, y),$$

$$(\alpha \cup \beta)(x, y) = \alpha(x, y) \vee \beta(x, y).$$

Definition 2.4 [12] A relation α on the set R is called left compatible if $(a, b) \in \alpha$ implies $(x + a, x + b) \in \alpha$ and $(xa, xb) \in \alpha$, for all a, b, x of R, and is called right compatible if $(a, b) \in \alpha$ implies $(a + x, b + x) \in \alpha$ and $(ax, bx) \in \alpha$, for all a, b, x of R.

Remark 2.5 For any relation α on the set R

(i) It is called compatible if $(a, b) \in \alpha$ and $(c, d) \in \alpha$ implies $(a + c, b + d) \in \alpha$ and $(ac, bd) \in \alpha$, for all a, b, c, d of R,

(ii) A left (right) compatible equivalence relation on R is called a left (right) congruence relation on R,

(iii) A compatible equivalence relation on R is called a congruence relation on R.

As is well known (see [3]), a relation α on R is a congruence relation if and only if it is both a left and a right congruence relation on R.

3. Fuzzy congruence relations

In this section, we introduce the notion of fuzzy congruence relations on rings and give some properties about fuzzy congruence relations.

Definition 3.1 [3] A fuzzy set α of $R \times R$ is called a fuzzy relation on R. A fuzzy relation α on R is called a fuzzy equivalence relation if it satisfies the following conditions:

(i) $\alpha(x, x) = 1$ for all x of R (fuzzy reflexive),

(ii) $\alpha(x, y) = \alpha(y, x)$ for all of R (fuzzy symmetric),

(iii) $\alpha(x, y) \geq \bigvee_{z \in R} [\alpha(x, z) \wedge \alpha(z, y)]$ for all x, y of R (fuzzy transitive).

We note that α is fuzzy transitive if and only if $\alpha \supset \alpha \circ \alpha$.

Definition 3.2 [12] A fuzzy relation α on R is called a fuzzy left compatible relation if $\alpha(x+a, x+b) \geq \alpha(a,b)$ and $\alpha(xa, xb) \geq \alpha(a,b)$ for all x, a, b of R, and is called a fuzzy right compatible relation if $\alpha(a+x, b+x) \geq \alpha(a,b)$ and $\alpha(ax, bx) \geq \alpha(a,b)$ for all x, a, b of R. It is called a fuzzy compatible relation if $\alpha(a+c, b+d) \geq \alpha(a,b) \wedge \alpha(c,d)$ and $\alpha(ac, bd) \geq \alpha(a,b) \wedge \alpha(c,d)$.

Remark 3.3 A fuzzy relation on R is called a fuzzy compatible relation if and only if it is both a left and a right fuzzy compatible relation on R.

Proposition 3.4 Let α and β be any fuzzy compatible relations on R. Then $\alpha \circ \beta$ is also a fuzzy compatible relation on R.

Proof. For every $a, b, x \in R$. Since α and β are fuzzy compatible relations, we have

$$\begin{aligned}(\alpha \circ \beta)(x+a, x+b) &= \bigvee_{z \in R} [\alpha(x+a, z) \wedge \beta(z, x+b)] \\ &\geq [\alpha(x+a, x+z) \wedge \beta(x+z, x+b)] \\ &\geq [\alpha(a,z) \wedge \beta(z,b)].\end{aligned}$$

Then we have

$$\begin{aligned}(\alpha \circ \beta)(x+a, x+b) &\geq \bigvee_{z \in R} [\alpha(a,z) \wedge \beta(z,b)] \\ &= (\alpha \circ \beta)(a,b),\end{aligned}$$

$$\begin{aligned}(\alpha \circ \beta)(xa, xb) &= \bigvee_{z \in R} [\alpha(xa, z) \wedge \beta(z, xb)] \\ &\geq [\alpha(xa, xz) \wedge \beta(xz, xb)] \\ &\geq [\alpha(a,z) \wedge \beta(z,b)].\end{aligned}$$

Hence

$$\begin{aligned}(\alpha \circ \beta)(xa, xb) &\geq \bigvee_{z \in R} [\alpha(a,z) \wedge \beta(z,b)] \\ &= (\alpha \circ \beta)(a,b).\end{aligned}$$

This means that $\alpha \circ \beta$ is a fuzzy left compatible relation. It can be seen in a similar way that $\alpha \circ \beta$ is a fuzzy right compatible relation. Thus we obtain that $\alpha \circ \beta$ is a fuzzy compatible relation.

Definition 3.5 [12] A fuzzy equivalence relation α on R is called a fuzzy congruence relation if the following conditions are satisfied for all x, y, z, t of R

(i) $\alpha(x+y, z+t) \geq \alpha(x,z) \wedge \alpha(y,t)$,

(ii) $\alpha(xy, zt) \geq \alpha(x,z) \wedge \alpha(y,t)$.

We denote the set of all fuzzy congruence relations on R by $FC(R)$.

Example 3.6 Let $\mathbb{Z}$ be the set of all integers. Then $\mathbb{Z}$ is a ring with respect to the usual addition and multiplication of numbers. The fuzzy relation α on $\mathbb{Z}$ defined by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0.5 & \text{if } x \neq y \text{ and both } x, y \text{ are either or odd,} \\ 0 & \text{otherwise.} \end{cases}$$

is a fuzzy congruence relation on $\mathbb{Z}$.

Proposition 3.7 [12] *Let α be a fuzzy congruence relation on R. Then for all $x, y, z \in R$ we have the following results:*

(i) $\alpha(x,y) \geq \alpha(x+z, y+z) \wedge \alpha(xz, yz) \wedge \alpha(zx, zy)$,

(ii) $\alpha(-x,-y) = \alpha(x,y)$.

Proposition 3.8 [3] *Let α and β be fuzzy congruence relations on R. Then $\alpha \circ \beta$ is a fuzzy congruence relation on R if and only if $\alpha \circ \beta = \beta \circ \alpha$.*

Let α be a fuzzy relation on R. For each $\lambda \in [0,1]$, we put

$$R_\alpha(\lambda) = \{(a,b) : (a,b) \in R \times R, \alpha(a,b) \geq \lambda\}.$$

This set is called the λ-level set of α.

Theorem 3.9 *A fuzzy relation α is a fuzzy congruence relation on R if and only if for each $\lambda \in [0,1]$, the λ-level set $R_\alpha(\lambda)$ is a congruence relation on R.*

Proof. Since α is a fuzzy congruence relation on R, then $\alpha(x,x) = 1$, for every $x \in R$, we have $(x,x) \in R_\alpha(\lambda)$, which means $R_\alpha(\lambda)$ is a reflexive relation.

For all $(x,y) \in R_\alpha(\lambda), \alpha(x,y) = \alpha(y,x) \geq \lambda$, i.e. $(y,x) \in R_\alpha(\lambda)$, $R_\alpha(\lambda)$ is a symmetric relation.

For each $(x,y),(z,y) \in R_\alpha(\lambda)$, $\alpha(x,z) \geq \lambda$, $\alpha(z,y) \geq \lambda$, since $\alpha(x,y) \geq \bigvee_{z \in R} [\alpha(x,z) \wedge \alpha(z,y)]$, then

$$\alpha(x,y) \geq \lambda, \text{ i.e., } (x,y) \in R_\alpha(\lambda).$$

Therefore, $R_\alpha(\lambda)$ is a transitive relation. Consequently, $R_\alpha(\lambda)$ is a equivalence relation on R.

For every $(x,y),(a,b) \in R_\alpha(\lambda)$, since α is a fuzzy congruence relation on R, then we have $\alpha(a+x,b+y) \geq \alpha(a,b) \wedge \alpha(x,y) \geq \lambda$, $\alpha(ax,by) \geq \alpha(a,b) \wedge \alpha(x,y) \geq \lambda$, i.e., $(a+x,b+y) \in R_\alpha(\lambda)$ and $(ax,by) \in R_\alpha(\lambda)$.

Thus $R_\alpha(\lambda)$ is a compatible relation on R, which implies, $R_\alpha(\lambda)$ is a congruence relation on R.

Conversely, let $\lambda \in [0,1]$, since $R_\alpha(\lambda)$ is a congruence relation on R, then for all $x \in R, \alpha(x,x) \geq \lambda$, implies $\alpha(x,x) = 1$, i.e., α is a fuzzy reflexive relation. For each $x, y \in R$, if $\alpha(x,y) \neq \alpha(y,x)$, let $\alpha(x,y) = \lambda_1, \alpha(y,x) = \lambda_2$, if $\lambda_1 > \lambda_2$, then $(y,x) \notin R_\alpha(\lambda_1)$, but $(x,y) \in R_\alpha(\lambda_1)$, since $R_\alpha(\lambda_1)$ is a congruence relation, hence $(y,x) \in R_\alpha(\lambda_1)$, contradiction.

So $\alpha(x,y) = \alpha(y,x)$. When $\lambda_1 < \lambda_2$, the proof of it is similarly. Hence α is fuzzy a symmetric relation. For each $x, y, z \in R$, let $\alpha(x,z) = t_1, \alpha(z,y) = t_2$, if $t_1 \leq t_2$ then $(x,z), (z,y) \in R_\alpha(t_1)$. Since $R_\alpha(t_1)$ is a congruence relation on R, hence $(x,y) \in R_\alpha(t_1)$, and $\alpha(x,y) \geq t_1 = \bigvee_{z \in R} [\alpha(x,z) \wedge \alpha(z,y)]$, thus α is a fuzzy transitive relation. When $t_1 \geq t_2$, the proof of it is similarly, hence α is a fuzzy equivalence relation on R.

For each $a, b, x, y \in R$, let $\alpha(a,b) = s, \alpha(x,y) = t$, if $s \leq t$, then $(a,b) \in R_\alpha(s), (x,y) \in R_\alpha(t) \leq R_\alpha(s)$. Since $R_\alpha(s)$ is a congruence relation on R, hence $\alpha(a+x, b+y) \in R_\alpha(s), (ax, by) \in R_\alpha(s)$, we have $\alpha(a+x, b+y) \geq s = \alpha(a,b) \wedge \alpha(x,y), \alpha(ax, by) \geq s = \alpha(a,b) \wedge \alpha(x,y)$. Hence α is a fuzzy compatible relation on R. When $s \geq t$, the proof is similar. It follows that α is a fuzzy congruence relation on R. This completes the proof.

4. Quotient rings induced by fuzzy congruence relations

In this section, we introduce the notion of quotient rings by fuzzy congruence relations and give the Fuzzy First, Second and Third Isomorphism Theorems of rings by means of fuzzy congruence relations.

Definition 4.1 Let α be a fuzzy congruence relation on R. For every element $x \in R$, we define a subset

$$\alpha_x = \{y \in R | \alpha(x,y) = 1\}$$

of R and $R/\alpha = \{\alpha_x | x \in R\}$.

Theorem 4.2 *If α is a fuzzy congruence relation of R, then R/α is a ring under the binary operations defined by*

$$\alpha_x + \alpha_y = \alpha_{x+y} \text{ and } \alpha_x \alpha_y = \alpha_{xy}$$

for any $x, y \in R$.

Proof. First, we show that the above binary operations are well-defined. In fact, if $\alpha_x = \alpha_{x'}$ and $\alpha_y = \alpha_{y'}$, then we have $\alpha(x,x') = 1$ and $\alpha(y,y') = 1$. Since $\alpha(x,x') \leq \alpha(x+y, x'+y)$ and $\alpha(y,y') \leq \alpha(x'+y, x'+y')$ so

$$\begin{aligned} \alpha(x+y, x'+y') &\geq \bigvee_{z \in R} [\alpha(x+y, z) \wedge \alpha(z, x'+y')] \\ &\geq \alpha(x+y, x'+y) \wedge \alpha(x'+y, x'+y') \\ &\geq \alpha(x,x') \wedge \alpha(y,y') \\ &= 1. \end{aligned}$$

Then we have $\alpha(x+y, x'+y') = 1$. This means $\alpha_{x+y} = \alpha_{x'+y'}$.
Again, since $\alpha(x, x') \leq \alpha(xy, x'y)$ and $\alpha(y, y') \leq \alpha(x'y, x'y')$,

$$\begin{aligned}
\alpha(xy, x'y') &\geq \bigvee_{z \in R} [\alpha(xy, z) \wedge \alpha(z, x'y')] \\
&\geq \alpha(xy, x'y) \wedge \alpha(x'y, x'y') \\
&\geq \alpha(x, y') \wedge \alpha(y, y') \\
&= 1.
\end{aligned}$$

Therefore, $\alpha(xy, x'y') = 1$, this means $\alpha_{xy} = \alpha_{x'y'}$. Hence addition and multiplication are well-defined. Because it is a routine matter to verify that the set R/α is a ring under the binary defined above and we omit its proof.

Notation 4.3 If α is a fuzzy congruence relation on R, then R/α is a ring which has the zero element α_0.

Lemma 4.4 *Let R, R' be rings and f be a homomorphism from R to R'. If α' is a fuzzy congruence relation of R', then the map $f^{-1}(\alpha')$ defined by*

$$f^{-1}(\alpha')(x, y) = \alpha'(f(x), f(y))$$

for all $x, y \in R$ is a fuzzy congruence relation of R.

Proof. For all $x, y, z, \in R$, then

$$f^{-1}(\alpha')(x, x) = 1,\ f^{-1}(\alpha')(x, y) = \alpha'(f(x), f(y)) = \alpha'(f(y), f(x)) = f^{-1}(\alpha')(y, x),$$

which means $f^{-1}(\alpha')$ is a fuzzy reflexive relation and fuzzy symmetric relation of R. Since

$$\begin{aligned}
f^{-1}(\alpha')(x, y) &= \alpha'(f(x), f(y)) \\
&\geq \bigvee_{f(z) \in R'} [\alpha'(f(x), f(z)) \wedge \alpha'(f(z), f(y))] \\
&\geq \alpha'(f(x), f(z)) \wedge \alpha'(f(z), f(y)) \\
&= f^{-1}(\alpha')(x, z) \wedge f^{-1}(\alpha')(z, y) \\
&\geq \bigvee_{z \in R} [f^{-1}(\alpha')(x, z) \wedge f^{-1}(\alpha')(z, y)].
\end{aligned}$$

This means $f^{-1}(\alpha')$ is a fuzzy transitive relation of R. So $f^{-1}(\alpha')$ is a fuzzy equivalence relation of R. Again

$$\begin{aligned}
f^{-1}(\alpha')(z + x, z + y) &= \alpha'(f(z + x), f(z + y)) \\
&= \alpha'(f(z) + f(x), f(z) + f(y)) \\
&\geq \alpha'(f(x), f(y)) \\
&= f^{-1}(\alpha')(x, y),
\end{aligned}$$

$$\begin{aligned}
f^{-1}(\alpha')(zx, zy) &= \alpha'(f(zx), f(zy)) \\
&= \alpha'(f(z)f(x), f(z)f(y)) \\
&\geq \alpha'(f(x), f(y)) \\
&= f^{-1}(\alpha')(x, y).
\end{aligned}$$

This means that $f^{-1}(\alpha')$ is a fuzzy left compatible relation of R. By the same argument, we can see that $f^{-1}(\alpha')$ a fuzzy right compatible relation of R. Thus we obtain that $f^{-1}(\alpha')$ is a fuzzy congruence relation of R. This completes the proof.

Theorem 4.5 (Fuzzy First Isomorphism Theorem) *Let R, R' be rings, f be an epimorphism from R to R', and α' be a fuzzy congruence relation of R. Then*

$$R/f^{-1}(\alpha') \cong R'/\alpha'.$$

Proof. It follows from Theorem 4.2 and Lemma 4.4, $R/f^{-1}(\alpha')$ and R'/α' are both rings. We define a map h from $R/f^{-1}(\alpha')$ to R'/α' by

$$h(f^{-1}(\alpha')_x) = \alpha'_{f(x)} \text{ for all } x \in R.$$

(i) h is well-defined: If $f^{-1}(\alpha')_x = f^{-1}(\alpha')_y$, then $f^{-1}(\alpha')(x, y) = 1$. By Definition 4.1, we have $\alpha'(f(x), f(y)) = 1$, which implies $\alpha'_{f(x)} = \alpha'_{f(y)}$. Therefore, h is well-defined.

(ii) h is a homomorphism: $h(f^{-1}(\alpha')_x + f^{-1}(\alpha')_y) = h(f^{-1}(\alpha')_{x+y}) = \alpha'_{f(x+y)}$ $= \alpha'_{f(x)+f(y)} = \alpha'_{f(x)} + \alpha'_{f(y)}$; $h(f^{-1}(\alpha')_x f^{-1}(\alpha')_y) = h(f^{-1}(\alpha')_{xy}) = \alpha'_{f(xy)}$ $= \alpha'_{f(x)f(y)} = \alpha'_{f(x)}\alpha'_{f(y)}$. This implies that h is a homomorphism.

(iii) h is an epimorphism: For any $\alpha'_y \in R'/\alpha'$, since f is epimorphic, there exists $x \in R$ such that $f(x) = y$. So $h(f^{-1}(\alpha')_x) = \alpha'_{f(x)} = \alpha'_y$.

(iv) h is a monomorphism: Suppose that $h(f^{-1}(\alpha')_x) = h(f^{-1}(\alpha')_y)$, then $\alpha'_{f(x)} = \alpha'_{f(y)}$, which implies $\alpha'(f(x), f(y)) = 1$. From Lemma 4.4, we have $f^{-1}(\alpha')(x, y) = 1$. Hence $f^{-1}(\alpha')_x = f^{-1}(\alpha')_y$. This means h is a monomorphism.

In conclusion, $R/f^{-1}(\alpha') \cong R'/\alpha'$.

Corollary 4.6 *Let α be a fuzzy congruence relation of R, then the mapping $f : R \to R/\alpha$ defined by $f(x) = \alpha_x$ for all $x \in R$, is a homomorphism.*

Lemma 4.7 *Let α be a fuzzy congruence relation of R, we denote*

$$R_\alpha = \{y \in R | \alpha(0, y) = 1\}.$$

Then R_α is an ideal of R.

Proof. It is clear.

Lemma 4.8 *Let I be an ideal of R, α and β are fuzzy congruence relations of R.*

(i) *If α is restricted to I, then α is a fuzzy congruence relation of I,*

(ii) *$\alpha \cap \beta$ is fuzzy congruence relation of R,*

(iii) *I/α is an ideal of R/α.*

Proof. (i) Obviously.

(ii) For any $x, y \in R$, since $(\alpha \cap \beta)(x, y) = \alpha(x, y) \wedge \beta(x, y)$, $\alpha \cap \beta$ is a fuzzy reflexive relation and a fuzzy symmetric relation, we only show $\alpha \cap \beta$ is a fuzzy transitive relation. Since

$$\begin{aligned}(\alpha \cap \beta)(x, y) &= \alpha(x, y) \wedge \beta(x, y) \\ &\geq \alpha(x, z) \wedge \alpha(z, y) \wedge \beta(x, z) \wedge \beta(z, y) \\ &= (\alpha \cap \beta)(x, z) \wedge (\alpha \cap \beta)(z, y) \\ &\geq \bigvee_{z \in R} [(\alpha \cap \beta)(x, z) \wedge (\alpha \cap \beta)(z, y)].\end{aligned}$$

It follows from Definition 3.1 that $\alpha \cap \beta$ is a fuzzy transitive relation. Therefore $\alpha \cap \beta$ is a fuzzy equivalence relation of R.

Again, for every $a \in R$, since

$$\begin{aligned}(\alpha \cap \beta)(a + x, a + y) &= \alpha(a + x, a + y) \wedge \beta(a + x, a + y) \\ &\geq \alpha(x, y) \wedge \beta(x, y) \\ &= (\alpha \cap \beta)(x, y),\end{aligned}$$

$$\begin{aligned}(\alpha \cap \beta)(ax, ay) &= \alpha(ax, ay) \wedge \beta(ax, ay) \\ &\geq \alpha(x, y) \wedge \beta(x, y) \\ &= (\alpha \cap \beta)(x, y).\end{aligned}$$

This means $\alpha \cap \beta$ is a fuzzy left compatible relation. Similarly $\alpha \cap \beta$ is a fuzzy right compatible relation. Hence $\alpha \cap \beta$ is a fuzzy congruence relation of R.

(iii) First, we show that $\{\alpha_a \mid a \in I\}$ is an ideal of R/α. For any $\alpha_a, \alpha_b \in \{\alpha_a \mid a \in I\}$, where $a, b \in I$. Since I is an ideal, $a - b \in I$, hence $\alpha_a - \alpha_b = \alpha_{a-b} \in \{\alpha_a \mid a \in I\}$. For any $\alpha_a \in \{\alpha_a \mid a \in I\}, \alpha_x \in R/\alpha$, where $a \in I$ and $x \in R$, then $ax, xa \in I$, hence $\alpha_a \alpha_x = \alpha_{ax} \in \{\alpha_a \mid a \in I\}$ and $\alpha_x \alpha_a = \alpha_{xa} \in \{\alpha_a \mid a \in I\}$. Thus $\{\alpha_a \mid a \in I\}$ is an ideal of R/α.

Next, we define $\varphi : I/\alpha \to R/\alpha$ by

$$(\alpha|I)_a \mapsto \alpha_a \ \text{ for all } \ (\alpha|I)_a \in I/\alpha.$$

It is easy to verify that $I/\alpha \cong \{\alpha_a \mid a \in I\}$ under the mapping. Hence, we may regard I/α as an ideal of R/α in isomorphic sense, completing the proof.

Theorem 4.9 (Fuzzy Second Isomorphism Theorem) *Let α, β be two fuzzy congruence relations of a ring R with $\alpha_0 \subseteq \beta_0$. Then*

$$(R_\alpha + R_\beta)/\beta \cong R_\alpha/\alpha \cap \beta.$$

Proof. By Lemma 4.8, β is a fuzzy congruence relation of $R_\alpha + R_\beta$ and $\alpha \cap \beta$ is a fuzzy congruence relation of R_α. Thus $(R_\alpha + R_\beta)/\beta$ and $R_\alpha/\alpha \cap \beta$ are both rings. For any $x \in R_\alpha + R_\beta$, then $x = a + b$, where $a \in R_\alpha, b \in R_\beta$, it implies $\alpha(0, a) = 1$ and $\beta(0, b) = 1$.

Define $f:(R_\alpha+R_\beta)/\beta\cong R_\alpha/\alpha\cap\beta$ by

$$f(\beta_x)=(\alpha\cap\beta)_a.$$

If $\beta_x=\beta_{x'}$, where $x'=a'+b', a'\in R_\alpha, b'\in R_\beta$, then we have $\alpha(0,a')=1$, $\beta(0,b')$=1 and $\beta(x,x')=\beta(a+b,a'+b')=1$. Since $\alpha(a,a')\geq\alpha(a,0)\wedge\alpha(0,a')=1$, so $\alpha(a,a')=1$. Similarly, $\beta(b,b')=1$. By Definition 4.1 and Lemma 4.7 and $\alpha_0\subseteq\beta_0$, we have $R_\alpha\subseteq R_\beta$. Therefore, $a,a'\in R_\beta$, which implies $\beta(0,a)=1$ and $\beta(0,a')=1$. Since $\beta(a,a')\geq\beta(a,0)\wedge\beta(0,a')=1$, $\beta(a,a')=1$. From Definition 2.3, $(\alpha\cap\beta)(a,a')=\alpha(a,a')\wedge\beta(a,a')=1$, which implies $(\alpha\cap\beta)_a=(\alpha\cap\beta)_{a'}$. Hence f is well-defined.

For any $\beta_x,\beta_y\in(R_\alpha+R_\beta)/\beta$, where $x=a+b$, $y=a_1+b_1$, $a,a_1\in R_\alpha$ and $b,b_1\in R_\beta$, $x+y=(a+a_1)+(b+b_1)$, $xy=(a+a_1)(b+b_1)=aa_1+(ab_1+ba_1+bb_1)=aa_1+b'$, $b'=ab_1+ba_1+bb_1\in R_\beta$. We have $f(\beta_x+\beta_y)=f(\beta_{x+y})=(\alpha\cap\beta)_{a+a_1}=(\alpha\cap\beta)_a+(\alpha\cap\beta)_{a_1}=f(\beta_x)+f(\beta_y)$, $f(\beta_x\beta_y)=f(\beta_{xy})=(\alpha\cap\beta)_{aa_1}=(\alpha\cap\beta)_a(\alpha\cap\beta)_{a_1}=f(\beta_x)f(\beta_y)$. Hence f is a homomorphism.

For any $(\alpha\cap\beta)_a\in R_\alpha/(\alpha\cap\beta)$, taking $b\in R_\beta$, $\exists x=a+b\in R_\alpha+R_\beta$, then $\beta_x\in(R_\alpha+R_\beta)/\beta$ and $f(\beta_x)=(\alpha\cap\beta)_a$. Hence f is an epimorphism.

For any $x,y\in R_\alpha+R_\beta$, where $x=a+b, y=a_1+b_1, a,a_1\in R_\alpha$ and $b,b_1\in R_\beta$, if $(\alpha\cap\beta)_a=(\alpha\cap\beta)_{a_1}$, then $(\alpha\cap\beta)(a,a_1)=1$, i.e, $\alpha(a,a_1)\wedge\beta(a,a_1)=1$, which implies $\alpha(a,a_1)=1$ and $\beta(a,a_1)=1$. Since $b,b_1\in R_\beta$, we have $\beta(0,b)=1$ and $\beta(0,b_1)=1$. Hence $\beta(b,b_1)\geq\beta(b,0)\wedge\beta(0,b')=1$, and $\beta(b,b_1)=1$.

Then

$$\begin{aligned}\beta(a+b,a_1+b_1)&\geq\beta(a+b,a_1+b)\wedge\beta(a_1+b,a_1+b_1)\\&\geq\beta(a,a_1)\wedge\beta(b,b_1)\\&=1.\end{aligned}$$

Thus $\beta(a+b,a_1+b_1)=1,\beta(x,y)=1$, this means $\beta_x=\beta_y$. Therefore f is a monomorphism. Hence, we have shown that$(R_\alpha+R_\beta)/\beta\cong R_\alpha/\alpha\cap\beta$, completing the proof.

Notation 4.10 Let α and β be fuzzy relations of a ring R. We denote $\alpha\leq\beta$, if $\alpha(x,y)\leq\beta(x,y)$ for all $x,y\in R$.

Theorem 4.11 (Fuzzy Third Isomorphism Theorem) *Let α,β be two fuzzy congruence relations of a ring R with $\alpha\leq\beta$. Then*

$$(R/\alpha)/(R_\beta/\alpha)\cong R/\beta.$$

Proof. By Lemma 4.7 and Lemma 4.8, R_β/α is an ideal of R/α.

Define $f:R/\alpha\to R/\beta$ by $f(\alpha_x)=\beta_x$ for all $x\in R$. If $\alpha_x=\alpha_y$, then $\alpha(x,y)=1$. Since $\alpha\leq\beta$, so $\beta(x,y)\geq\alpha(x,y)=1$, thus $\beta(x,y)=1,\beta_x=\beta_y$. Hence f is well-defined.

$f(\alpha_x+\alpha_y)=f(\alpha_{x+y})=\beta_{x+y}=\beta_x+\beta_y=f(\alpha_x)+f(\alpha_y)$,
$f(\alpha_x\alpha_y)=f(\alpha_{xy})=\beta_{xy}=\beta_x\beta_y=f(\alpha_x)f(\alpha_y)$,

they mean f is a homomorphism. For any $\beta_x \in R/\beta$, there exist $\alpha_x \in R/\alpha$ such that $f(\alpha_x) = \beta_x$, so f is an epimorphism.

Now, we show $kerf = R_\beta/\alpha$. In fact, $kerf = \{\alpha_x \in R/\alpha \mid f(\alpha_x) = \beta_0\} = \{\alpha_x \in R/\alpha \mid \beta_x = \beta_0\} = \{\alpha_x \in R/\alpha \mid \beta(0,x) = 1\} = \{\alpha_x \in R/\alpha | x \in R_\beta\} = R_\beta/\alpha$. Therefore, $(R/\alpha)/(R_\beta/\alpha) \cong R/\beta$. The proof is complete.

5. Fuzzy ideal and fuzzy congruence relations

In this section, we discuss the relationships between fuzzy ideals and fuzzy congruence relations on R. In particular, we show that there exists a one-to-one mapping from all fuzzy ideals and all fuzzy congruence relations of R.

Theorem 5.1 *Let μ be a fuzzy ideal of R. Then the fuzzy relation α_μ on R, defined by*

$$\alpha_\mu(a,b) = \mu(a-b)$$

, for all (a,b) of $R \times R$, is a fuzzy congruence relation on R.

Proof. For all $a, b \in R$, then we have $\alpha_\mu(a,a) = \mu(0) = 1$, α_μ is a fuzzy reflexive relation; $\alpha_\mu(a,b) = \mu(a-b)$, $\alpha_\mu(b,a) = \mu(b-a)$, so $\alpha_\mu(a,b) = \alpha_\mu(b,a)$, which means α_μ is a fuzzy symmetric relation;

$$\bigvee_{x\in R}[\alpha_\mu(a,x) \wedge \alpha_\mu(x,b)] = \bigvee_{x\in R}[\mu(a-x) \wedge \mu(x-b)] \le \bigvee_{x\in R}\mu(a-b) = \alpha_\mu(a,b),$$

α_μ is a fuzzy transitive relation. This means that α_μ is a fuzzy equivalence relation on R. For all $a, b, x, y \in R$, since

$$\begin{aligned} \alpha_\mu(x,a) \wedge \alpha_\mu(y,b) &= \mu(x-a) \wedge \mu(y-b) \\ &\le \mu[(x+y)-(a+b)] \\ &= \alpha_\mu(x+y, a+b), \end{aligned}$$

and

$$\begin{aligned} \alpha_\mu(xy, ab) &= \mu(xy-ab) \\ &= \mu[(x-a)y + a(y-b)] \\ &\ge \mu[(x-a)y] \wedge \mu[a(y-b)] \\ &\ge \mu(x-a) \wedge \mu(y-b) \\ &= \alpha_\mu(x,a) \wedge \alpha_\mu(y,b). \end{aligned}$$

We obtain that α_μ is a fuzzy congruence relation on R.

Corollary 5.2 *Let μ and ν be two fuzzy ideals on a ring R, then*

(i) $\alpha_{\mu\cap\nu} = \alpha_\mu \cap \alpha_\nu$,

(ii) $\alpha_{\mu+\nu} = \alpha_\mu \circ \alpha_\nu$.

Proof. (i) For each $x, y \in R$,

$$\begin{aligned}\alpha_{(\mu\cap\nu)}(x,y) &= (\mu\cap\nu)(x-y)\\ &= \mu(x-y)\wedge\nu(x-y)\\ &= \alpha_\mu(x,y)\wedge\alpha_\nu(x,y)\\ &= (\alpha_\mu\cap\alpha_\nu)(x,y).\end{aligned}$$

(ii) For every $x, y \in R$,

$$\begin{aligned}(\alpha_\mu\circ\alpha_\nu)(x,y) &= \bigvee_{z\in R}[\alpha_\mu(x,z)\wedge\alpha_\nu(z,y)]\\ &= \bigvee_{z\in R}[\mu(x-z)\wedge\nu(z-y)]\\ &= (\mu+\nu)(x-y)\\ &= \alpha_{(\mu+\nu)}(x,y).\end{aligned}$$

Hence $\alpha_{\mu+\nu} = \alpha_\mu \circ \alpha_\nu$. The proof is complete.

Theorem 5.3 *Let α be any fuzzy congruence relation on a ring R, then the fuzzy set μ_α of R, defined by*

$$\mu_\alpha(a) = \alpha(0, a)$$

is a fuzzy ideal of R for all $a \in R$.

Proof. For all $a, b \in R$, since α is a fuzzy congruence relation, then we have

$$\begin{aligned}\mu_\alpha(a+b) &= \alpha(0, a+b)\\ &= \alpha(0+0, a+b)\\ &\geq \alpha(0,a)\wedge\alpha(0,b)\\ &= \mu_\alpha(a)\wedge\mu_\alpha(b).\end{aligned}$$

Hence $\mu_\alpha(a+b) \geq \mu_\alpha(a)\wedge\mu_\alpha(b)$. Again, $\mu_\alpha(-a) = \alpha(0,-a) = \alpha(0,a) = \mu_\alpha(a)$ and $\mu_\alpha(ab) = \alpha(0,ab) \geq \alpha(0,a)\wedge\alpha(0,b) = \mu_\alpha(a)\wedge\mu_\alpha(b)$. Hence $\mu_\alpha(ab) \geq \mu_\alpha(a)\wedge\mu_\alpha(b)$. Since $\mu_\alpha(0) = \alpha(0,0) = 1$, μ_α is a fuzzy subring of R. Moreover, $\mu_\alpha(a) = \alpha(0,a) = \alpha(0,a)\wedge\alpha(b,b) \leq \alpha(0,ab) = \mu_\alpha(ab)$, hence $\mu_\alpha(ab) \geq \mu_\alpha(a)$. Similarly $\mu_\alpha(ab) \geq \mu_\alpha(b)$. Therefore, we obtain that μ_α is a fuzzy ideal of R. The proof is complete.

Corollary 5.4 *Let α and β be two fuzzy congruence relations on a ring R, then*

(i) $\mu_{\alpha\cap\beta} = \mu_\alpha \cap \mu_\beta$,

(ii) $\mu_{\alpha\circ\beta} = \mu_\alpha + \mu_\beta$.

Proof. (i) For each $x, y \in R$,

$$\begin{aligned}\mu_{(\alpha\cap\beta)}(x) &= (\alpha\cap\beta)(0,x)\\ &= \alpha(0,x)\wedge\beta(0,x)\\ &= \mu_\alpha(x)\wedge\mu_\beta(x)\\ &= (\mu_\alpha\cap\mu_\alpha)(x).\end{aligned}$$

(ii) Since

$$\begin{aligned}\mu_{(\alpha\circ\beta)}(x) &= (\alpha\circ\beta)(0,x)\\ &= \bigvee_{y\in R}[\alpha(0,y)\wedge\beta(y,x)]\\ &= \bigvee_{z\in R}[\mu_\alpha(y)\wedge\mu_\beta(x-y)]\\ &= (\mu_\alpha+\mu_\alpha)(x).\end{aligned}$$

This completes the proof.

Proposition 5.5 *Let R be a ring, then there exists a one-to-one mapping from $FI(R)$ onto $FC(R)$.*

Proof. We define a mapping φ from $FI(R)$ to $FC(R)$ as follows:

$$\varphi(\mu) = \alpha_\mu$$

for every μ of $FI(R)$. By Theorem 5.1, φ is well-defined. If $\alpha_\mu = \alpha_\nu$, then $\alpha_\mu(a,b) = \alpha_\nu(a,b)$, for all $(a,b) \in R\times R$, so $\mu(a-b) = \nu(a-b)$, hence $\mu = \nu$, this means φ is injective. For each $\alpha \in FC(R)$, $a, b \in R$, by Theorem 5.3, $\varphi(\mu_\alpha)(a,b) = \alpha_{\mu_\alpha}(a,b) = \mu_\alpha(a-b) = \alpha(0,a-b) = \alpha(a,b)$. Thus φ is surjective. Consequently, φ is a one-to-one mapping from $FI(R)$ onto $FC(R)$.

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