THE TRIPARTITE RAMSEY NUMBERS $r_t(C_4; 2)$ AND $r_t(C_4; 3)$

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Abstract. The k-colored tripartite Ramsey numbers $r_t(G; k)$ is the smallest positive integer n such that any k-coloring of lines of a complete tripartite graph $K_{n,n,n}$ there always exists a monochromatic subgraph isomorphic to G. When G is C_4 it is known, but unpublished in a journal, that $r_t(C_4; 2) = 3$. In this paper we simplify the proof of $r_t(C_4; 2) = 3$ and show the new result that $r_t(C_4; 3) = 7$.

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1. Introduction

A graph G is *n-partite*, $n \geq 1$, if it is possible to partition $V(G)$ into n subsets $V_1, V_2, \ldots, V_n$ (called partite sets) such that every element of $E(G)$ joins a vertex of V_i to a vertex of $V_j, i \neq j$. For $n = 2$, such graphs are called *bipartite graphs*. For $n = 3$, such graphs are called *tripartite graphs*. A *complete n-partite G* is an *n-partite* graph with partite sets $V_1, V_2, \ldots, V_n$ having the added property that if $u \in V_i$ and $v \in V_j, i \neq j$, then $uv \in E(G)$. When $|V_i| = p_i$, we denote the complete n-partite graph by $K_{p_1,p_2,\ldots,p_n}$.

Consider a complete bipartite graph $K_{s,s}$ of order $p = 2s$. Let each line of $K_{s,s}$ be colored by using either red or blue color. We shall call such a $K_{s,s}$ 2-colored.

Consider a subgraph $K_{m,n}$ of 2-colored $K_{s,s}$. If all lines of $K_{m,n}$ have red(blue) color, we shall say that the $K_{s,s}$ contains a red(blue) $K_{m,n}$. The smallest number s of points such that $K_{s,s}$ always contains red $K_{m,n}$ or blue $K_{m,n}$ is called bipartite Ramsey number and denoted by $r_b(K_{m,n}; 2)$ or $r_b(K_{m,n}, K_{m,n})$.

According to the definition of bipartite Ramsey numbers in this paper, V. Longani [7], has found that

$$r_b(K_{1,n}, K_{1,n}) = 2n - 1 \ (n = 1, 2, 3, ...),$$
$$r_b(K_{2,2}, K_{2,2}) = 5,$$
$$r_b(K_{2,3}, K_{2,3}) = 9,$$

and L.W. Beineke and A.J. Schwenk [1], have also found that

$$r_b(K_{2,2}, K_{2,2}) = 5,$$
$$r_b(K_{3,3}, K_{3,3}) = 17.$$

The 3-colored bipartite Ramsey number $r_b(G; 3)$ is the smallest integer n such that any 3-coloring of lines of a complete bipartite graph $K_{n,n}$ there always exists monochromatic subgraph isomorphic to G. In [4], W. Goddard, M.A. Henning, and O.R. Hellermann showed that $r_b(C_4; 3) = 11$.

Consider a complete tripartite graph $K_{s,s,s}$. Let each line of the $K_{s,s,s}$ be colored by using one of k colors. We call such a $K_{s,s,s}$ as k-colored. For 2-coloring, the smallest number s of points such that the $K_{s,s,s}$ always contains monochromatic $K_{m,n}$ is called tripartite Ramsey number $r_t(K_{m,n}; 2)$ or $r_t(K_{m,n}, K_{m,n})$.

In [5], K. Leamyoo and V. Longani, have found that $r_t(K_{2,2}, K_{2,2}) = 3$ (or $r_t(C_4; 2) = 3$). Also in [2], [3], S. Buada and V. Longani, have shown that $r_t(K_{2,3}, K_{2,3}) = 5$, $r_t(K_{2,4}, K_{2,4}) = 7$.

For 3-coloring, the tripartite Ramsey number of the graph G, denoted by $r_t(G; 3)$ (or $r_t(G, G, G)$) is the minimum integer n such that for any 3-coloring of the lines of $K_{n,n,n}$ there always exists monochromatic subgraph G.

2. Main results

The proof that $r_t(C_4; 2) = 3$ in [5] is rather lengthy and has not been published in a journal. In this section we provide a new shorter proof that $r_t(C_4; 2) = 3$ and prove a new result that $r_t(C_4; 3) = 7$.

For a $K_{3,3,3}$, consider all twenty seven lines that are adjacent to all points of a V_i. We shall call the lines that are adjacent to V_i as the lines of V_i. Before proving Theorem 1, we prove Lemma 1 first.

Lemma 1. *Let $K_{3,3,3}$ be a 2-colored complete tripartite graph and each V_1, V_2, and V_3 be the set of three non-adjacent points of the $K_{3,3,3}$. There exists at least one V_i of which the numbers of red lines and blue lines of the V_i are not equal.*

Proof. Consider the three V_i's. Suppose there are nine red lines and nine blue lines of each V_i.

Since there are totally nine red lines of V_1, consider when there are n ($n \geq 0$) red lines which join points of V_1 and V_2, and so there are $9 - n$ red lines which join points of V_1 and V_3. Since for V_2 there are also exactly nine red lines of V_2, therefore there are $9 - n$ red lines which join points of V_2 and V_3.

Now we can see that there are $(9 - n) + (9 - n)$ red lines of V_3. Since there are exactly nine red lines of V_3, therefore

$$\begin{aligned}(9 - n) + (9 - n) &= 9 \\ n &= 4.5.\end{aligned}$$

This is not possible. Therefore, there exist some V_i's of which the numbers of red lines and blue lines of the V_i's are not equal. ■

In order to prove Theorem 1 and Theorem 2, it would be convenient to represent a 2-colored $K_{m,n}$ by an $m \times n$ matrix $B = [b_{ij}]$.

Given a 2-colored $K_{m,n}$ with V_1 and V_2 as its partite sets of size m and n, respectively. Let $V_1 = \{r_1, r_2, \ldots, r_m\}$ and $V_2 = \{c_1, c_2, \ldots, c_n\}$. For the corresponding $B = [b_{ij}]$ of the $K_{m,n}$, we let $b_{ij} = 1$ if the line $r_i c_j$ is red, and let $b_{ij} = 0$ if the line $r_i c_j$ is blue. For example, the following (a) and (b) in Figure 2.1 illustrate the 2-colored $K_{4,5}$ and its corresponding matrix B. Here, we use dark lines to indicate red lines and dash lines to indicate blue lines.

c_1 c_2 c_3 c_4 c_5 V_1

(a)

V_2

r_1 r_2 r_3 r_4

(b)

$$\begin{array}{c} \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} \begin{array}{c} \begin{array}{ccccc} c_1 & c_2 & c_3 & c_4 & c_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Figure 2.1

Theorem 1. $r_t(C_4; 2) = 3$

Proof. Consider the 2-colored $K_{2,2,2}$ graph illustrated in Figure 2.2.

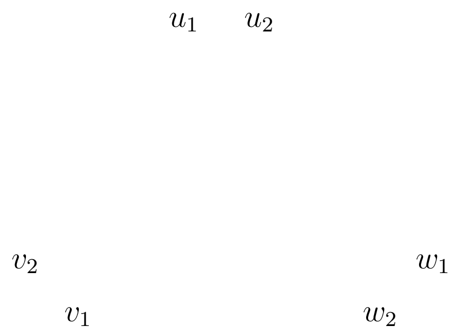


Figure 2.2

It can be seen that the $K_{2,2,2}$ contains neither red C_4 nor blue C_4. Therefore $r_t(C_4; 2) > 2$. That is

$$r_t(C_4; 2) \geq 3 \tag{2.1}$$

Let $K_{3,3,3}$ be a 2-colored complete tripartite graph. Consider the set V_1, V_2 and V_3 of three non-adjacent points of the $K_{3,3,3}$

$$\begin{aligned} V_1 &= \{u_1, u_2, u_3\} \\ V_2 &= \{v_1, v_2, v_3\} \\ V_3 &= \{w_1, w_2, w_3\}. \end{aligned}$$

From Lemma 1, we can assume that from V_1, the numbers of red lines are greater than the numbers of blue lines, that is the numbers of red lines are equal to ten or greater. We only need to consider the case when the numbers of red lines of V_1 is ten and show that in such case the $K_{3,3,3}$ always contain red C_4. For the cases when the number of red lines is greater than ten, the results follow immediately.

Let $V(G_1) = V_1$ and $V(G_2) = V_2 \cup V_3$. For $V(G_1)$, let u_1, u_2, u_3 be respectively replaced by r_1, r_2, r_3. Also for $V(G_2)$, let v_1, v_2, v_3, w_1, w_2, w_3 be respectively replaced by c_1, c_2, c_3, c_4, c_5, c_6. That is,

$$\begin{aligned} V(G_1) &= \{r_1, r_2, r_3\} \\ V(G_2) &= \{c_1, c_2, c_3, c_4, c_5, c_6\}. \end{aligned}$$

By ignoring the lines between V_2 and V_3 and consider the defined $V(G_1)$ and $V(G_2)$. The $K_{3,3,3}$ is now reduced to 2-colored $K_{3,6}$. In order to prove the theorem we only need to show that this $K_{3,6}$ always contains red C_4.

We find the value of $r_t(C_4; 2)$ by considering the 2-colored $K_{3,6}$. If there are $m, n, s, t, u (1 \leq m, n \leq 3$ and $1 \leq s, t, u \leq 6)$ such that some submatrices

$$\begin{bmatrix} b_{ms} & b_{mt} \\ b_{ns} & b_{nt} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{2.2}$$

then the $K_{3,6}$ contains red C_4.

Let d_1, d_2, d_3 be degrees of red lines of r_1, r_2, r_3 respectively. We can choose r_i's such that $d_1 \geq d_2 \geq d_3$. Here we have the conditions that $d_1 + d_2 + d_3 = 10$ and $0 \leq d_i \leq 6$, $i = 1, 2, 3$. Next, we consider two main cases.

Case 1. $d_1 + d_2 \geq 8$.

Here, the possible $d_1 \geq d_2$ are $4 \geq 4$, $5 \geq 3$, $5 \geq 4$, $5 \geq 5$, $6 \geq 3$, $6 \geq 4$, $6 \geq 5$, $6 \geq 6$. It is easy to show that for all of these cases the $K_{3,6}$ always contains red C_4. Consider cases $d_1 = 4$, and $d_2 = 4$, for example. For a case in Table 1, parts of the matrix involving r_1 and r_2 could be

	c_1	c_2	c_3	c_4	c_5	c_6
r_1	1	1	**1**	**1**		
r_2			**1**	**1**	1	1

Table 1:

from which submatrix of the form (2.5) always appears; that is the $K_{3,6}$ contains red C_4.

Case 2. $d_1 + d_2 < 8$.

With the conditions for d_i, there is only one subcase to consider $d_1 = 4$, $d_2 = 3$, $d_3 = 3$. We consider two more possibilities: Subcase 2.1 and Subcase 2.2.

Subcase 2.1. When there are two or more points of c_i's each of which is joined by red lines to both of r_1 and r_2, see Table 2, then we see that the $K_{3,6}$ contains red C_4.

	c_1	c_2	c_3	c_4	c_5	c_6
r_1	1	1	**1**	**1**		
r_2			**1**	**1**	1	

Table 2:

Subcase 2.2. One point of c_i's is joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by four red lines to c_1, c_2, c_3, c_4 and r_2 is joined by three red lines to c_4, c_5, c_6. Consider the three red lines joining r_3. Either at least two of three red lines are joined, from r_3, to some points among c_1, c_2, c_3, c_4 or at least two of these three red lines are joined to some points among c_4, c_5, c_6. In either case, we see that red C_4 is contained in the $K_{3,6}$, see Table 3 for example.

	c_1	c_2	c_3	c_4	c_5	c_6
r_1	**1**	**1**	1	1		
r_2				1	1	1
r_3	**1**	**1**				1

Table 3:

Hence the $K_{3,3,3}$ will always contain red C_4. Therefore,

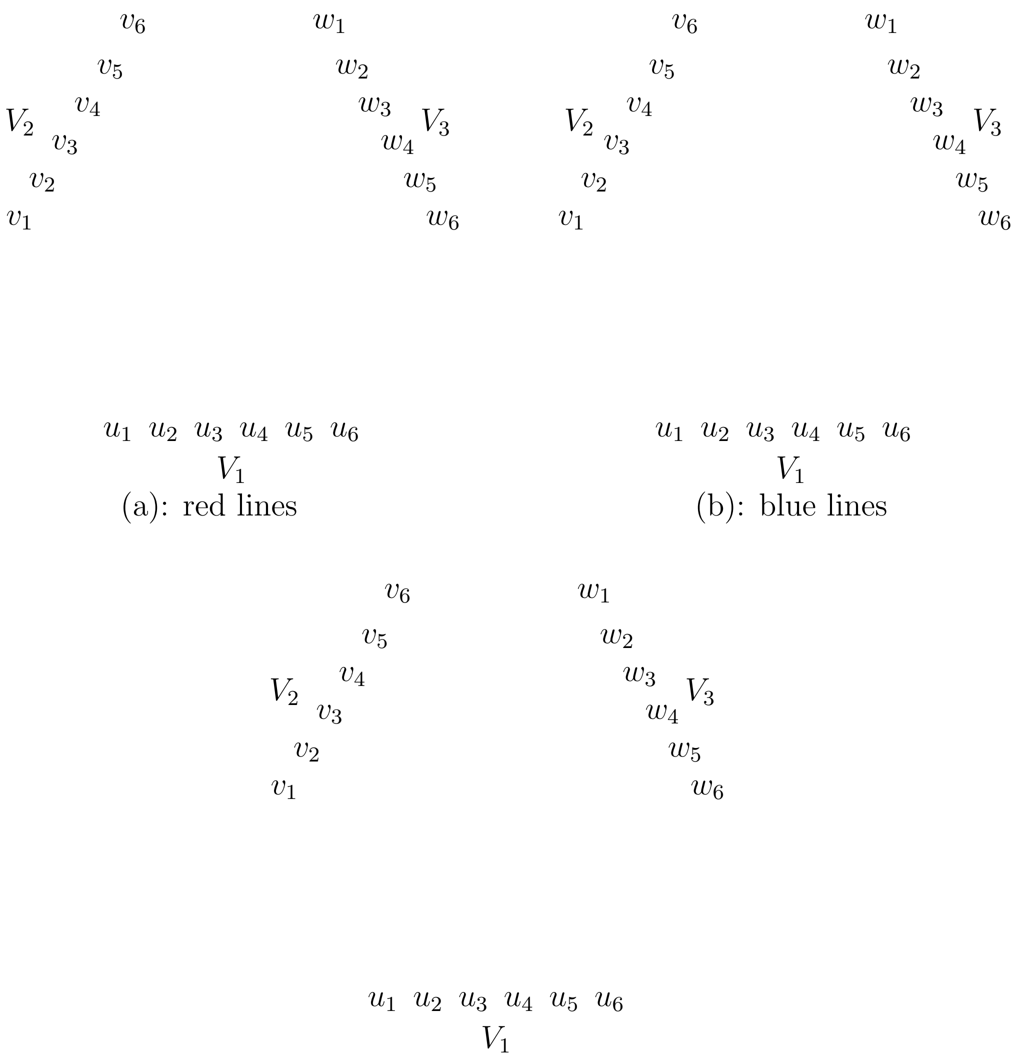
$$r_t(C_4; 2) \leq 3. \tag{2.3}$$

By (2.1) and (2.3), we have $r_t(C_4;2) = 3$ as required. ∎

Next, we show that $r_t(C_4;3) = 7$.

Theorem 2. $r_t(C_4;3) = 7$.

Proof. Consider the subgraphs of a $K_{6,6,6}$ illustrated in Figure 2.3. Here, (a), (b), and (c) represent subgraphs of the 3-colored $K_{6,6,6}$ with red, blue, and green lines, respectively.



(a): red lines

(b): blue lines

(c): green lines

Figure 2.3

It can be verified that the $K_{6,6,6}$ contains no monochromatic C_4. Therefore, $r_t(C_4;3) > 6$. That is,

$$r_t(C_4;3) \geq 7. \tag{2.4}$$

Let $K_{7,7,7}$ be a 3-colored complete tripartite graph with $p = 21$. Consider the set V_1, V_2 and V_3 of seven non-adjacent points of the $K_{7,7,7}$:

$$V_1 = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\},$$
$$V_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\},$$
$$V_3 = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}.$$

We shall call the lines that are adjacent to V_i as the lines of V_i. Since there are totally ninety eight lines of V_1, there are at least thirty three lines of V_1 with the same, say red, color. Consider thirty three of these red lines.

Let $V(G_1) = V_1$ and $V(G_2) = V_2 \cup V_3$. For $V(G_1)$, let u_1, u_2, u_3, u_4, u_5, u_6, u_7 be respectively replaced by r_1, r_2, r_3, r_4, r_5, r_6, r_7. Also for $V(G_2)$, let v_1, v_2, v_3, v_4, v_5, v_6, v_7, w_1, w_2, w_3, w_4, w_5, w_6, w_7 be respectively replaced by c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}. That is,

$$V(G_1) = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7\}$$
$$V(G_2) = \{c_1, c_2, \dots, c_{14}\}.$$

By ignoring the lines between V_2 and V_3 and consider the defined $V(G_1)$ and $V(G_2)$ the $K_{7,7,7}$ is now reduced to 3-colored $K_{7,14}$. In order to prove the theorem we only need to show that this $K_{7,14}$ always contains red C_4.

We find an upper bound of $r_t(C_4; 3)$ by considering the 3-colored $K_{7,14}$.

If there are $m, n, s, t (1 \leq m, n \leq 7$ and $1 \leq s, t \leq 14)$ such that some submatrices

$$\begin{bmatrix} b_{ms} & b_{mt} \\ b_{ns} & b_{nt} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{2.5}$$

then the $K_{7,14}$ contains red $K_{2,2}$ or C_4.

Let d_1, d_2, d_3, d_4, d_5, d_6, d_7 be degrees of red lines of r_1, r_2, r_3, r_4, r_5, r_6, r_7 respectively. We can choose r_i's such that $d_1 \geq d_2 \geq d_3 \geq d_4 \geq d_5 \geq d_6 \geq d_7$. Here we have the conditions that

$$d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 = 33$$

and $0 \leq d_i \leq 14$, $i = 1, 2, 3, 4, 5, 6, 7$.

Next, we consider two main cases: Case 1 and Case 2.

Case 1. $d_1 + d_2 + d_3 \geq 18$.

Here, the possible $d_1 \geq d_2 \geq d_3$ are $6 \geq 6 \geq 6$, $7 \geq 6 \geq 5$, $7 \geq 6 \geq 6$, $7 \geq 7 \geq 5$, $7 \geq 7 \geq 6$, $7 \geq 7 \geq 7$, $8 \geq 5 \geq 5$, $8 \geq 6 \geq 5$, $8 \geq 6 \geq 6$, $8 \geq 7 \geq 5$, $8 \geq 7 \geq 6$, $8 \geq 7 \geq 7$, $8 \geq 8 \geq 6$, $8 \geq 8 \geq 6$, $8 \geq 8 \geq 7$, $8 \geq 8 \geq 8$, $9 \geq 5 \geq 5$, $9 \geq 6 \geq 5$, $9 \geq 6 \geq 6$, $9 \geq 7 \geq 5$, $9 \geq 7 \geq 6$, $9 \geq 7 \geq 7$, $9 \geq 8 \geq 5$, $10 \geq 5 \geq 5$, $10 \geq 6 \geq 5$, $10 \geq 6 \geq 6$, $10 \geq 7 \geq 5$, $11 \geq 5 \geq 5$, $11 \geq 6 \geq 5$, $11 \geq 6 \geq 5$, $12 \geq 5 \geq 5$.

It is easy to show that for all of these cases the $K_{7,14}$ always contains red C_4.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	1								
r_2						1	1	1	1	**1**	**1**			
r_3	1									**1**	**1**	1	1	1

Table 4:

Consider the cases $d_1 = 6$, $d_2 = 6$ and, $d_3 = 6$, for example. For a case in Table 4, parts of the matrix involving r_1, r_2 and, r_3 could be
from which submatrix of the form (2.2) always appears, that is the $K_{7,14}$, and so the $k_{7,7,7}$ contains red C_4.

Case 2. $d_1 + d_2 + d_3 < 18$.

With the conditions for d_i, there are only four subcases to consider.

Subcase 2.1. $d_1 = 7, d_2 = 5, d_3 = 5, d_4 = 4, d_5 = 4, d_6 = 4, d_7 = 4$.

When there are two or more points c_i's each of which is joined to both of r_1 and r_2 by red lines, we can see that the $K_{7,14}$ contains red C_4, see Table 5 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	1	**1**	**1**							
r_2						**1**	**1**	1	1	1				

Table 5:

We consider two more possibilities 2.1.1 and 2.1.2.

2.1.1. One c_i is joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by seven red lines to $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ and r_2 is joined by five red lines to c_7, c_8, c_9, c_{10}, c_{11}. We now consider four sub-possibilities (1), (2), (3), and (4).

(1) c_{12}, c_{13}, c_{14} are not joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

If c_{12}, c_{13}, c_{14} are not joined to r_3 for example, then either at least three red lines from r_3 are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least three red lines from r_3 are joined to points among c_8, c_9, c_{10}, c_{11}. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 6 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1	1	1							
r_2							1	1	1	1	1			
r_3	**1**	**1**	1					1	1					

Table 6:

(2) One of c_{12}, c_{13}, c_{14} is joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

For example, suppose r_3 is joined to c_{12} by red line, then there are four other red lines joining r_3. From these four red lines, either at least two red lines from r_3 are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least two red lines from r_3 are joined to points among c_8, c_9, c_{10}, c_{11}. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 7 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1	1	1							
r_2							1	1	1	1	1			
r_3	**1**	**1**								1	1	1		

Table 7:

(3) Two of c_{12}, c_{13}, c_{14} are joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

For example, suppose r_3 is joined to c_{12}, c_{13} by red lines, then there are three other red lines joining r_3. From these three red lines, then either at least two red lines from r_3 are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least two red lines from r_3 are joined to points among c_8, c_9, c_{10}, c_{11}. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 8 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1	1	1							
r_2							1	1	1	1	1			
r_3	**1**	**1**									1	1	1	

Table 8:

(4) All c_{12}, c_{13}, c_{14} are joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

Suppose r_3 is joined to c_{12}, c_{13}, and c_{14}. Consider r_4 and the four red lines joining r_4. From these four red lines, either at least two red lines are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least two red lines are joined to points among c_8, c_9, c_{10}, c_{11} or at least two red lines are joined to points among c_{12}, c_{13}, c_{14}. In any case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 9 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	**1**	**1**	1	1	1	1							
r_2							1	1	1	1	1			
r_3	1										1	1	1	1
r_4		**1**	**1**							1				1

Table 9:

2.1.2. None of c_i's are joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by seven red lines to $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ and r_2 is joined by five red lines to c_8, c_9, c_{10}, c_{11}, c_{12}. We now consider three sub-possibilities (1), (2), and (3).

(1) c_{13}, c_{14} are not joined to some r_i's ($i = 3, 4, 5, 6, 7$) by red lines.

If c_{13}, c_{14} are not joined to r_3 for example, then either at least three red lines from r_3 are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least three red lines from r_3 are joined to points among c_8, c_9, c_{10}, c_{11}, c_{12}. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 10 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	**1**	1	1	1	1							
r_2								1	1	1	1	1		
r_3	**1**	**1**	**1**					1	1					

Table 10:

(2) One of c_{13}, c_{14} is joined to some r_i's ($i = 3, 4, 5, 6, 7$) by red lines.

For example, suppose r_3 is joined to c_{13} by red line, then there are four other red lines joining r_3. From these four red lines, either at least two red lines are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least two red lines are joined to points among c_8, c_9, c_{10}, c_{11}, c_{12}. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 11 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1	1	1							
r_2								1	1	1	1	1		
r_3	**1**	**1**									1	1	1	

Table 11:

(3) Both of c_{13}, c_{14} are joined to some r_i's ($i = 3, 4, 5, 6, 7$) by red lines.

Suppose r_3 is joined to c_{13}, and c_{14}, then there are three other red lines joining r_3. From these three red lines, either at least two red lines are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least two red lines are joined to points among c_8, c_9, c_{10}, c_{11}, c_{12}. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 12 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1	1	1							
r_2								1	1	1	1	1		
r_3	**1**	**1**										1	1	1

Table 12:

Subcase 2.2. $d_1 = 6, d_2 = 6, d_3 = 5, d_4 = 4, d_5 = 4, d_6 = 4, d_7 = 4$.

When there are two or more points c_i's each of which is joined to both of r_1 and r_2 by red lines, then we can see that the $K_{7,14}$ contains red C_4, see Table 13 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	**1**	**1**								
r_2					**1**	**1**	1	1	1	1				

Table 13:

We consider two more possibilities 2.2.1 and 2.2.2.

2.2.1. One c_i is joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by six red lines to $c_1, c_2, c_3, c_4, c_5, c_6$ and r_2 is joined by six red lines to c_6, c_7, c_8, c_9, c_{10}, c_{11}. Similar to the cases in 2.1.1 of subcase 2.1, we have that the $K_{7,14}$ contains red C_4.

2.2.2. None of c_i's are joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by six red lines to $c_1, c_2, c_3, c_4, c_5, c_6$ and r_2 is joined by six red lines to c_7, c_8, c_9, c_{10}, c_{11}, c_{12}. Similar to the cases in 2.1.2 of subcase 2.1, we have that the $K_{7,14}$ contains red C_4.

Subcase 2.3. $d_1 = 6, d_2 = 5, d_3 = 5, d_4 = 5, d_5 = 4, d_6 = 4, d_7 = 4$.

When there are two or more points c_i's each of which is joined to both of r_1 and r_2 by red lines, then we can see that the $K_{7,14}$ contains red C_4, see Table 14 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	1	**1**	**1**								
r_2					**1**	**1**	1	1	1					

Table 14:

We consider two more possibilities 2.3.1 and 2.3.2.

2.3.1. One c_i is joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by six red lines to $c_1, c_2, c_3, c_4, c_5, c_6$ and r_2 is joined by five red lines to c_6, c_7, c_8, c_9, c_{10}. We now consider five sub-possibilities (1), (2), (3), (4), and (5).

(1) c_{11}, c_{12}, c_{13}, c_{14} are not joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

If c_{11}, c_{12}, c_{13}, c_{14} are not joined to r_3 for example, then either at least three red lines from r_3 are joined to points among c_1, c_2, c_3, c_4, c_5 or at least three red lines are joined to points among c_6, c_7, c_8, c_9, c_{10}. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 15 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1	1								
r_2						1	1	1	1	1				
r_3	**1**	**1**	1						1	1				

Table 15:

(2) One of c_{11}, c_{12}, c_{13}, c_{14} is joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

For example, suppose r_3 is joined to c_{11} by red line, then there are four other red lines joining r_3. From these four red lines, then either at least two red lines from r_3 are joined to points among c_1, c_2, c_3, c_4, c_5 or at least two red lines are joined to points among c_6, c_7, c_8, c_9, c_{10}. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 16 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1	1								
r_2						1	1	1	1	1				
r_3	**1**	**1**							1	1	1			

Table 16:

(3) Two of c_{11}, c_{12}, c_{13}, c_{14} are joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

For example, suppose r_3 is joined to c_{11}, c_{12} by red lines, then there are three other red lines joining r_3. From these three red lines, then either at least two red lines from r_3 are joined to points among c_1, c_2, c_3, c_4, c_5 or at least two red lines are joined to points among c_6, c_7, c_8, c_9, c_{10}. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 17 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1	1								
r_2						1	1	1	1	1				
r_3	**1**	**1**								1	1	1		

Table 17:

(4) Three of c_{11}, c_{12}, c_{13}, c_{14} are joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

Suppose r_3 is joined to c_{11}, c_{12}, c_{13} by red lines. Consider r_4 and the five red lines joining r_4. From these five red lines, if r_4 is joined to c_{14}, then either at least two red lines are joined to points among c_1, c_2, c_3, c_4, c_5 or at least two red lines are joined to points among c_6, c_7, c_8, c_9, c_{10} or at least two red lines are joined to points among c_{11}, c_{12}, c_{13}. In any case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 18 for example. For the case when r_4 is not joined to c_{14}, we have that red C_4 is also formed.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	**1**	**1**	1	1	1								
r_2						1	1	1	1	1				
r_3	1									1	1	1	1	
r_4		**1**	**1**				1						1	1

Table 18:

(5) All of c_{11}, c_{12}, c_{13}, c_{14} are joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

Suppose r_3 is joined to c_{11}, c_{12}, c_{13}, c_{14} by red lines. Consider r_4 and the five red lines joining r_4. From these five red lines, then either at least two red lines are joined to points among c_1, c_2, c_3, c_4, c_5 or at least two red lines are joined to points among c_6, c_7, c_8, c_9, c_{10} or at least two red lines are joined to points among c_{11}, c_{12}, c_{13}, c_{14}. In any case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 19 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	**1**	**1**	1	1	1								
r_2						1	1	1	1	1				
r_3	1										1	1	1	1
r_4		**1**	**1**				1						1	1

Table 19:

2.3.2. None of c_i's are joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by six red lines to $c_1, c_2, c_3, c_4, c_5, c_6$ and r_2 is joined by five red lines to c_7, c_8, c_9, c_{10}, c_{11}. Similar to the cases in 2.1.1 of subcase 2.1, we have that the $K_{7,14}$ contains red C_4.

Subcase 2.4. $d_1 = 5, d_2 = 5, d_3 = 5, d_4 = 5, d_5 = 5, d_6 = 4, d_7 = 4.$

When there are two or more points c_i's each of which is joined to both of r_1 and r_2 by red lines, we can see that the $K_{7,14}$ contains red C_4, see Table 20 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	1	**1**	**1**									
r_2				**1**	**1**	1	1	1						

Table 20:

We consider two more possibilities 2.4.1 and 2.4.2.

2.4.1. One c_i is joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by five red lines to c_1, c_2, c_3, c_4, c_5 and r_2 is joined by five red lines to c_5, c_6, c_7, c_8, c_9. We now consider six sub-possibilities (1), (2), (3), (4), (5), and (6).

(1) $c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$ are not joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

If $c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$ are not joined to r_3 for example, then either at least three red lines from r_3 are joined to points among c_1, c_2, c_3, c_4 or at least three red lines are joined to points among c_5, c_6, c_7, c_8, c_9. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 21 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1									
r_2					1	1	1	1	1					
r_3	**1**	**1**	1					1	1					

Table 21:

(2) One of $c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$ is joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

For example, suppose r_3 is joined to c_{10} by red line, then there are four other red lines joining r_3. From these four red lines, then either at least two red lines from r_3 are joined to points among c_1, c_2, c_3, c_4 or at least two red lines are joined to points among c_5, c_6, c_7, c_8, c_9. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 22 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1									
r_2					1	1	1	1	1					
r_3	**1**	**1**						1	1	1				

Table 22:

(3) Two of $c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$ are joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

For example, suppose r_3 is joined to c_{10}, c_{11} by red lines, then there are three other red lines joining r_3. From these three red lines, then either at least two red lines from r_3 are joined to points among c_1, c_2, c_3, c_4 or at least two red lines are joined to points among c_5, c_6, c_7, c_8, c_9. In either case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 23 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1									
r_2					1	1	1	1	1					
r_3	**1**	**1**							1	1	1			

Table 23:

(4) Three of c_{10}, c_{11}, c_{12}, c_{13}, c_{14} are joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

Suppose r_3 is joined to c_{10}, c_{11}, c_{12} by red lines. Consider r_4 and the five red lines joining r_4. From these five red lines, if r_4 is not joined to c_{13} and c_{14}, then either at least two red lines are joined to points among c_1, c_2, c_3, c_4 or at least two red lines are joined to points among c_5, c_6, c_7, c_8, c_9 or at least two red lines are joined to points among c_{10}, c_{11}, c_{12}. In any case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 24 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	**1**	**1**	1	1									
r_2					1	1	1	1	1					
r_3	1								1	1	1	1		
r_4		**1**	**1**			1	1			1				

Table 24:

If r_4 is joined to one of c_{13} and c_{14}, say c_{13}, then there are four other red lines joining r_4. From these four red lines, either at least two red lines are joined to points among c_1, c_2, c_3, c_4 or at least two red lines are joined to points among c_5, c_6, c_7, c_8, c_9 or at least two red lines are joined to points among c_{10}, c_{11}, c_{12}. In any case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 25 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	**1**	**1**	1	1									
r_2					1	1	1	1	1					
r_3	1								1	1	1	1		
r_4		**1**	**1**			1				1			1	

Table 25:

If r_4 is joined to c_{13} and c_{14}, then consider r_5 and the five red lines joining r_5. From these five red lines, then either at least two red lines are joined to points among c_1, c_2, c_3, c_4 or at least two red lines are joined to points among c_5, c_6, c_7, c_8, c_9 or at least two red lines are joined to points among c_{10}, c_{11}, c_{12} or at least two red lines are joined to points c_{13} and c_{14}. In any case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 26 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	1	**1**	**1**	1									
r_2					1	1	1	1	1					
r_3	1								1	1	1	1		
r_4		1				1				1			1	1
r_5			**1**	**1**			1				1		1	

Table 26:

(5) Four of c_{10}, c_{11}, c_{12}, c_{13}, c_{14} are joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

Suppose r_3 is joined to c_{10}, c_{11}, c_{12}, c_{13} by red lines. Consider r_4 and the five red lines joining r_4. From these five red lines, if r_4 is not joined to c_{14}, then either at least two red lines are joined to points among c_1, c_2, c_3, c_4 or at least two red lines are joined to points among c_5, c_6, c_7, c_8, c_9 or at least two red lines are joined to points among c_{10}, c_{11}, c_{12}, c_{13}. In any case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 27 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	**1**	**1**	1	1									
r_2					1	1	1	1	1					
r_3	1									1	1	1	1	
r_4		**1**	**1**			1				1	1			

Table 27:

If r_4 is joined to c_{14}, then there are four other red lines joining r_4. From these four red lines,then either at least two red lines are joined to points among c_1, c_2, c_3, c_4 or at least two red lines are joined to points among c_5, c_6, c_7, c_8, c_9 or at least two red lines are joined to points among c_{10}, c_{11}, c_{12}, c_{13}. In any case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 28 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	1	**1**	**1**	1	1									
r_2					1	1	1	1	1					
r_3	1									1	1	1	1	
r_4		**1**	**1**			1				1				1

Table 28:

(6) All of c_{10}, c_{11}, c_{12}, c_{13}, c_{14} are joined to some r_i's $(i = 3, 4, 5, 6, 7)$ by red lines.

Suppose r_3 is joined to c_{10}, c_{11}, c_{12}, c_{13}, c_{14} by red lines. Consider r_4 and the five red lines joining r_4. From these five red lines, then either at least two red lines are joined to points among c_1, c_2, c_3, c_4 or at least two red lines are joined to points among c_5, c_6, c_7, c_8, c_9 or at least two red lines are joined to points among c_{10}, c_{11}, c_{12}, c_{13}, c_{14}. In any case, we can see that red C_4 is contained in the $K_{7,14}$, see Table 29 for example.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
r_1	**1**	**1**	1	1	1									
r_2					1	1	1	1	1					
r_3										1	1	1	1	1
r_4	**1**	**1**				1				1	1			

Table 29:

2.4.2. None c_i's are joined by red lines to both of r_1 and r_2.

Suppose that r_1 is joined by five red lines to c_1, c_2, c_3, c_4, c_5 and r_2 is joined by five red lines to c_6, c_7, c_8, c_9, c_{10}.

Similar to the cases in 2.3.1 of subcase 2.3, red C_4 is always contained in the $K_{7,14}$.

Therefore,

$$r_t(C_4; 3) \leq 7. \tag{2.6}$$

From the inequalities (2.4) and (2.6), we have $r_t(C_4; 3) = 7$ as required. ∎

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