NEW CHARACTERIZATIONS OF SOLUBILITY OF FINITE GROUPS

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Abstract. A subgroup H of a group G is said to be S-supplemented in G if there exists a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} denotes the subgroup of H generated by all those subgroups of H which are S-permutable in G. In this paper, two new characterizations of solubility of finite groups are presented in terms of S-supplemented subgroups of primes power orders, where primes belong to $\{3, 5\}$. In particular, a counterexample is given to show that the conjecture, proposed by Heliel at the end of [A.A. Heliel, *A note on c-supplemented subgroups of finite groups*, Comm. Algebra, 42 (2014), 1650-1656] and related to c-supplemented subgroups of primes power orders, is negative.

Keywords: Finite groups, c-supplemented subgroups, S-supplemented subgroups, minimal subgroups, Sylow subgroups.

AMS Mathematics Subject Classification (2010): 20D10, 20D15, 20D20.

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All groups considered are finite.

Following Ballester-Bolinches, Wang and Guo [2], [12], a subgroup H of a group G is said to be c-supplemented in G if G has a subgroup T such that $G = HT$ and $H \cap T \leq H_G$, where H_G denotes the largest normal subgroup of G contained in H. In [1], Asaad and Ramadan prove that a group G is soluble provided that every minimal subgroup of G is c-supplemented in G. Recently, in [6], Heliel has generalized this result and proved the following theorems.

Theorem A. *If each subgroup of prime odd order of a group G is c-supplemented in G, then G is soluble.*

Theorem B. *A group G is soluble if and only if every Sylow subgroup of G of odd order is c-supplemented in G.*

In connection with the above two results, the following conjecture is posed at the end of [6].

Conjecture. *Let G be a group such that every non-cyclic Sylow subgroup P of odd order of G has a subgroup D such that $1 < |D| \leq |P|$ and all subgroups H of P with $|H| = |D|$ are c-supplemented in G. Then, is G soluble?*

In this short note, we first present a counterexample to show that the answer to this conjecture is negative in general and then give a generalization of Theorems A and B.

Example. Let $G = A_5 \times H$, where A_5 is the alternating group of degree 5 and H is an elementary group of order p^n with $p > 5$ and $n \geq 2$. Then G satisfies the condition of the preceding conjecture, but G is insoluble.

Next, we generalize Theorems A and B as the following two results respectively.

Theorem C. *Let G be a group and $\pi = \pi(G) \cap \{3, 5\}$. If every subgroup of G of order p with $p \in \pi$ is c-supplemented in G, then G is soluble.*

Theorem D. *Let G be a group and $\pi = \pi(G) \cap \{3, 5\}$. Then G is soluble if and only if every Sylow p-subgroup of G with $p \in \pi$ is c-supplemented in G and $L_2(8)$ is not involved in G.*

Here, we say that a group K is involved in a group G if K is isomorphic to a homomorphic image of a subgroup H of G. Note that such a homomorphic image is often called a section of G.

Recall that a subgroup H of a group G is said to be S-supplemented in G if G has a subgroup T such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} denotes the subgroup of H generated by all those subgroups of H which are S-quasinormal (permutable with all Sylow subgroups of G) in G (see Skiba [11] or [10]). By the definition, all c-supplemented subgroups are also S-supplemented subgroups. Hence, Theorems C and D are special cases of the following results.

Theorem E. *Let G be a group and $\pi = \pi(G) \cap \{3, 5\}$. If every subgroup of G of order p with $p \in \pi$ is S-supplemented in G, then G is soluble.*

Theorem F. *Let G be a group and $\pi = \pi(G) \cap \{3, 5\}$. Then G is soluble if and only if every Sylow p-subgroup of G with $p \in \pi$ is S-supplemented in G and $L_2(8)$ is not involved in G.*

In order to prove these two results, we need the following lemmas.

Lemma 1. [3, Theorem 5.4] *Let G be a group such that $(|G|, 15) = 1$. Then G is soluble.*

Lemma 2. [10, Lemma 2.10] *Let G be a group and $H \leq K \leq G$.*

(1) *If H is S-supplemented in G, then H is S-supplemented in K.*
(2) *Suppose that H is normal G. Then K/H is S-supplemented in G/H if and only if K is S-supplemented in G.*
(3) *Suppose that H is normal in G. Then the subgroup EH/H is S-supplemented in G/H for every S-supplemented subgroup E of G satisfying $(|E|, |H|) = 1$.*

Lemma 3. *Let P be a nontrivial normal p-subgroup of a group G with p odd. If all cyclic subgroups of P of order p are S-supplemented in G, then each G-chief factor below P is cyclic.*

Proof. This follows directly from Theorem A in [11]. ∎

Lemma 4. [5, Theorem 1] *Let G be a nonabelian simple group with H a subgroup of G such that $|G : H| = p^a$. Then one of the following holds.*

(1) *$G = A_n$ and $H \simeq A_{n-1}$ with $n = p^a$.*
(2) *$G = L_n(q)$ and H is the stabilizer of a line or hyperplane.*
Then $|G : H| = (q^n - 1)/(q - 1) = p^a$.
(3) *$G = L_2(11)$ and $H \simeq A_5$.*
(4) *$G = M_{23}$ and $H \simeq M_{22}$ or $G = M_{11}$ and $H \simeq M_{10}$.*
(5) *$G = U_4(2)$ and H is the parabolic subgroup of index 27.*

Lemma 5. [9, §5] *Let H be a Hall π-subgroup of the finite simple group G and $3 \notin \pi$. Then either H has a Sylow tower or $H = G = {}^2B_2(q)$.*

Lemma 6. *Let $G = L_n(q)$ and H a subgroup of G such that $|G : H| = 3^a$, where $a \geq 1$. Then $G = L_2(8)$ and $H \simeq 2^3.Z_7$ with index 9.*

Proof. This is a special case of Theorem 1.1 in [8]. ∎

Proof of Theorem E. Suppose the result is false and let G be a counterexample of minimal order. Then

(1) *Every proper subgroup of G is soluble.*

It follows from Lemmas 2 and 1 and the choice of G.

(2) *G is not a nonabelian simple group.*

Assume that G is a nonabelian simple group. Then G is a minimal simple group by (1). Let H be a subgroup of G of order $p \in \pi$. If H is S-quasinormal in G, then $H \leq O_p(G)$, a contradiction. Suppose G has a subgroup T such that $G = HT$ and $H \cap T = 1$. If $p = 3$, then we deduce that G is soluble. If $p = 5$, then G is isomorphic to A_5. However, the subgroups of A_5 of order 3 are not S-supplemented, which contradicts our initial assumption for G. Hence G cannot be a nonabelian simple group.

(3) $\overline{G} = G/\Phi(G)$ *is a minimal simple group.*

By (2), suppose that N is any nontrivial proper normal subgroup of G. Let M be any maximal subgroup of G. By (1), both N and M are soluble. If N is not contained in M, then $G = MN$ and so $G/N \simeq M/M \cap N$ is soluble. It follows that G is soluble, a contradiction. Hence $N \leq \Phi(G)$ and (3) holds.

(4) *Final contradiction.*

By (3), $\overline{G}$ is isomorphic to one of the following simple groups (see Huppert [7, Ch.II, Remark 7.5]):

(i) $L_2(p)$, $p > 3$ is a prime, and 5 does not divide $p^2 - 1$;

(ii) $L_2(3^r)$, r is an odd prime;

(iii) $L_2(2^r)$, r is a prime;

(iv) $Sz(2^r)$, r is an odd prime;

(v) $L_3(3)$.

Suppose first that $\overline{G}$ is isomorphic to one of the simple groups in (i)-(iv). By [7, Ch.II, Theorem 8.27] and [13, p.117, Theorem 4.1], every Sylow p-subgroup of $\overline{G}$ is cyclic, where $p = 3$ or 5. We claim that p does not divide the order of $\Phi(G)$. Otherwise, let P be the Sylow p-subgroup of $\Phi(G)$ and G_p be a Sylow p-subgroup of G. Then G_p/P is cyclic. By Lemma 2 and Lemma 3, every G-chief factor below P is cyclic. It follows that $G/C_G(P)$ is supersoluble (see [4, Corollary 3.2.9]). Thus, $G = C_G(P)$ by (1). Hence $P \leq Z(G)$ and so G_p is abelian. Furthermore, $G = G'$ according to Step (3). Therefore, we have that $P \cap Z(G) \cap G' = P$, a contradiction by [7, Ch.VI, Theorem 14.3]. Thus, the order of $\Phi(G)$ cannot be divisible by p. Let $H/\Phi(G)$ be a subgroup of $\overline{G}$ with order p. Then $H/\Phi(G) = \langle x \rangle \Phi(G)/\Phi(G)$ for some element x of G of order p. By Lemma 2, $H/\Phi(G)$ is S-supplemented in $\overline{G}$. Arguing as in (2), we deduce a contradiction.

Now, suppose that $\overline{G}$ is isomorphic to $PSL_3(3)$. We show that $\Phi(G)$ is a $3'$-group. If not, let P be the Sylow 3-subgroup of $\Phi(G)$. As above, we see that $P \leq Z(G)$. If all subgroups of G of order 3 are contained in P, then by [7, Ch.IV, Theorem 5.5], G is 3-nilpotent, which implies that G is soluble. Thus, for some element x of order 3 in G, $x \notin P$. By the hypothesis, $H = \langle x \rangle$ is S-supplemented in G. Then G has a subgroup T such that $G = HT$ and $H \cap T \leq H_{sG}$. If $H \cap T = 1$, then T is a proper subgroup of G of index 3. It is easy to see that $\Phi(G) \leq T$ and therefore $\overline{G}$ has a subgroup of index 3, a contradiction. Suppose that H is S-quasinormal in G. Then $H \leq O_p(G) \leq \Phi(G)$ and consequently $H \leq P$, a contradiction. Hence 3 does not divide the order of $\Phi(G)$. As in the foregoing paragraph, we derive a contraction, completing the proof.

Proof of Theorem F. If G is soluble, then every Sylow subgroup of G is complemented in G and thereby is S-supplemented in G. In addition, $L_2(8)$ is clearly not involved in G. Hence the necessity holds.

Now, we suppose that all Sylow p-subgroups of G with $p \in \pi$ are S-supplemented in G and G does not involve $L_2(8)$. We proceed by induction on the order of G. Let P be an arbitrary Sylow p-subgroup of G, where $p \in \pi$. Then, by the hypothesis, there exists a subgroup T in G such that $G = PT$ and $P \cap T \leq P_{sG}$. If

$P_{sG} \neq 1$, then $P_{sG} \leq O_p(G)$, where $O_p(G)$ denotes the largest normal p-subgroup of G. Consider the factor group $G/O_p(G)$. Then, it is easy to see that $G/O_p(G)$ satisfies the hypothesis and so $G/O_p(G)$ is soluble by induction. It follows that G is soluble. Hence we may assume that P_{sG} is trivial, which means that P is complemented in G.

Next, we argue that G is not a nonabelian simple group. If not, then G is a nonabelian simple group such that every Sylow p-subgroup of G with $p \in \pi$ is complemented in G by the foregoing discussion. Hence G has a subgroup H such that $|G : H| = p^a$, where $p = 3$ or 5. Thus, G is isomorphic to one of the groups listed in Lemma 4. If G is isomorphic to one of the following:

$$L_2(11),\ M_{23},\ M_{11},\ U_4(2),$$

then $\pi = \{3, 5\}$. By the preceding paragraph, G has two subgroups with two different indices 3^a and 5^b, where $a \geq 1$ and $b \geq 1$, which is impossible (see [5, pp. 304]). If $G = A_n$ and $H \simeq A_{n-1}$, then, by Lemma 5, we have that $n = 5$. But the Sylow 3-subgroups of A_5 are not complemented in A_5, a contradiction. At last, assume that $G = L_n(q)$. Then, by Lemma 6, G must be isomorphic to $L_2(8)$, contrary to our assumption for G. Thus, we have shown that G is not a nonabelian simple group.

Let N be a minimal normal subgroup of G. Then N is nontrivial. If N is an elementary abelian group, then, by Lemma 2, G/N satisfies the hypothesis and so G/N is soluble by induction. Thereby G is soluble. Suppose that N is insoluble. Denote $\pi' = \pi(N) \cap \{3, 5\}$. Obviously, $\pi' \subseteq \pi$. Then, by Lemma 1, $\pi' \neq \emptyset$. Let P be any Sylow p-subgroup of G with $p \in \pi'$ and set $L = PN$. Then $P \cap N$ is complemented in N by the first paragraph and therefore $P \cap N$ is S-supplemented in N. Note that $P \cap N$ is a Sylow p-subgroup of N. Thus, we see that N satisfies the hypothesis and so N is soluble by induction.

This contradiction completes the proof. ∎

Remark.

(1) The converse of Theorem E is not true in general. The alternating group A_4 of degree 4 is such a counterexample because every involution of A_4 is not S-supplemented in A_4 as A_4 has no subgroup of order 6.

(2) In Theorem F, the condition "G does not involve $L_2(8)$" can not be removed. In fact, $L_2(8)$ is a counterexample. In $L_2(8)$, $\pi = \pi(L_2(8)) \cap \{3, 5\} = \{3\}$ and every Sylow 3-subgroup of $L_2(8)$ is complemented in $L_2(8)$. Of course, every Sylow 3-subgroup of $L_2(8)$ is S-supplemented in $L_2(8)$. But $L_2(8)$ is a nonabelian simple group.

With respect to Theorem E and Theorem F, the following problem seems interesting.

Problem. *Let G be a group and $\pi = \pi(G) \cap \{3, 5\}$. Suppose that for every Sylow p-subgroup P of G with $p \in \pi$, G has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are S-supplemented in G. What can we say about the structure of G?*

Acknowledgements. The first author is very grateful to Dr. Xiaoyu Chen for providing him the reference [6] and several conversations on the topic discussed in this article.

This paper was supported by NSFC (11171364, 11271301), the Scientific Research Foundation of Chongqing Municipal Science and Technology Commission (cstc2013jcyjA00034), the Scientific and Technological Research Program of Chongqing Municipal Education Commission (KJ131204), the Scientific Research Foundation of Yongchuan Science and Technology Commission (Ycstc, 2013nc8006) and the Scientific Research Foundation of Chongqing University of Arts and Sciences (R2012SC21).

References

[1] Asaad, M., Ramadan, M., *Finite groups whose minimal subgroups are c-supplemented*, Comm. Algebra, 36 (2008), 1034-1040.

[2] Ballester-Bolinches, A., Wang, Y.M., Guo, X.Y., *C-supplemented subgroups of finite groups*, Glasgow Math. J., 42 (2000), 383-389.

[3] Chen, Z.M., *Inner-Outer-Σ-groups and minimal non-Σ-groups*, Southwest Normal University Press, Beibei, 1988. (in Chinese)

[4] Guo, W.B., *The Theory of Classes of Groups*, Science Press-Kluwer Academic Publishers, Beijing-New York-Dorlrecht-Boston-London, 2000.

[5] Guralnick, R.M., *Subgroups of prime power index in a smple group*, J. Algebra, 81 (1983), 304-311.

[6] Heliel, A.A., *A note on c-supplemented subgroups of finite groups*, Comm. Algebra, 42 (2014), 1650-1656.

[7] Huppert, B., *Endliche Gruppen.* I, Springer-Verlag, Heidelberg-New York, 1967.

[8] Li, C.H., *The primitive permutation groups of certain degrees*, J. Pure Appl. Math., 115 (1997), 275-287.

[9] Revin, D.O., Vdovin, E.P., *Hall subgroups of finite groups*, in: Contemp. Math., 402 (2006), 229-263.

[10] Skiba, A.N., *On weakly s-permutable subgroups of finite groups*, J. Algebra, 315 (2007), 192-209.

[11] Skiba, A.N., *On two questions of L.A. Shemetkov concerning hypercyclically embedded subgroups of finite groups*, J. Group Theory, 13 (2010), 841-850.

[12] Wang, Y.M., *Finite groups with some subgroups of Sylow subgroups c-supplemented*, J. Algebra, 224 (2000), 467-478.

[13] Wilson, R.A., *The finite simple groups*, Springer, London-Dordrecht-Heidelberg-New York, 2009.

Accepted: 02.08.2014