WEAK OPEN SETS ON SIMPLE EXTENSION IDEAL TOPOLOGICAL SPACE

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Abstract. In this paper we intend to introduce a new class of sets known as e-$\mathcal{I}^+$-open sets, defined in the light of simple extension topology and ideal topology. This set is investigated and found to be a weaker form of e-$\mathcal{I}$-open sets. We have also generalized this concept and studied its properties.

Keywords: ideal topological space, e-open, e-$\mathcal{I}$-open sets, simple extension to topology, e-$\mathcal{I}^+$-open.

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1. Introduction

Levine [9] in 1964, defined one topology,τ^+, to be *simple extension* of another topology, τ, on the same set X by $\tau^+(B) = \{O \cup (\acute{O} \cap B) | O, \acute{O} \in \tau\}$ for some $B \notin \tau$. He investigated the question of whether (X, τ^+) has certain properties possesses by (X, τ), the properties included regularity, complement, and normality. By the definition of simple expansion we infer that all topologies are simple expansion topologies.

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An ideal $\mathcal{I}$ on a topological space $(X,\mathcal{I})$ is a nonempty collection of subsets of X which satisfies the following conditions:

$$A \in \mathcal{I} \text{ and } B \subset A \text{ implies } B \in \mathcal{I};\ A \in \mathcal{I} \text{ and } B \in \mathcal{I} \text{ implies } A \cup B \in \mathcal{I}.$$

Applications to various fields were further investigated by Jankovic and Hamlett [7] Dontchev et al. [4]; Mukherjee et al. [10]; Arenas et al. [3]; et al. Nasef and Mahmoud [11] etc. Given a topological space $(X,\tau,\mathcal{I})$ with an ideal $\mathcal{I}$ on X and if $\wp(X)$ is the set of all subsets of X. Then operator $(.)^* : \wp(X) \to \wp(X)$, called a local function [13, 7] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subseteq X$,

$$A^*(\mathcal{I},\tau) = \{x \in X \mid U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\},$$

where $\tau(x) = \{U \in \tau \mid x \in U\}$.

A Kuratowski closure operator $Cl^*(x) = A \cup A^*(\mathcal{I},\tau)$.

When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I},\tau)$. X^* is often a proper subset of X.

A subset A of an ideal space $(X,\tau,\mathcal{I})$ is said to be R-I-open (resp. R-I-closed) [15] if $A = Int(Cl^*(A))$ $(resp.\, A = Cl^*(Int(A))$. A point $x \in X$ is called $\delta - I$-cluster point of A if $Int(Cl^*(U)) \cap A \neq \emptyset$ for each open set V containing x. The family of all δ-$\mathcal{I}$-cluster points of A is called the δ-$\mathcal{I}$-closure of A and is denoted by $\delta Cl_I(A)$. The set δ-$\mathcal{I}$-interior of A is the union of all R-I-open sets of X contained in A and its denoted by $\delta Int_I(A)$. A is said to be δ-$\mathcal{I}$-closed if $\delta Cl_I(A) = A$ [15].

The subject of ideals in topological spaces has been studied by Kuratowski [8] and Vaidyanathaswamy [14]. Jankovic and Hamlett [7] introduced the notation of $\mathcal{I}$-open sets in ideal topological space, and investigated further properties of ideal space. Further Abd El-Monsef et al. [2] investigated $\mathcal{I}$-open sets and $\mathcal{I}$-continuous functions. Hatir [6] introduced the notion of $semi^*$-$\mathcal{I}$-open sets and obtained a decomposition of $\mathcal{I}$-continuity. The notion of pre^*-$\mathcal{I}$-open sets to obtain decomposition of continuity was introduced by E. Ekici and T. Noiri [5]. In addition to this, the concept of e-$\mathcal{I}$-open sets and e-$\mathcal{I}$-continuous functions have been introduced by [1].

Definition 1.1. A subset A of an ideal topological space $(X,\tau,\mathcal{I})$ is called

1. $semi^*$-$\mathcal{I}$-open [6] if $A \subset Cl(\delta Int_I(A))$.

2. pre^*-$\mathcal{I}$-open [5] if $A \subseteq Int(\delta Cl_I(A))$.

3. $\delta\alpha$-$\mathcal{I}$-open [6] if $A \subset Int(Cl(\delta Int_I(A)))$.

4. $\delta\beta_I$-open [6] if $A \subset Int(Cl(\delta Int_I(A)))$.

5. e-$\mathcal{I}$-open [1] if $A \subset Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A))$.

In this paper we have made an attempt to extend these concept of $\mathcal{I}$-openness, *semi**-$\mathcal{I}$-openness, *pre**-$\mathcal{I}$-openness, e-$\mathcal{I}$-openness, $\delta\alpha$-$\mathcal{I}$-openness, $\delta\beta_I$-openness in simple extension topology.

2. e-$\mathcal{I}$-Open In Simple Extension

In all below definitions the interior $Int(A)$ refers to the interior in usual topology, δCl_I^+ denote the family of all δ-$\mathcal{I}^+$-cluster points of A, where. A point $x \in X$ is called δ-$\mathcal{I}^+$-cluster point of A if $Int(Cl^{+*}(U)) \cap A \neq \emptyset$ for each open set V containing x, $Cl^{*+}(A)$ is denoted the closure with respect to the ideal topological space under simple extension. And δInt_I^+ is the union of all R-I^+-open sets of X contained in A. Here a new local function is defined on the simple ideal topological space (SEITS) and its denoted as $A^{+*} = \{x \in X \mid U \cap A \notin \mathcal{I} \text{ for every } U \in \tau^+(B)\}$ as known as extend local functions with respect to τ^+ and $\mathcal{I}$. Also we defined a closure operator as $Cl^{+*}(A) = A \cup A^{+*}$. A subset A of $(X, \tau^+, \mathcal{I})$ is called $*+$ perfect if $A = A^{*+}$. The family of all e-$\mathcal{I}^+$-open defined by EI^+O.

Definition 2.1. Let A be a subset of simple extension ideal topological space (SEITS), then A is said to be

(1) $\mathcal{I}^+$ open set [12] if $A \subset Int(A^{*+})$.

(2) e^+-open if $A \subset Int(\delta Cl^+(A)) \cup Cl(\delta Int^+(A))$.

(3) R-I^+-open if $A = Int(Cl^{+*}(A))$.

(4) *semi**-$\mathcal{I}^+$-open if $A \subset Cl(\delta Int_I^+(A))$.

(5) *pre**-$\mathcal{I}^+$-open if $A \subseteq Int(\delta Cl_I^+(A))$.

(6) $\delta\alpha$-$\mathcal{I}^+$-open if $A \subset Int(Cl(\delta Int_I^+(A)))$.

(7) $\delta\beta_I^+$-open if $A \subset Int(Cl(\delta Int_I^+(A)))$.

(8) e-$\mathcal{I}^+$-open if $A \subset Cl(\delta Int_I^+(A)) \cup Int(\delta Cl_I^+(A))$.

Theorem 2.2. *Let $(X, \tau^+, \mathcal{I})$ be an simple extension ideal topological space (SEITS) the following hold:*

(1) *Every open is e-$\mathcal{I}^+$-open,*

(2) *Every e-$\mathcal{I}^+$-open is e-$\mathcal{I}$-open,*

(3) *Every $\mathcal{I}^+$-open is e-$\mathcal{I}^+$-open.*

Proof. (1) Let A be any subset of $(X, \tau^+, \mathcal{I})$ if A is open in τ we have:

$$\begin{aligned} A &= Int(A) \\ &\subset Int(\delta Cl_I^{+*}(A)) \\ &\subset Int(\delta Cl_I^{+*}(A)) \cup Cl(\delta Int_I^+(A))) \end{aligned}$$

Then A is e-$\mathcal{I}^+$-open.

(2) By the definition of e-$\mathcal{I}^+$-open and e-$\mathcal{I}$-open and since $Cl^{+*}(A) \subset Cl^*(A)$, then $\delta Cl_I^{+*}(A) \subset \delta Cl_I^*(A)$, under theses conditions every e-$\mathcal{I}^+$-open is e-$\mathcal{I}$-open.

(3) Obvious. ∎

Remark 2.3. From the above Theorem we know the class of e-$\mathcal{I}^+$-open sets is properly placed between an open set and e-$\mathcal{I}$-open set. But the converse no need to be true.

Example 2.4. Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ and an ideal $\mathcal{I} = \{\varnothing, \{b\}\}$,$B = \{b\}$, $\tau^+(B) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then the set $A = \{a, c\}$ is e-$\mathcal{I}^+$-open, but it is not open in the topology τ and τ^+.

Example 2.5. Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and an ideal $\mathcal{I} = \{\varnothing, \{c\}\}$,$B = \{b, c\}$, $\tau^+(B) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Here $\{a, c\}$ is e-$\mathcal{I}$-open, but it is not e-$\mathcal{I}^+$-open.

Proportion 2.6. *For any simple extension ideal topological space (SEITS) $(X, \tau^+, \mathcal{I})$ and $A \subset X$ we have:*

(1) *If $I = \emptyset$, then A is e-$\mathcal{I}^+$-open if and only if A is e^+-open.*

(2) *If $I = \wp(X)$, then A is e-$\mathcal{I}^+$-open if and only if $A \in \tau$.*

(3) *If $I = N$, then A is e-$\mathcal{I}^+$-open if and only if A is e^+-open, where N the ideal of nowhere dense.*

Proof. (1) Let $I = \emptyset$ and $A \subset X$. We have $\delta Cl_I^+(A)) = \delta Cl^+(A))$, $\delta Int_I^+(A)) = \delta Int^+(A))$ and $A^{+*} = Cl^+(A)$. on other hand, $Cl^{+*}(A) = A^{+*} \cup A = Cl^+(A)$. Hence $A^{+*} = Cl^+(A) = Cl^{+*}(A)$. Thus (1) follows immediately.

(2) Let $I = P(X)$ then $A^{+*} = \emptyset$, for any $A \subset X$. Since A is e-$\mathcal{I}^+$-open, we have

$$\begin{aligned} A &\subset Cl(\delta Int_I^+(A)) \cup Int(\delta Cl_I^+(A)) \\ &= Int[Int(Cl(\delta Int_I^+(A))) \cup \delta Cl_I^+(A)] \\ &\subset Int[Cl(\delta Int_I^+(A)) \cup \delta Cl_I^+(A)] \\ &\subset Int[\delta Cl_I^+(\delta Int_I^+(A \cup A))] \\ &\subset Int[\delta Cl_I^+(\delta Int_I^+(A))] \\ &\subset Int[Cl(Int(A))] \end{aligned}$$

This show $A \in \tau$.

(3)$\Leftarrow$ Every e-$\mathcal{I}^+$-open is e^+-open.

Let A be e-$\mathcal{I}^+$-open then, $A \subset Cl(\delta Int_I^+(A)) \cup Int(\delta Cl_I^+(A))$. By using this fact when $I = \emptyset$ part (1), $A^{+*} = Cl^+(A) = Cl^{+*}(A)$, we have $\delta Cl_I^+(A) = \delta Cl^+(A)$, $\delta Int_I^+(A) = \delta Int^+(A)$, since $\delta Cl_I^+(A)$ is the family of all δ-$\mathcal{I}^+$-cluster point of A, and $\delta Int_I^+(A)$ the union of all R-$\mathcal{I}^+$-open set of X we have respectively,

$$\begin{aligned}\emptyset \neq Int(Cl^{*+}(U)) \cap A &= Int(U^{*+} \cup U) \cap A = Int(Cl^+(U) \cup U) \cap A \\ &= Int(Cl^+(U)) \cap A \neq \emptyset\end{aligned}$$

From this we get $\delta Cl_I^+(A) = \delta Cl^+(A)$, and

$$\begin{aligned}A = Int(Cl^{*+}(A)) &= Int(A^{*+} \cup A) = Int[Cl^+(A) \cup A] \\ &= Int(Cl^+(A)) = A\end{aligned}$$

From this we get $\delta Int_I^+(A) = \delta Int^+(A)$. This show that

$$A \subset Cl(\delta Int_I^+(A)) \cup Int(\delta Cl_I^+(A)) \subset Cl(\delta Int^+(A)) \cup Int(\delta Cl^+(A)).$$

Now, let us consider $I = N$ and A is e^+-open.

$\Rightarrow$ If $I = N$ then $A^{+*} = Cl^{+*}(Int(Cl^{+*}A))$.

Since A is e^+-open then $A \subset Cl(\delta Int^+(A)) \cup Int(\delta Cl^+(A))$. Then

$$\begin{aligned}\emptyset &\neq Int(Cl^+(U)) \cap A = Int(U^+ \cup U) \cap A \\ &= Int(Cl^+(Int(Cl^+(U)) \cup U) \cap A \subset Int(Cl^{+*}(Int(Cl^{+*}(U))) \cup U) \cap A \\ &= Int(U^{*+} \cup U) \cap A = Int(Cl^{+*}(U)) \cap A \neq \emptyset\end{aligned}$$

From this we get $\delta Cl^+(A) \subset \delta Cl_I^+(A)$, and

$$\begin{aligned}A &= Int(Cl^+(A)) = Int(A^+ \cup A) = Int[Cl^+(Int(Cl^+(A))) \cup A] \\ &\subset Int[Cl^{+*}(Int(Cl^{+*}(A))) \cup A] = Int(A^{*+} \cup A) = Int(Cl^{+*}(A)) = A\end{aligned}$$

From this we get $\delta Int^+(A) \subset \delta Int_I^+(A)$.
A is e-$\mathcal{I}^+$-open. Hence the proof. ∎

Proposition 2.7. *Let A be a subset of (SITES) $(X, \tau^+, \mathcal{I})$ then the following properties hold:*

1. *Every semi*-$\mathcal{I}^+$-open is e-$\mathcal{I}^+$-open,*
2. *Every pre*-$\mathcal{I}^+$-open is e-$\mathcal{I}^+$-open,*
3. *Every e-$\mathcal{I}^+$-open is $\delta\beta_I^+$-open.*
4. *Every $\delta\alpha$-$\mathcal{I}^+$-open is $\delta\beta_I^+$-open.*

Proof. (1) and (2) are obvious from the definition of e-$\mathcal{I}^+$-open set.

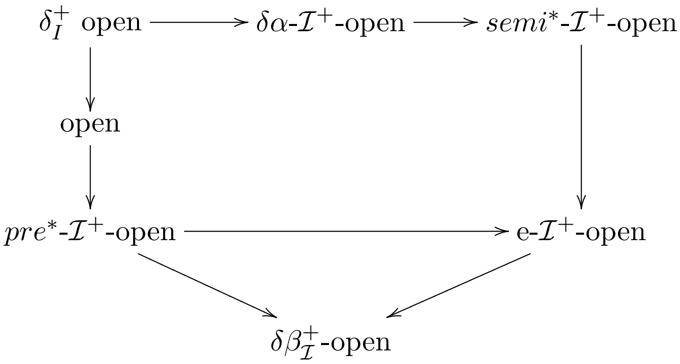
(3) Let A be e-$\mathcal{I}^+$-open. Then we have,

$$\begin{aligned} A &\subset Cl(\delta Int_I^+(A)) \cup Int(\delta Cl_I^+(A)) \\ &\subset Cl(Int(\delta Int_I^+(A))) \cup Int(Int(\delta Cl_I^+(A))) \\ &\subset Cl(Int(\delta Int_I^+(A)) \cup Int(\delta Cl_I^+(A))) \\ &\subset Cl[Int(\delta Int_I^+(A)) \cup \delta Cl_I^+(A)] \\ &\subset Cl[Int(\delta Cl_I^+(A \cup A))] \\ &= Cl(Int(\delta Cl_I^+(A))). \end{aligned}$$

This show that A is an $\delta\beta_I^+$-open set.

(4) proof is obvious. ∎

Remark 2.8. From above the following implication,



A is called δ_I^+ open if for each $x \in A$, there exist a R-$\mathcal{I}^+$-open set G such that $x \in G \subset A$. None of these implications is reversible as shown by examples given below.

Example 2.9. Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$, $B = \{b, c\}$, $\tau^+(B) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Then the set $A = \{a, c\}$ is e-$\mathcal{I}^+$-open, but it is not pre^*-$\mathcal{I}^+$-open.

Example 2.10. Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{c\}\}$ and an ideal $\mathcal{I}$=$\{\emptyset, \{b\}\}$. Let B=$\{a\}$, then $\tau^+(B) = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$. Here the set $A = \{b, d\}$ is e-$\mathcal{I}^+$-open, but it is not $semi^*$-$\mathcal{I}^+$-open. Because $Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A)) = Cl(\{a\}) \cup Int(X) = \{a, b\} \cup X = X \supset A$ and hence A is e-$\mathcal{I}^+$-open. Since $Cl(\delta Int_I(A)) = Cl(\{a\}) = \{a, b\} \not\supseteq A$. So A is not $semi^*$-$\mathcal{I}^+$-open.

Example 2.11. Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and an ideal $\mathcal{I} = \{\emptyset, \{b\}\}$. Let $B = \{b, c\}$, $\tau^+(B) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Here $A = \{a, c\}$ is e-$\mathcal{I}^+$-open, but it is not $\delta\alpha_{\mathcal{I}}^+$-open. Because $Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A)) = Cl(\{a\}) \cup Int(X) = \{a\} \cup X = X \supset A$ and hence A is e-$\mathcal{I}^+$-open. Since $Int(Cl(\delta Int_I(A))) = Int(Cl(\{a\})) = \{a\} \not\supseteq A$. So A is not $\delta\alpha_{\mathcal{I}}^+$-open.

Theorem 2.12. *Let $(X, \tau, \mathcal{I})$ an ideal in topological space and A, B subsets of X. Then, for local functions the following properties hold:*

(1) *If* $A \subset B$, *then* $A^{*+} \subset B^{*+}$,

(2) *For another ideal* $J \supset I$ *on* X, $A^{*+}(J) \subset A^{*+}(\mathcal{I})$,

(3) $A^{*+} \subset Cl(A)$,

(4) $A^{*+}(\mathcal{I}) = Cl(A^{*+}) \subset Cl(A)$ (*i.e* A^{*+}

(5) $(A^{*+})^{*+} \subset A^{*+}$,

(6) $(A \cup B)^{*+} = A^{*+} \cup B^{*+}$,

(7) A^{*+}-$B^{*+} = (A - B)^{*+} - B^{*+} \subset (A - B)^{*+}$,

(8) *If* $U \in \tau$, *then* $U \cap A^{*+} = U \cap (U \cap A)^{*+} \subset (U \cap A)^{*+}$,

(9) *If* $I \in \mathcal{I}$, *then* $(A\text{-}I)^{*+} \subset A^{*+} = (A \cup I)^{*+}$,

Proof. Obvious using the Definition of A^{*+}. ∎

Proposition 2.13. *Let* $(X, \tau^+, \mathcal{I})$ *be SEITS and let* A, $U \subseteq X$. *If* A *is* e-$\mathcal{I}^+$*-open set and* $U \in \tau$. *Then* $A \cap U$ *is an* e-I^+*-open.*

Proof. By assumption $A \subset Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A))$ and $U \subseteq Int(U)$. By Theorem 2.12 (8) we have,

$$\begin{aligned} A \cap U \subset & (Cl(\delta Int_I^+(A)) \cup Int(\delta Cl_I^+(A))) \cap Int(U) \\ & \subset (Cl(\delta Int_I^+(A)) \cap Int(U)) \cup (Int(\delta Cl_I^+(A)) \cap Int(U)) \\ & \subset (Cl(\delta Int_I^+(A)) \cap Cl(Int(U))) \cup (Int(\delta Cl_I^+(A)) \cap Cl(Int(U))) \\ & \subset (Cl(\delta Int_I^+(A)) \cap Int(U)) \cup (Int(Cl(\delta Cl_I^+(A)) \cap Cl(Cl(Int(U))))) \\ & \subset Cl(\delta Int_I^+(A \cap U) \cup (Int(Cl(\delta Cl_I^+(A)) \cap Cl(Int(U)))) \\ & \subset Cl(\delta Int_I^+(A \cap U)) \cup (Int(Cl(\delta Cl_I^+(A)) \cap Int(U))) \\ & \subset Cl(\delta Int_I^+(A \cap U)) \cup (Int(\delta Cl_I^+(A \cap U))). \end{aligned}$$

Thus $A \cap U$ is e-$\mathcal{I}^+$-open. ∎

Proposition 2.14. *Let* $(X, \tau^+, \mathcal{I})$ *be SEITS then the following hold.*

(1) *The union of any family of* e-$\mathcal{I}^+$*-open sets is an* e-$\mathcal{I}^+$*-open set.*

(2) *The intersection of arbitrary family of* e-$\mathcal{I}^+$*-closed sets is* e-$\mathcal{I}^+$*-closed.*

(3) *If* $A \in EI^+O(X, \tau^+, \mathcal{I})$ *and* $B \in \tau$, *then* $A \cap B \in EI^+O(X, \tau^+, \mathcal{I})$.

Proof. (1) Let $\{A_\alpha | \alpha \in \Delta\}$ be a family of e-$\mathcal{I}^+$-open set, $A_\alpha \subset Cl(\delta Int_I^+(A_\alpha)) \cup Int(\delta Cl_I^+(A_\alpha))$. Hence

$$\begin{aligned} \cup_\alpha A_\alpha & \subset \cup_\alpha [Cl(\delta Int_I^+(A_\alpha)) \cup Int(\delta Cl_I^+(A_\alpha))] \\ & \subset \cup_\alpha [Cl(\delta Int_I^+(A_\alpha))] \cup \cup_\alpha [Int(\delta Cl_I^+(A_\alpha))] \\ & \subset [Cl(\cup_\alpha(\delta Int_I^+(A_\alpha))] \cup [Int(\cup_\alpha(\delta Cl_I^+(A_\alpha))] \\ & \subset [Cl(\cup_\alpha(\delta Int_I^+(A_\alpha))] \cup [Int(\cup_\alpha(\delta Cl_I^+(A_\alpha))] \\ & \subset [Cl(\delta Int_I^+(\cup_\alpha A_\alpha))] \cup [Int(\delta Cl_I^+(\cup_\alpha A_\alpha))]. \end{aligned}$$

$U_\alpha A_\alpha$ is e-$\mathcal{I}^+$-open.

(2) Let $\{B_\alpha/\alpha \in \Delta\}$ be a family of e-$\mathcal{I}^+$-closed set. Then $\{B^c_\alpha/\alpha \in \Delta\}$ be a family of e-$\mathcal{I}^+$-open set. By (1) $\cup^c_\alpha A_\alpha$ is e-$\mathcal{I}^+$-open. Hence $(\cap_\alpha A_\alpha)^c = \cup^c_\alpha A_\alpha$ is e-$\mathcal{I}^+$-open $(\cap_\alpha A_\alpha)$ is e-$\mathcal{I}^+$-closed set. Hence the proof.

(3) Let $A \in EI^+O(X, \tau^+, \mathcal{I})$ and $B \in \tau$ then $A \subset Cl(\delta Int^+_I(A)) \cup Int(\delta Cl^+_I(A))$ and

$$\begin{aligned} A \cap B &\subset [Cl(\delta Int^+_I(A)) \cup Int(\delta Cl^+_I(A))] \cap B \\ &\subset [Cl(\delta Int^+_I(A)) \cap B] \cup [Int(\delta Cl^+_I(A)) \cap B] \\ &\subset [Cl(\delta Int^+_I(A \cap B))] \cup [Int(\delta Cl^+_I(A \cap B))]. \end{aligned}$$

This proof come from the fact $\delta Int^+_I(A)$ is the union of all R-$\mathcal{I}^+$-open of X contend in A. Then

$$\begin{aligned} A = Int(Cl^{*+}(A)) \Rightarrow A \cap B &= Int(Cl^{*+}(A)) \cap B \\ &= Int(A^{*+} \cup A) \cap B \\ &= Int[(A \cap B) \cup (A^{*+} \cap B)] \\ &\subset Int[Cl^{*+}(A \cap B)] = A \cap B \end{aligned}$$

Hence $Cl(\delta Int^+_I(A)) \cap B \subset Cl(\delta Int^+_I(A \cap B))$, and other part is obvious. ■

Let $(X, \tau^+, \mathcal{I})$ be a SEITS and A be a subset of X, we denoted the relative topology [12] on A by τ^+/A and $\mathcal{I}/A = \{A \cap I : I \in \mathcal{I}\}$ is clearly ideal on A.

Lemma 2.15. *Let $(X, \tau^+, \mathcal{I})$ be a SEITS and A, B subset of X such that $B \subset A$. Then $B^{+*}(\tau^+|_A, \mathcal{I}|_A) = B^{+*}(\tau^+, \mathcal{I}) \cap A$.*

Proposition 2.16. *Let $(X, \tau^+, \mathcal{I})$ be a SEITS and let A, $U \subseteq X$. If $V \in EI^+O(X, \tau^+, \mathcal{I})$ set and $U \in \tau$. Then $U \cap V \in EIO(U, \tau^+|_U, \mathcal{I}|_U)$.*

Proof. Since U is open, we have $Int_U(A) = Int(A)$ for any subset A of U. By using this fact and Theorem (2.12). We get the proof. ■

Definition 2.17. [12] A point $x \in X$ is said to be $\mathcal{I}^+$ limit point of A if for every $\mathcal{I}^+$ open set U in X, $U \cap (A \backslash x) \neq \emptyset$. The set of all $\mathcal{I}^+$ limit point of A is called the $\mathcal{I}^+$ derived set of A denoted by $D^+_I(A)$.

Definition 2.18. Let A be a subset of X.

(1) The intersection of all e-$\mathcal{I}^+$-closed containing A is called the e-I^+-closure of A and its denoted by $Cl^{I+}_e(A)$,

(2) The e-$\mathcal{I}^+$-interior of A, denoted by $Int^{I+}_e(A)$, is defined by the union of all e-I^+-open sets contained in A.

Definition 2.19. Let A be a subset of $(X, \tau^+, \mathcal{I})$. A point $x \in X$ is said to be $\mathcal{I}^+$ limit point of A if for every e-$\mathcal{I}^+$ open set U in X, $U \cap (A \backslash x) \neq \emptyset$. The set of all $e - \mathcal{I}^+$ limit point of A is called the $e - \mathcal{I}^+$ derived set of A denoted by $D^+_{eI}(A)$.

Since every open is pre^*-$\mathcal{I}^+$ open set and every pre^*-$\mathcal{I}^+$ open set is e-$\mathcal{I}^+$ open set we have. $D^+_{eI}(A) \subset D(A)$ for any $A \subset X$. Moreover, since every closed set is e-$\mathcal{I}^+$-closed set we have $A \subset Cl^{I+}_e(A) \subset Cl(A)$.

Lemma 2.20. *If $D^+_{eI}(A) = D(A)$, then we have $Cl^{I+}_e(A) = Cl(A)$*

Proof. Straightforward. ∎

Corollary 2.21. *If $D(A) \subset D^+_{eI}(A)$ for every subset $A \subset X$. Then for any subset C and B of X, we have $Cl^{I+}_e(B \cup C) = Cl^{I+}_e(B) \cup Cl^{I+}_e(C)$.*

Theorem 2.22. *If A be a subset of $(X, \tau^+, \mathcal{I})$, then $x \in Cl^{I+}_e(A)$ if and only if every e-$\mathcal{I}^+$ open set U containing x intersect A.*

Proof. Let us prove that $x \in Cl^{I+}_e(A)$ if and only if there exists e-$\mathcal{I}^+$ open set U containing x which does not intersect A, hence $x \notin Cl^{I+}_e(A) \Rightarrow x \notin X \backslash Cl^{I+}_e(A)$ which does not intersect A.

Conversely, let U be e-$\mathcal{I}^+$-open set U containing x which does not intersect A. Then $(X \backslash U)$ is e-$\mathcal{I}^+$-open set U containing A and $x \in (X \backslash U)$ but $Cl^{I+}_e(A) \subset X \backslash U$. ∎

Theorem 2.23. $Cl^{I+}_e(A) = A \cup D^+_{eI}(A)$.

Proof. If $x \in D^+_{eI}(A)$. Then, for every e-$\mathcal{I}^+$ open set U containing x, we have $U \cap (A \backslash x) \neq \emptyset$. Therefore $x \in Cl^{I+}_e(A)$, i.e.,

$$A \cup D^+_{eI}(A) \subseteq Cl^{I+}_e(A) \tag{$*$}$$

Conversely, let $x \in Cl^{I+}_e(A)$. If $x \in A$, then $x \in A \cup D^+_{eI}(A)$. Let $x \notin A$, since $x \in Cl^{I+}_e(A)$ every e-$\mathcal{I}^+$-open set U containing x intersects A. But $x \notin A \Rightarrow U \cap (A \backslash x) \neq \emptyset$. Therefore $x \in Cl^{I+}_e(A)$, i.e.,

$$Cl^{I+}_e(A) \subseteq A \cup D^+_{eI}(A) \tag{$**$}$$

From $(*)$ and $(**)$, we get $Cl^{I+}_e(A) = A \cup D^+_{eI}(A)$. ∎

3. Generalized e-$\mathcal{I}^+$-Closed Sets

Definition 3.1. A subset A of a SEITS $(X, \tau^+, \mathcal{I})$ is said to be gEI^+-closed if $Cl^{I+}_e(A) \subset U$ whenever $A \subset U$ and $U \in \tau^+$.

The set of all gEI^+ closed sets of X is denoted as $GEI^+C(X)$.

Example 3.2. *Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ and an ideal $\mathcal{I} = \{\emptyset, \{b\}\}$, $B = \{b\}$, $\tau^+(B) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then the sets $\{\emptyset, X, \{a\}, \{a, c\}, \{a, b\}\}$ are e-$\mathcal{I}^+$-open, and gEI^+-closed sets are $\{\emptyset, X, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$.*

Since every $\mathcal{I}^+$-closed set is e-$\mathcal{I}^+$-closed we have $Cl^{I+}_e(A) \subseteq I^+Cl(A)$.

Theorem 3.3. Let $(X, \tau^+, \mathcal{I})$ be an simple extension ideal topological space (SEITS) then the following hold:

(1) *Every $\mathcal{I}^+$-closed set is gEI^+-closed.*

(2) *Every e-$\mathcal{I}^+$-closed set is gEI^+-closed.*

Proof. (1) Since A is $\mathcal{I}^+$-closed set we have $A = I^+Cl(A) \subseteq U$ by the above note we have $Cl_e^{I+}(A) \subseteq I^+Cl(A)$, then $Cl_e^{I+}(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau^+$. Hence the proof.

(2) Let A be e-$\mathcal{I}^+$-closed set. Then $A = Cl_e^{I+}(A) \subseteq U$. Hence A is gEI^+-closed. But the converse need not be true. ∎

Example 3.4. *Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ and an ideal $\mathcal{I} = \{\emptyset, \{b\}\}$, $B = \{b\}$, $\tau^+ = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then the set $\{a, c\}$ is gEI^+-closed sets but not e-$\mathcal{I}^+$-closed.*

Theorem 3.5. *If A be gEI^+-closed of a SEITS $(X, \tau^+, \mathcal{I})$, $Cl_e^{I+}(A) \backslash A$ does not contains any nonempty closed set.*

Proof. (1) Let S be closed set such that $S \subseteq Cl_e^{I+}(A) \backslash A$. Then $(X \backslash S)$ is open and

$$S \subseteq Cl_e^{I+}(A) \cap A^c. \qquad (*)$$

$S \subseteq A^c \Rightarrow A \subset (X \backslash S)$. Since A is gEI^+-closed we have $Cl_e^{I+}(A) \subseteq (X \backslash S)$. Hence

$$S \subseteq X \backslash Cl_e^{I+}(A). \qquad (**)$$

From $(*)$ and $(**)$, we have $S \subseteq Cl_e^{I+}(A) \cap X \backslash Cl_e^{I+}(A) = \emptyset$. Hence $Cl_e^{I+}(A) \backslash A$. ∎

Theorem 3.6. *If A be gEI^+-closed set of a SEITS $(X, \tau^+, \mathcal{I})$ and $A \subseteq B \subseteq Cl_e^{I+}(A)$ then B is also gEI^+-closed.*

Proof. Let A be gEI^+-closed set and $A \subseteq B \subseteq Cl_e^{I+}(A)$. Then $Cl_e^{I+}(A) \subseteq Cl_e^{I+}(B) \subseteq Cl_e^{I+}(A)$ which implies that $Cl_e^{I+}(A) = Cl_e^{I+}(B)$ let us now consider U to be open set in $(X, \tau^+, \mathcal{I})$ containing B. Then $A \subseteq U$ and A is gEI^+-closed. $\Rightarrow Cl_e^{I+}(A) \subseteq U \Rightarrow Cl_e^{I+}(B) \subseteq U$. Then B is gEI^+-closed set. ∎

Theorem 3.7. *A gEI^+-closed set A is also e-I^+-closed if and only if $Cl_e^{I+}(A) \backslash A$ is closed.*

Proof. Let A be e-I^+-closed $A = Cl_e^{I+}(A)$. If $Cl_e^{I+}(A) \backslash A = \emptyset$ which is closed.

Conversely, let $Cl_e^{I+}(A) \backslash A$ is closed. By Theorem (3.6) we know that $Cl_e^{I+}(A) \backslash A$ does not contains any nonempty closed set. Therefor $Cl_e^{I+}(A) \backslash A = \emptyset \Rightarrow Cl_e^{I+}(A) = A$. Hence A is e-I^+-closed. ∎

Theorem 3.8. *If A and B are gEI^+-closed sets such that $D(A) \subseteq D_{eI}^+(A)$ and $D(B) \subseteq D_{eI}^+(B)$. Then $A \cup B$ is gEI^+-closed.*

Proof. Let U be an open set such that $A \cup B \subseteq U$. Then since A and B are gEI^+-closed sets we have $Cl_e^{I+}(A) \subseteq U$ $Cl_e^{I+}(B) \subseteq U$. Since $D(A) \subseteq D_{eI}^+(A)$, thus $D(A) = D_{eI}^+(A)$ and by Lemma (2.20) $Cl(A) = Cl_e^{I+}(A)$ and, similarly, $Cl(B) = Cl_e^{I+}(B)$. Thus

$$Cl_e^{I+}(A \cup B) \subseteq Cl(A \cup B) = Cl(A) \cup Cl(B) = Cl_e^{I+}(A) \cup Cl_e^{I+}(B) \subseteq U.$$

This implies $A \cup B$ is gEI^+-closed. ∎

Proposition 3.9. *If A and B are gEI^+-closed sets such that $D(A) \subseteq D^+_{eI}(A)$ and $D(B) \subseteq D^+_{eI}(B)$. Then $A \cup B$ is gEI^+-closed.*

Proof. Let U be an open set such that $A \cup B \subseteq U$. Then since A and B are gEI^+-closed sets we have $Cl^{I+}_e(A) \subseteq U$ $Cl^{I+}_e(B) \subseteq U$. Since $D(A) \subseteq D^+_{eI}(A)$, thus $D(A) = D^+_{eI}(A)$ and by Lemma (2.20) $Cl(A) = Cl^{I+}_e(A)$ and similarly $Cl(B) = Cl^{I+}_e(B)$. Thus

$$Cl^{I+}_e(A \cup B) \subseteq Cl(A \cup B) = Cl(A) \cup Cl(B) = Cl^{I+}_e(A) \cup Cl^{I+}_e(B) \subseteq U.$$

This implies $A \cup B$ is gEI^+-closed. ■

Definition 3.10. Let $B \subseteq A \subseteq X$. The set B is said to be gEI^+-closed relative to A if ${}_ACl^{I+}_e(B) \subseteq U$ whenever $B \subseteq U$ and U is open in A, where ${}_ACl^{I+}_e(B) = A \cap Cl^{I+}_e(B)$.

Theorem 3.11. *If $B \subseteq A \subseteq X$ and A is gEI^+-closed and open, then B is gEI^+-closed relative to A if and only if B is gEI^+-closed in X.*

Proof. Let A be a gEI^+-closed and open. Let B is gEI^+-closed relative to A. Since A be an gEI^+-closed and open then $Cl^{I+}_e(A) \subseteq A$. Therefore, $Cl^{I+}_e(B) \subseteq Cl^{I+}_e(A) \subseteq A$. Therefore, ${}_ACl^{I+}_e(B) \subseteq Cl^{I+}_e(B) \cap A = Cl^{I+}_e(B)$. Now, let U be open in X and $B \subseteq U$. Then, $U \cap A$ is open in A and $B \subseteq U \cap A$. Since B is gEI^+-closed relative to A we have ${}_ACl^{I+}_e(B) \subseteq U \cap A$. Hence ${}_ACl^{I+}_e(B) \subseteq U \cap A \subseteq U$. Therefore, B is gEI^+-closed. Conversely, let B is gEI^+-closed in X. Consider U is an open in A and $B \subseteq U$. Then $U = V \cap A$ where V is open in $(X, \tau^+, \mathcal{I})$. Now $B \subseteq V and\ B$ is gEI^+-closed in X. This implies $Cl^{I+}_e(B) \cap A \subset V \cap A = U$, i.e., $Cl^{I+}_e(B) \cap A \subseteq U$. ■

Definition 3.12. A set A is said to be gEI^+-open if and only if $(X \backslash A)$ is gEI^+-closed. The family of all gEI^+-open subset of X is denoted by $GEI^+O(X)$. The largest gEI^+-open set contained in X is called the gEI^+-interior of A and is denoted by $gEI^+(Int(A))$ also A is gEI^+-open if and only if $gEI^+(Int(A)) = A$.

Proposition 3.13. *Let $(X, \tau^+, \mathcal{I})$ be an simple extension ideal topological space (SEITS) then Statement $Cl^{I+}_e(X \backslash A) = X \backslash Cl^{I+}_e(A)$ hold.*

Proof. Let $x \in Cl^{I+}_e(X \backslash A)$.
⇔ every e-$\mathcal{I}^+$-open set U containing x intersects $(X \backslash A)$.
⇔ there is no e-$\mathcal{I}^+$-open set U containing x and contained in A.
⇔ $x \in X \backslash Cl^{I+}_e(A)$. ■

Theorem 3.14. *A subset A of a SETIS $(X, \tau^+, \mathcal{I})$ is gEI^+-open if and only if $S \subseteq Int^{I+}_e(A)$ where S is closed and $S \subseteq A$.*

Proof. Let A be gEI^+-open and suppose that S is closed and $S \subseteq A$. Then $(X \backslash A)$ is gEI^+-closed and $(X \backslash A) \subset (X \backslash S)$. Now, $(X \backslash S)$ is open and $(X \backslash A)$ is gEI^+-closed. Therefore, $Cl^{I+}_e(X \backslash A) \subseteq (X \backslash S)$. By Proportion (3.13) $Cl^{I+}_e(X \backslash A) = X \backslash Int^{I+}_e(A)$. Hence $X \backslash Int^{I+}_e(A) \subseteq (X \backslash S)$. i.e., $S \subseteq Int^{I+}_e(A)$.

Conversely, let $S \subseteq Int^{I+}_e(A)$ where S is closed and $S \subseteq A$. Now, to prove A is gEI^+-open is the same as proving $(X \backslash A)$ gEI^+-closed. Let G be an open set containing $(X \backslash A)$ then $S = (X \backslash G)$ is closed set such that $S \subseteq A$. Therefore, $S \subseteq Int^{I+}_e(A)$. i.e., $Cl^{I+}_e(X \backslash A) = (X \backslash Cl^{I+}_e(X \backslash A)) \subset (X \backslash S)$ $Cl^{I+}_e(X \backslash A) \subseteq G$. Therefore, $(X \backslash A)$ gEI^+-closed. i.e., A is gEI^+-open. Hence the proof. ■

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