THE TRIPARTITE RAMSEY NUMBERS $r_t(C_4; 2)$ AND $r_t(C_4; 3)$

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Abstract. The $k$-colored tripartite Ramsey numbers $r_t(G; k)$ is the smallest positive integer $n$ such that any $k$-coloring of lines of a complete tripartite graph $K_{n,n,n}$ there always exists a monochromatic subgraph isomorphic to $G$. When $G$ is $C_4$ it is known, but unpublished in a journal, that $r_t(C_4; 2) = 3$. In this paper we simplify the proof of $r_t(C_4; 2) = 3$ and show the new result that $r_t(C_4; 3) = 7$.

Keywords and phrases: tripartite Ramsey numbers, bipartite Ramsey numbers, Ramsey numbers, tripartite graphs, bipartite graphs.

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1. Introduction
A graph \( G \) is \( n \)-partite, \( n \geq 1 \), if it is possible to partition \( V(G) \) into \( n \) subsets \( V_1, V_2, \ldots, V_n \) (called partite sets) such that every element of \( E(G) \) joins a vertex of \( V_i \) to a vertex of \( V_j \), \( i \neq j \). For \( n = 2 \), such graphs are called bipartite graphs. For \( n = 3 \), such graphs are called tripartite graphs. A complete \( n \)-partite \( G \) is an \( n \)-partite graph with partite sets \( V_1, V_2, \ldots, V_n \) having the added property that if \( u \in V_i \) and \( v \in V_j, i \neq j \), then \( uv \in E(G) \). When \( |V_i| = p_i \), we denote the complete \( n \)-partite graph by \( K_{p_1,p_2,\ldots,p_n} \).

Consider a complete bipartite graph \( K_{s,s} \) of order \( p = 2s \). Let each line of \( K_{s,s} \) be colored by using either red or blue color. We shall call such a \( K_{s,s} \) 2-colored.

Consider a subgraph \( K_{m,n} \) of 2-colored \( K_{s,s} \). If all lines of \( K_{m,n} \) have red(blue) color, we shall say that the \( K_{s,s} \) contains a red(blue) \( K_{m,n} \). The smallest number \( s \) of points such that \( K_{s,s} \) always contains red \( K_{m,n} \) or blue \( K_{m,n} \) is called bipartite Ramsey number and denoted by \( r_b(K_{m,n};2) \) or \( r_b(K_{m,n},K_{m,n}) \).

According to the definition of bipartite Ramsey numbers in this paper, V. Longani [7], has found that

\[
\begin{align*}
  r_b(K_{1,n},K_{1,n}) &= 2n - 1 \quad (n = 1, 2, 3, \ldots), \\
  r_b(K_{2,2},K_{2,2}) &= 5, \\
  r_b(K_{2,3},K_{2,3}) &= 9,
\end{align*}
\]

and L.W. Beineke and A.J. Schwenk [1], have also found that

\[
\begin{align*}
  r_b(K_{2,2},K_{2,2}) &= 5, \\
  r_b(K_{3,3},K_{3,3}) &= 17.
\end{align*}
\]

The 3-colored bipartite Ramsey number \( r_b(G;3) \) is the smallest integer \( n \) such that any 3-coloring of lines of a complete bipartite graph \( K_{n,n} \) there always exists monochromatic subgraph isomorphic to \( G \). In [4], W. Goddard, M.A. Henning, and O.R. Hellermann showed that \( r_b(C_4;3) = 11 \).

Consider a complete tripartite graph \( K_{s,s,s} \). Let each line of the \( K_{s,s,s} \) be colored by using one of \( k \) colors. We call such a \( K_{s,s,s} \) \( k \)-colored. For 2-coloring, the smallest number \( s \) of points such that the \( K_{s,s,s} \) always contains monochromatic \( K_{m,n} \) is called tripartite Ramsey number \( r_t(K_{m,n};2) \) or \( r_t(K_{m,n},K_{m,n}) \).

In [5], K. Leamyoo and V. Longani, have found that \( r_l(K_{2,2},K_{2,2}) = 3 \) (or \( r_l(C_4;2) = 3 \)). Also in [2], [3], S. Buada and V. Longani, have shown that \( r(l(K_{2,3},K_{2,3}) = 5, r(l(K_{2,4},K_{2,4}) = 7 \).

For 3-coloring, the tripartite Ramsey number of the graph \( G \), denoted by \( r_t(G;3) \) (or \( r_t(G,G,G) \)) is the minimum integer \( n \) such that for any 3-coloring of the lines of \( K_{n,n,n} \) there always exists monochromatic subgraph \( G \).

2. Main results
The proof that \( r_t(C_4;2) = 3 \) in [5] is rather lengthy and has not been published in a journal. In this section we provide a new shorter proof that \( r_t(C_4;2) = 3 \) and prove a new result that \( r_t(C_4;3) = 7 \).
For a $K_{3,3,3}$, consider all twenty seven lines that are adjacent to all points of a $V_i$. We shall call the lines that are adjacent to $V_i$ as the lines of $V_i$. Before proving Theorem 1, we prove Lemma 1 first.

**Lemma 1.** Let $K_{3,3,3}$ be a 2-colored complete tripartite graph and each $V_1$, $V_2$, and $V_3$ be the set of three non-adjacent points of the $K_{3,3,3}$. There exists at least one $V_i$ of which the numbers of red lines and blue lines of the $V_i$ are not equal.

**Proof.** Consider the three $V_i$’s. Suppose there are nine red lines and nine blue lines of each $V_i$.

Since there are totally nine red lines of $V_1$, consider when there are $n$ ($n \geq 0$) red lines which join points of $V_1$ and $V_2$, and so there are $9 - n$ red lines which join points of $V_1$ and $V_3$. Since for $V_2$ there are also exactly nine red lines of $V_2$, therefore there are $9 - n$ red lines which join points of $V_2$ and $V_3$.

Now we can see that there are $(9 - n) + (9 - n)$ red lines of $V_3$. Since there are exactly nine red lines of $V_3$, therefore

$$ (9 - n) + (9 - n) = 9 $$

$$ n = 4.5. $$

This is not possible. Therefore, there exist some $V_i$’s of which the numbers of red lines and blue lines of the $V_i$’s are not equal.

In order to prove Theorem 1 and Theorem 2, it would be convenient to represent a 2-colored $K_{m,n}$ by an $m \times n$ matrix $B = [b_{ij}]$.

Given a 2-colored $K_{m,n}$ with $V_1$ and $V_2$ as its partite sets of size $m$ and $n$, respectively. Let $V_1 = \{r_1, r_2, \ldots, r_m\}$ and $V_2 = \{c_1, c_2, \ldots, c_n\}$. For the corresponding $B = [b_{ij}]$ of the $K_{m,n}$, we let $b_{ij} = 1$ if the line $r_i c_j$ is red, and let $b_{ij} = 0$ if the line $r_i c_j$ is blue. For example, the following (a) and (b) in Figure 2.1 illustrate the 2-colored $K_{3,5}$ and its corresponding matrix $B$. Here, we use dark lines to indicate red lines and dash lines to indicate blue lines.

(a) \[
\begin{array}{ccccc}
\text{c}_1 & \text{c}_2 & \text{c}_3 & \text{c}_4 & \text{c}_5 \\
\text{V}_1
\end{array} \]

(b) \[
\begin{array}{cccc}
\text{r}_1 & \text{r}_2 & \text{r}_3 & \text{r}_4 \\
\text{V}_2
\end{array} \\
\begin{array}{ccccc}
\text{c}_1 & \text{c}_2 & \text{c}_3 & \text{c}_4 & \text{c}_5 \\
\text{r}_1 & 1 & 0 & 1 & 0 & 1 \\
\text{r}_2 & 0 & 1 & 1 & 1 & 0 \\
\text{r}_3 & 1 & 0 & 0 & 0 & 1 \\
\text{r}_4 & 0 & 1 & 1 & 0 & 0 \\
\end{array}
\]

Figure 2.1
Theorem 1. \( r_t(C_4; 2) = 3 \)

**Proof.** Consider the 2-colored \( K_{2,2,2} \) graph illustrated in Figure 2.2.

\[
\begin{array}{cc}
  u_1 & u_2 \\
  v_2 & w_1 \\
  v_1 & w_2
\end{array}
\]

**Figure 2.2**

It can be seen that the \( K_{2,2,2} \) contains neither red \( C_4 \) nor blue \( C_4 \). Therefore \( r_t(C_4; 2) > 2 \). That is

\( (2.1) \quad r_t(C_4; 2) \geq 3 \)

Let \( K_{3,3,3} \) be a 2-colored complete tripartite graph. Consider the set \( V_1, V_2 \) and \( V_3 \) of three non-adjacent points of the \( K_{3,3,3} \)

\[
V_1 = \{u_1, u_2, u_3\} \\
V_2 = \{v_1, v_2, v_3\} \\
V_3 = \{w_1, w_2, w_3\}.
\]

From Lemma 1, we can assume that from \( V_1 \), the numbers of red lines are greater than the numbers of blue lines, that is the numbers of red lines are equal to ten or greater. We only need to consider the case when the numbers of red lines of \( V_1 \) is ten and show that in such case the \( K_{3,3,3} \) always contain red \( C_4 \). For the cases when the number of red lines is greater than ten, the results follow immediately.

Let \( V(G_1) = V_1 \) and \( V(G_2) = V_2 \cup V_3 \). For \( V(G_1) \), let \( u_1, u_2, u_3 \) be respectively replaced by \( r_1, r_2, r_3 \). Also for \( V(G_2) \), let \( v_1, v_2, v_3, w_1, w_2, w_3 \) be respectively replaced by \( c_1, c_2, c_3, c_4, c_5, c_6 \). That is,

\[
V(G_1) = \{r_1, r_2, r_3\} \\
V(G_2) = \{c_1, c_2, c_3, c_4, c_5, c_6\}.
\]

By ignoring the lines between \( V_2 \) and \( V_3 \) and consider the defined \( V(G_1) \) and \( V(G_2) \). The \( K_{3,3,3} \) is now reduced to 2-colored \( K_{3,6} \). In order to prove the theorem we only need to show that this \( K_{3,6} \) always contains red \( C_4 \).

We find the value of \( r_t(C_4; 2) \) by considering the 2-colored \( K_{3,6} \). If there are \( m, n, s, t, u(1 \leq m, n \leq 3 \) and \( 1 \leq s, t, u \leq 6 \) such that some submatrices

\( (2.2) \quad \begin{bmatrix} b_{ms} & b_{mt} \\ b_{ns} & b_{nt} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \)

then the \( K_{3,6} \) contains red \( C_4 \).
Let $d_1, d_2, d_3$ be degrees of red lines of $r_1, r_2, r_3$ respectively. We can choose $r_i$'s such that $d_1 \geq d_2 \geq d_3$. Here we have the conditions that $d_1 + d_2 + d_3 = 10$ and $0 \leq d_i \leq 6, i = 1, 2, 3$. Next, we consider two main cases.

**Case 1.** $d_1 + d_2 \geq 8$.

Here, the possible $d_1 \geq d_2$ are $4 \geq 4, 5 \geq 3, 5 \geq 4, 5 \geq 5, 6 \geq 3, 6 \geq 4, 6 \geq 5, 6 \geq 6$. It is easy to show that for all of these cases the $K_{3,6}$ always contains red $C_4$. Consider cases $d_1 = 4$, and $d_2 = 4$, for example. For a case in Table 1, parts of the matrix involving $r_1$ and $r_2$ could be

<table>
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<tr>
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<tbody>
<tr>
<td>$r_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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Table 1:

from which submatrix of the form (2.5) always appears; that is the $K_{3,6}$ contains red $C_4$.

**Case 2.** $d_1 + d_2 < 8$.

With the conditions for $d_i$, there is only one subcase to consider $d_1 = 4, d_2 = 3, d_3 = 3$. We consider two more possibilities: Subcase 2.1 and Subcase 2.2.

**Subcase 2.1.** When there are two or more points of $c_i$’s each of which is joined by red lines to both of $r_1$ and $r_2$, see Table 2, then we see that the $K_{3,6}$ contains red $C_4$.

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<tr>
<td>$r_1$</td>
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<td>$r_2$</td>
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Table 2:

**Subcase 2.2.** One point of $c_i$’s is joined by red lines to both of $r_1$ and $r_2$.

Suppose that $r_1$ is joined by four red lines to $c_1, c_2, c_3, c_4$ and $r_2$ is joined by three red lines to $c_4, c_5, c_6$. Consider the three red lines joining $r_3$. Either at least two of these red lines are joined, from $r_3$, to some points among $c_1, c_2, c_3, c_4$ or at least two of these three red lines are joined to some points among $c_4, c_5, c_6$. In either case, we see that red $C_4$ is contained in the $K_{3,6}$, see Table 3 for example.

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</thead>
<tbody>
<tr>
<td>$r_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
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<tr>
<td>$r_2$</td>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$r_3$</td>
<td>1</td>
<td>1</td>
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<td></td>
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</tr>
</tbody>
</table>

Table 3:

Hence the $K_{3,3,3}$ will always contain red $C_4$. Therefore,

(2.3) \[ r_t(C_4; 2) \leq 3. \]
By (2.1) and (2.3), we have \( r_t(C_4; 2) = 3 \) as required.

Next, we show that \( r_t(C_4; 3) = 7 \).

**Theorem 2.** \( r_t(C_4; 3) = 7 \).

**Proof.** Consider the subgraphs of a \( K_{6,6,6} \) illustrated in Figure 2.3. Here, (a), (b), and (c) represent subgraphs of the 3-colored \( K_{6,6,6} \) with red, blue, and green lines, respectively.

It can be verified that the \( K_{6,6,6} \) contains no monochromatic \( C_4 \). Therefore, \( r_t(C_4; 3) > 6 \). That is,

\[
(2.4) \quad r_t(C_4; 3) \geq 7.
\]
Let \( K_{7,7,7} \) be a 3-colored complete tripartite graph with \( p = 21 \). Consider the set \( V_1, V_2 \) and \( V_3 \) of seven non-adjacent points of the \( K_{7,7,7} \):

\[
\begin{align*}
V_1 &= \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}, \\
V_2 &= \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}, \\
V_3 &= \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}.
\end{align*}
\]

We shall call the lines that are adjacent to \( V_i \) as the lines of \( V_i \). Since there are totally ninety eight lines of \( V_1 \), there are at least thirty three lines of \( V_1 \) with the same, say red, color. Consider thirty three of these red lines.

Let \( V(G_1) = V_1 \) and \( V(G_2) = V_2 \cup V_3 \). For \( V(G_1) \), let \( u_1, u_2, u_3, u_4, u_5, u_6, u_7 \) be respectively replaced by \( r_1, r_2, r_3, r_4, r_5, r_6, r_7 \). Also for \( V(G_2) \), let \( v_1, v_2, v_3, v_4, v_5, v_6, v_7, w_1, w_2, w_3, w_4, w_5, w_6, w_7 \) be respectively replaced by \( c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14} \). That is,

\[
\begin{align*}
V(G_1) &= \{r_1, r_2, r_3, r_4, r_5, r_6, r_7\}, \\
V(G_2) &= \{c_1, c_2, \ldots, c_{14}\}.
\end{align*}
\]

By ignoring the lines between \( V_2 \) and \( V_3 \) and consider the defined \( V(G_1) \) and \( V(G_2) \) the \( K_{7,7,7} \) is now reduced to 3-colored \( K_{7,14} \). In order to prove the theorem we only need to show that this \( K_{7,14} \) always contains red \( C_4 \).

We find an upper bound of \( r_i(C_4; 3) \) by considering the 3-colored \( K_{7,14} \).

If there are \( m, n, s, t(1 \leq m, n \leq 7 \) and \( 1 \leq s, t \leq 14) \) such that some submatrices

\[
\begin{bmatrix}
\begin{array}{c}
\text{b}_{ms} & \text{b}_{mt} \\
\text{b}_{ns} & \text{b}_{nt}
\end{array}
\end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\]

then the \( K_{7,14} \) contains red \( K_{2,2} \) or \( C_4 \).

Let \( d_1, d_2, d_3, d_4, d_5, d_6, d_7 \) be degrees of red lines of \( r_1, r_2, r_3, r_4, r_5, r_6, r_7 \) respectively. We can choose \( r_i \)'s such that \( d_1 \geq d_2 \geq d_3 \geq d_4 \geq d_5 \geq d_6 \geq d_7 \). Here we have the conditions that

\[
d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 = 33
\]

and \( 0 \leq d_i \leq 14, i = 1, 2, 3, 4, 5, 6, 7 \).

Next, we consider two main cases: Case 1 and Case 2.

**Case 1.** \( d_1 + d_2 + d_3 \geq 18 \).

Here, the possible \( d_1 \geq d_2 \geq d_3 \) are \( 6 \geq 6 \geq 6, 7 \geq 6 \geq 5, 7 \geq 6 \geq 6, 7 \geq 7 \geq 5, 7 \geq 7 \geq 6, 7 \geq 7 \geq 7, 8 \geq 5 \geq 5, 8 \geq 6 \geq 5, 8 \geq 6 \geq 6, 8 \geq 7 \geq 5, 8 \geq 7 \geq 6, 8 \geq 7 \geq 7, 8 \geq 8 \geq 6, 8 \geq 8 \geq 7, 8 \geq 8 \geq 8, 9 \geq 5 \geq 5, 9 \geq 6 \geq 5, 9 \geq 6 \geq 6, 9 \geq 7 \geq 5, 9 \geq 7 \geq 6, 9 \geq 7 \geq 7, 9 \geq 8 \geq 6, 10 \geq 5 \geq 5, 10 \geq 6 \geq 5, 10 \geq 6 \geq 6, 10 \geq 7 \geq 5, 11 \geq 5 \geq 5, 11 \geq 6 \geq 5, 11 \geq 6 \geq 6, 12 \geq 5 \geq 5 \).

It is easy to show that for all of these cases the \( K_{7,14} \) always contains red \( C_4 \).
Consider the cases \( d_1 = 6, d_2 = 6 \) and \( d_3 = 6 \), for example. For a case in Table 4, parts of the matrix involving \( r_1, r_2 \) and \( r_3 \) could be from which submatrix of the form (2.2) always appears, that is the \( K_{7,14} \), and so the \( k_{7,7,7} \) contains red \( C_4 \).

**Case 2.** \( d_1 + d_2 + d_3 < 18 \).

With the conditions for \( d_i \), there are only four subcases to consider.

**Subcase 2.1.** \( d_1 = 7, d_2 = 5, d_3 = 5, d_4 = 4, d_5 = 4, d_6 = 4, d_7 = 4 \).

When there are two or more points \( c_i \)‘s each of which is joined to both of \( r_1 \) and \( r_2 \) by red lines, we can see that the \( K_{7,14} \) contains red \( C_4 \), see Table 5 for example.

\[
\begin{array}{cccccccccccc}
  c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} \\
 r_1 & 1 & 1 & 1 & 1 & 1 & & & & & & & & \\
 r_2 & & & 1 & 1 & 1 & 1 & 1 & & & & & & \\
 r_3 & 1 & & & & & & & & & & & & \\
\end{array}
\]

**Table 4:**

We consider two more possibilities 2.1.1 and 2.1.2.

**2.1.1.** One \( c_i \) is joined by red lines to both of \( r_1 \) and \( r_2 \).

Suppose that \( r_1 \) is joined by seven red lines to \( c_1, c_2, c_3, c_4, c_5, c_6, c_7 \) and \( r_2 \) is joined by five red lines to \( c_7, c_8, c_9, c_{10}, c_{11} \). We now consider four sub-possibilities (1), (2), (3), and (4).

(1) \( c_{12}, c_{13}, c_{14} \) are not joined to some \( r_i \)’s (\( i = 3, 4, 5, 6, 7 \)) by red lines.

If \( c_{12}, c_{13}, c_{14} \) are not joined to \( r_3 \) for example, then either at least three red lines from \( r_3 \) are joined to points among \( c_1, c_2, c_3, c_4, c_5, c_6, c_7 \) or at least three red lines from \( r_3 \) are joined to points among \( c_8, c_9, c_{10}, c_{11} \). In either case, we can see that red \( C_4 \) is contained in the \( K_{7,14} \), see Table 6 for example.

\[
\begin{array}{cccccccccccc}
  c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} \\
 r_1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & & & \\
 r_2 & & & 1 & 1 & 1 & 1 & 1 & & & & & & \\
 r_3 & 1 & 1 & 1 & & & & 1 & 1 & & & & & \\
\end{array}
\]

**Table 6:**
2) One of $c_{12}$, $c_{13}$, $c_{14}$ is joined to some $r_i$'s $(i = 3, 4, 5, 6, 7)$ by red lines.

For example, suppose $r_3$ is joined to $c_{12}$ by red line, then there are four other red lines joining $r_3$. From these four red lines, either at least two red lines from $r_3$ are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least two red lines from $r_3$ are joined to points among $c_8, c_9, c_{10}, c_{11}$. In either case, we can see that red $C_4$ is contained in the $K_{7,14}$, see Table 7 for example.

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Table 7:

3) Two of $c_{12}$, $c_{13}$, $c_{14}$ are joined to some $r_i$'s $(i = 3, 4, 5, 6, 7)$ by red lines.

For example, suppose $r_3$ is joined to $c_{12}$, $c_{13}$ by red lines, then there are three other red lines joining $r_3$. From these three red lines, then either at least two red lines are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least two red lines are joined to points among $c_8, c_9, c_{10}, c_{11}$ or at least two red lines are joined to points among $c_{12}, c_{13}, c_{14}$. In any case, we can see that red $C_4$ is contained in the $K_{7,14}$, see Table 8 for example.

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Table 8:

4) All $c_{12}$, $c_{13}$, $c_{14}$ are joined to some $r_i$'s $(i = 3, 4, 5, 6, 7)$ by red lines.

Suppose $r_3$ is joined to $c_{12}$, $c_{13}$, and $c_{14}$. Consider $r_4$ and the four red lines joining $r_4$. From these four red lines, either at least two red lines are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least two red lines are joined to points among $c_8, c_9, c_{10}, c_{11}$ or at least two red lines are joined to points among $c_{12}, c_{13}, c_{14}$. In any case, we can see that red $C_4$ is contained in the $K_{7,14}$, see Table 9 for example.

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Table 9:
2.1.2. None of $c_i$'s are joined by red lines to both of $r_1$ and $r_2$.

Suppose that $r_1$ is joined by seven red lines to $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ and $r_2$ is joined by five red lines to $c_8, c_9, c_{10}, c_{11}, c_{12}$. We now consider three sub-possibilities (1), (2), and (3).

(1) $c_{13}, c_{14}$ are not joined to some $r_i$'s ($i = 3, 4, 5, 6, 7$) by red lines.

If $c_{13}, c_{14}$ are not joined to $r_3$ for example, then either at least three red lines from $r_3$ are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least three red lines from $r_3$ are joined to points among $c_8, c_9, c_{10}, c_{11}, c_{12}$. In either case, we can see that red $C_4$ is contained in the $K_{7,14}$, see Table 10 for example.

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Table 10:

(2) One of $c_{13}, c_{14}$ is joined to some $r_i$'s ($i = 3, 4, 5, 6, 7$) by red lines.

For example, suppose $r_3$ is joined to $c_{13}$ by red line, then there are four other red lines joining $r_3$. From these four red lines, either at least two red lines are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least two red lines are joined to points among $c_8, c_9, c_{10}, c_{11}, c_{12}$. In either case, we can see that red $C_4$ is contained in the $K_{7,14}$, see Table 11 for example.

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Table 11:

(3) Both of $c_{13}, c_{14}$ are joined to some $r_i$'s ($i = 3, 4, 5, 6, 7$) by red lines.

Suppose $r_3$ is joined to $c_{13}$, and $c_{14}$, then there are three other red lines joining $r_3$. From these three red lines, either at least two red lines are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ or at least two red lines are joined to points among $c_8, c_9, c_{10}, c_{11}, c_{12}$. In either case, we can see that red $C_4$ is contained in the $K_{7,14}$, see Table 12 for example.

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Table 12:
Subcase 2.2. \( d_1 = 6, d_2 = 6, d_3 = 5, d_4 = 4, d_5 = 4, d_6 = 4, d_7 = 4 \).

When there are two or more points \( c_i \)’s each of which is joined to both of \( r_1 \) and \( r_2 \) by red lines, then we can see that the \( K_{7,14} \) contains red \( C_4 \), see Table 13 for example.

\[
\begin{array}{cccccccccccc}
  & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} \\
 r_1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & & & & \\
r_2 & & 1 & 1 & 1 & 1 & 1 & & & & & & & & \\
\end{array}
\]

Table 13:

We consider two more possibilities 2.2.1 and 2.2.2.

2.2.1. One \( c_i \) is joined by red lines to both of \( r_1 \) and \( r_2 \).

Suppose that \( r_1 \) is joined by six red lines to \( c_1, c_2, c_3, c_4, c_5, c_6 \) and \( r_2 \) is joined by six red lines to \( c_6, c_7, c_8, c_9, c_{10}, c_{11} \). Similar to the cases in 2.1.1 of subcase 2.1, we have that the \( K_{7,14} \) contains red \( C_4 \).

2.2.2. None of \( c_i \)’s are joined by red lines to both of \( r_1 \) and \( r_2 \).

Suppose that \( r_1 \) is joined by six red lines to \( c_1, c_2, c_3, c_4, c_5, c_6 \) and \( r_2 \) is joined by six red lines to \( c_7, c_8, c_9, c_{10}, c_{11}, c_{12} \). Similar to the cases in 2.1.2 of subcase 2.1, we have that the \( K_{7,14} \) contains red \( C_4 \).

Subcase 2.3. \( d_1 = 6, d_2 = 5, d_3 = 5, d_4 = 4, d_5 = 4, d_6 = 4, d_7 = 4 \).

When there are two or more points \( c_i \)’s each of which is joined to both of \( r_1 \) and \( r_2 \) by red lines, then we can see that the \( K_{7,14} \) contains red \( C_4 \), see Table 14 for example.

\[
\begin{array}{cccccccccccc}
  & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} \\
 r_1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & & & & \\
r_2 & & 1 & 1 & 1 & 1 & 1 & & & & & & & & \\
\end{array}
\]

Table 14:

We consider two more possibilities 2.3.1 and 2.3.2.

2.3.1. One \( c_i \) is joined by red lines to both of \( r_1 \) and \( r_2 \).

Suppose that \( r_1 \) is joined by six red lines to \( c_1, c_2, c_3, c_4, c_5, c_6 \) and \( r_2 \) is joined by five red lines to \( c_6, c_7, c_8, c_9, c_{10} \). We now consider five sub-possibilities (1), (2), (3), (4), and (5).

(1) \( c_{11}, c_{12}, c_{13}, c_{14} \) are not joined to some \( r_i \)’s \( (i = 3, 4, 5, 6, 7) \) by red lines.

If \( c_{11}, c_{12}, c_{13}, c_{14} \) are not joined to \( r_3 \) for example, then either at least three red lines from \( r_3 \) are joined to points among \( c_1, c_2, c_3, c_4, c_5 \) or at least three red lines are joined to points among \( c_6, c_7, c_8, c_9, c_{10} \). In either case, we can see that red \( C_4 \) is contained in the \( K_{7,14} \), see Table 15 for example.
Table 15:

(2) One of $c_{11}, c_{12}, c_{13}, c_{14}$ is joined to some $r_i$'s ($i = 3, 4, 5, 6, 7$) by red lines.

For example, suppose $r_3$ is joined to $c_{11}$ by red line, then there are four other red lines joining $r_3$. From these four red lines, then either at least two red lines from $r_3$ are joined to points among $c_1, c_2, c_3, c_4, c_5$ or at least two red lines are joined to points among $c_6, c_7, c_8, c_9, c_{10}$. In either case, we can see that red $C_4$ is contained in the $K_{7,14}$, see Table 16 for example.

Table 16:

(3) Two of $c_{11}, c_{12}, c_{13}, c_{14}$ are joined to some $r_i$'s ($i = 3, 4, 5, 6, 7$) by red lines.

For example, suppose $r_3$ is joined to $c_{11}, c_{12}$ by red lines, then there are three other red lines joining $r_3$. From these three red lines, then either at least two red lines from $r_3$ are joined to points among $c_1, c_2, c_3, c_4, c_5$ or at least two red lines are joined to points among $c_6, c_7, c_8, c_9, c_{10}$. In either case, we can see that red $C_4$ is contained in the $K_{7,14}$, see Table 17 for example.

Table 17:

(4) Three of $c_{11}, c_{12}, c_{13}, c_{14}$ are joined to some $r_i$'s ($i = 3, 4, 5, 6, 7$) by red lines.

Suppose $r_3$ is joined to $c_{11}, c_{12}, c_{13}$ by red lines. Consider $r_4$ and the five red lines joining $r_4$. From these five red lines, if $r_4$ is joined to $c_{14}$, then either at least two red lines are joined to points among $c_1, c_2, c_3, c_4, c_5$ or at least two red lines are joined to points among $c_6, c_7, c_8, c_9, c_{10}$ or at least two red lines are joined to points among $c_{11}, c_{12}, c_{13}$. In any case, we can see that red $C_4$ is contained in the $K_{7,14}$, see Table 18 for example. For the case when $r_4$ is not joined to $c_{14}$, we have that red $C_4$ is also formed.
the tripartite Ramsey numbers \( r_t(C_4;2) \) and \( r_t(C_4;3) \)

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Table 18:

(5) All of \( c_{11}, c_{12}, c_{13}, c_{14} \) are joined to some \( r_i \)'s \((i = 3, 4, 5, 6, 7)\) by red lines.

Suppose \( r_3 \) is joined to \( c_{11}, c_{12}, c_{13}, c_{14} \) by red lines. Consider \( r_4 \) and the five red lines joining \( r_4 \). From these five red lines, then either at least two red lines are joined to points among \( c_1, c_2, c_3, c_4, c_5 \) or at least two red lines are joined to points among \( c_6, c_7, c_8, c_9, c_{10} \) or at least two red lines are joined to points among \( c_{11}, c_{12}, c_{13}, c_{14} \). In any case, we can see that red \( C_4 \) is contained in the \( K_{7,14} \), see Table 19 for example.

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Table 19:

2.3.2. None of \( c_i \)'s are joined by red lines to both of \( r_1 \) and \( r_2 \).

Suppose that \( r_1 \) is joined by six red lines to \( c_1, c_2, c_3, c_4, c_5, c_6 \) and \( r_2 \) is joined by five red lines to \( c_7, c_8, c_9, c_{10}, c_{11} \). Similar to the cases in 2.1.1 of subcase 2.1, we have that the \( K_{7,14} \) contains red \( C_4 \).

Subcase 2.4. \( d_1 = 5, d_2 = 5, d_3 = 5, d_4 = 5, d_5 = 5, d_6 = 4, d_7 = 4 \).

When there are two or more points \( c_i \)'s each of which is joined to both of \( r_1 \) and \( r_2 \) by red lines, we can see that the \( K_{7,14} \) contains red \( C_4 \), see Table 20 for example.

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Table 20:

We consider two more possibilities 2.4.1 and 2.4.2.

2.4.1. One \( c_i \) is joined by red lines to both of \( r_1 \) and \( r_2 \).

Suppose that \( r_1 \) is joined by five red lines to \( c_1, c_2, c_3, c_4, c_5 \) and \( r_2 \) is joined by five red lines to \( c_6, c_7, c_8, c_9 \). We now consider six sub-possibilities (1), (2), (3), (4), (5), and (6).
(1) $c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$ are not joined to some $r_i$’s ($i = 3, 4, 5, 6, 7$) by red lines.

If $c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$ are not joined to $r_3$ for example, then either at least three red lines from $r_3$ are joined to points among $c_1, c_2, c_3, c_4$ or at least three red lines are joined to points among $c_5, c_6, c_7, c_8, c_9$. In either case, we can see that red $C_4$ is contained in the $K_{7,14}$, see Table 21 for example.

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Table 21:

(2) One of $c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$ is joined to some $r_i$’s ($i = 3, 4, 5, 6, 7$) by red lines.

For example, suppose $r_3$ is joined to $c_{10}$ by red line, then there are four other red lines joining $r_3$. From these four red lines, then either at least two red lines from $r_3$ are joined to points among $c_1, c_2, c_3, c_4$ or at least two red lines are joined to points among $c_5, c_6, c_7, c_8, c_9$. In either case, we can see that red $C_4$ is contained in the $K_{7,14}$, see Table 22 for example.

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Table 22:

(3) Two of $c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$ are joined to some $r_i$’s ($i = 3, 4, 5, 6, 7$) by red lines.

For example, suppose $r_3$ is joined to $c_{10}, c_{11}$ by red lines, then there are three other red lines joining $r_3$. From these three red lines, then either at least two red lines from $r_3$ are joined to points among $c_1, c_2, c_3, c_4$ or at least two red lines are joined to points among $c_5, c_6, c_7, c_8, c_9$. In either case, we can see that red $C_4$ is contained in the $K_{7,14}$, see Table 23 for example.

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Table 23:
(4) Three of $c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$ are joined to some $r_i$’s ($i = 3, 4, 5, 6, 7$) by red lines.

Suppose $r_3$ is joined to $c_{10}, c_{11}, c_{12}$ by red lines. Consider $r_4$ and the five red lines joining $r_4$. From these five red lines, if $r_4$ is not joined to $c_{13}$ and $c_{14}$, then either at least two red lines are joined to points among $c_1, c_2, c_3, c_4$ or at least two red lines are joined to points among $c_5, c_6, c_7, c_8, c_9$ or at least two red lines are joined to points among $c_{10}, c_{11}, c_{12}$. In any case, we can see that red $C_4$ is contained in the $K_{7,14}$, see Table 24 for example.

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**Table 24:**

If $r_4$ is joined to one of $c_{13}$ and $c_{14}$, say $c_{13}$, then there are four other red lines joining $r_4$. From these four red lines, either at least two red lines are joined to points among $c_1, c_2, c_3, c_4$ or at least two red lines are joined to points among $c_5, c_6, c_7, c_8, c_9$ or at least two red lines are joined to points among $c_{10}, c_{11}, c_{12}$. In any case, we can see that red $C_4$ is contained in the $K_{7,14}$, see Table 25 for example.

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**Table 25:**

If $r_4$ is joined to $c_{13}$ and $c_{14}$, then consider $r_5$ and the five red lines joining $r_5$. From these five red lines, then either at least two red lines are joined to points among $c_1, c_2, c_3, c_4$ or at least two red lines are joined to points among $c_5, c_6, c_7, c_8, c_9$ or at least two red lines are joined to points among $c_{10}, c_{11}, c_{12}$ or at least two red lines are joined to points $c_{13}$ and $c_{14}$. In any case, we can see that red $C_4$ is contained in the $K_{7,14}$, see Table 26 for example.

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**Table 26:**
(5) Four of \(c_{10}, c_{11}, c_{12}, c_{13}, c_{14}\) are joined to some \(r_i\)'s \((i = 3, 4, 5, 6, 7)\) by red lines.

Suppose \(r_3\) is joined to \(c_{10}, c_{11}, c_{12}, c_{13}\) by red lines. Consider \(r_4\) and the five red lines joining \(r_4\). From these five red lines, if \(r_4\) is not joined to \(c_{14}\), then either at least two red lines are joined to points among \(c_1, c_2, c_3, c_4\) or at least two red lines are joined to points among \(c_{10}, c_{11}, c_{12}, c_{13}\). In any case, we can see that red \(C_4\) is contained in the \(K_{7,14}\), see Table 27 for example.

<table>
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Table 27:

If \(r_4\) is joined to \(c_{14}\), then there are four other red lines joining \(r_4\). From these four red lines, then either at least two red lines are joined to points among \(c_1, c_2, c_3, c_4\) or at least two red lines are joined to points among \(c_{10}, c_{11}, c_{12}, c_{13}\). In any case, we can see that red \(C_4\) is contained in the \(K_{7,14}\), see Table 28 for example.

<table>
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<tr>
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<tr>
<td>(r_3)</td>
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<tr>
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</tbody>
</table>

Table 28:

(6) All of \(c_{10}, c_{11}, c_{12}, c_{13}, c_{14}\) are joined to some \(r_i\)'s \((i = 3, 4, 5, 6, 7)\) by red lines.

Suppose \(r_3\) is joined to \(c_{10}, c_{11}, c_{12}, c_{13}, c_{14}\) by red lines. Consider \(r_4\) and the five red lines joining \(r_4\). From these five red lines, then either at least two red lines are joined to points among \(c_1, c_2, c_3, c_4\) or at least two red lines are joined to points among \(c_{10}, c_{11}, c_{12}, c_{13}, c_{14}\). In any case, we can see that red \(C_4\) is contained in the \(K_{7,14}\), see Table 29 for example.
THE TRIPARTITE RAMSEY NUMBERS $r_t(C_4;2)$ AND $r_t(C_4;3)$

<table>
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<tr>
<th></th>
<th>$c_1$</th>
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</thead>
<tbody>
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<tr>
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</tbody>
</table>

Table 29:

2.4.2. None $c_i$'s are joined by red lines to both of $r_1$ and $r_2$.

Suppose that $r_1$ is joined by five red lines to $c_1, c_2, c_3, c_4, c_5$ and $r_2$ is joined by five red lines to $c_6, c_7, c_8, c_9, c_{10}$.

Similar to the cases in 2.3.1 of subcase 2.3, red $C_4$ is always contained in the $K_{7,14}$.

Therefore,

(2.6) \[ r_t(C_4; 3) \leq 7. \]

From the inequalities (2.4) and (2.6), we have $r_t(C_4; 3) = 7$ as required.

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References


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