THE HOMOGENEOUS BALANCE METHOD AND ITS APPLICATIONS FOR FINDING THE EXACT SOLUTIONS FOR NONLINEAR EVOLUTION EQUATIONS

Elsayed M.E. Zayed[1]
Khaled A.E. Alurrfi

Department of Mathematics
Faculty of Science
Zagazig University
Zagazig
Egypt

Abstract. In this article, we apply the homogeneous balance method to find the exact solutions of some nonlinear evolution equations in mathematical physics, namely, the Kaup-Kupershmidt equation, the Ito equation, the Caudrey-Dodd-Gibbon equation, the Lax equation and the Sawada-Kotera equation. These equations have wide applications in quantum mechanics and non linear optics. The efficiency of this method for constructing these exact solutions is demonstrated.

Keywords: the homogeneous balance method, nonlinear evolution equations, exact solutions.

AMS Subject Classifications: 35K99, 35P05, 35P99.

1. Introduction

When a nonlinear evolution equation is analyzed, one of the most important question is the construction of the exact solutions of that equation. Searching for exact solutions of that equation plays an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena appears in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, geochemistry and so on. In the past several decades, exact solutions may help to find new phenomena. Many powerful methods for obtaining these exact solutions are presented, such as the inverse scattering transform [1], the Hirota method [3], the truncated Painleve expansion [9], the Backlund transform [1], [15], the exp-function method [20], [21], the simplest equation method [10], the Weierstrass elliptic function method [7], the Jacobi elliptic function method [13], [14], the tanh-function method [18], the (G'/G)-expansion method [22], the modified simple equation method [5], [25], the homogeneous balance method [8], [16], [17], [26] and so on. To realize these

[1]Corresponding author. E-mail address: e.m.e.zayed@hotmail.com

methods, one applies some special functions, then exact solutions read as a finite series in these special functions.

The objective of this article is to demonstrate efficiency of the homogeneous balance method for finding exact solutions of some nonlinear evolution equations in the mathematical physics, namely, the Kaup-Kupershmidt equation, the Ito equation, the Caudrey-Dodd-Gibbon equation, the Lax equation and the Sawada-Kotera equation. This article is organized as follows. In Section 2, we give the description of the homogeneous balance method. In Section 3, we apply this method to five nonlinear evolution equations indicated above. In Section 4, physical explanations of our obtained solutions are given. In Section 5, conclusions are given.

2. Description of the homogeneous balance method

Suppose we have a nonlinear evolution equation in the form

$$F(u, u_t, u_x, u_{xx}, ...) = 0, \tag{2.1}$$

where F is a polynomial in $u(x, t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method [8], [16], [17], [26]:

Step 1. Using the wave transformation

$$u(x, t) = u(\xi),\ \xi = kx + \omega t, \tag{2.2}$$

to reduce equation (2.1) to the following ODE:

$$P(u, u', u'', ...) = 0, \tag{2.3}$$

where P is a polynomial in $u(\xi)$ and its total derivatives, while k, ω are constants and $' = d/d\xi$

Step 2. We suppose that equation (2.3) has the formal solution

$$u(\xi) = \sum_{n=0}^{N} a_n Q(\xi)^n, \tag{2.4}$$

where $a_n (n = 0, 1, ...N)$ are constants to be determined, such that $a_N \neq 0$, and $Q(\xi)$ is the solution of the equation

$$Q'(\xi) = Q^2(\xi) - Q(\xi). \tag{2.5}$$

equation (2.5) has the solution

$$Q(\xi) = \frac{1}{1 \pm e^{\xi}}. \tag{2.6}$$

Step 3. We determine the positive integer N in equation (2.4) by considering the homogeneous balance between the highest order derivatives and the nonlinear terms in equation (2.3).

Step 4. Substitute equation (2.4) into equation (2.3), we calculate all the necessary derivatives $u', u'', ...$ of the function $u(\xi)$. As a result of this substitution, we get a polynomial of $Q^i, (i = 0, 1, 2, ...)$. In this polynomial we gather all terms of same powers and equating them to zero, we obtain a system of algebraic equations which can be solved by the Maple or Mathematica to get the unknown parameters a_n $(n = 0, 1, ..., N)$, k and ω. Consequently, we obtain the exact solutions of equation (2.1).

3. Applications

In this section, we apply the homogeneous balance method to find the exact solutions of the following nonlinear partial differential equations:

3.1. Example 1. The Kaup-Kupershmidt (KK) equation

This equation is well known [6], [11], [23], [24] and has the form

$$u_t + 20u^2u_x + 25u_xu_{xx} + 10uu_{3x} + u_{5x} = 0. \tag{3.1}$$

Let us now solve equation (3.1) by using the homogeneous balance method. To this end, we use the wave transformation (2.2) to reduce equation (3.1) to the following ODE:

$$\omega u' + 20ku^2u' + 25k^3u'u'' + 10k^3uu^{(3)} + k^5u^{(5)} = 0. \tag{3.2}$$

Balancing $u^{(5)}$ with u^2u' yields $N = 2$. Consequently, equation (3.2) has the formal solution

$$u = a_0 + a_1Q + a_2Q^2, \tag{3.3}$$

where a_0, a_1 and a_2 are constants to be determined such that $a_2 \neq 0$. From equation (3.3), we get

$$u' = (Q-1)Q\left(a_1 + 2Qa_2\right), \tag{3.4}$$

$$u'' = (Q-1)Q\left[(-1+2Q)a_1 + 2Q(-2+3Q)a_2\right], \tag{3.5}$$

$$u^{(3)} = (Q-1)Q\left[\left(1-6Q+6Q^2\right)a_1 + 2Q\left(4-15Q+12Q^2\right)a_2\right], \tag{3.6}$$

$$\begin{aligned} u^{(4)} = {} & (Q-1)Q[\left(-1+14Q-36Q^2+24Q^3\right)a_1 \\ & +2Q\left(-8+57Q-108Q^2+60Q^3\right)a_2], \end{aligned} \tag{3.7}$$

$$u^{(5)} = (Q-1)Q[\left(1-30Q+150Q^2-240Q^3+120Q^4\right)a_1 \tag{3.8}$$

$$+2Q\left(16-195Q+660Q^2-840Q^3+360Q^4\right)a_2]. \tag{3.9}$$

Substituting (3.3)-(3.8) into (3.2) and equating all the coefficients of powers of $Q(\xi)$ to zero, we obtain

$$-k^5a_1 - \omega a_1 - 10k^3a_0a_1 - 20ka_0^2a_1 = 0, \tag{3.10}$$

$$\begin{aligned}&31k^5a_1 + \omega a_1 + 70k^3a_0a_1 + 20ka_0^2a_1 - 35k^3a_1^2 - \\ &- 40ka_0a_1^2 - 32k^5a_2 - 2\omega a_2 - 80k^3a_0a_2 - 40ka_0^2a_2 = 0,\end{aligned} \tag{3.11}$$

$$\begin{aligned}&- 180k^5a_1 - 120k^3a_0a_1 + 170k^3a_1^2 + 40ka_0a_1^2 - 20ka_1^3 + 422k^5a_2 \\ &+ 2\omega a_2 + 380k^3a_0a_2 + 40ka_0^2a_2 - 240k^3a_1a_2 - 120ka_0a_1a_2 = 0,\end{aligned} \tag{3.12}$$

$$\begin{aligned}&390k^5a_1 + 60k^3a_0a_1 - 245k^3a_1^2 + 20ka_1^3 - 1710k^5a_2 - 540k^3a_0a_2 \\ &+ 1000k^3a_1a_2 + 120ka_0a_1a_2 - 80ka_1^2a_2 - 280k^3a_2^2 - 80ka_0a_2^2 = 0,\end{aligned} \tag{3.13}$$

$$\begin{aligned}&- 360k^5a_1 + 110k^3a_1^2 + 3000k^5a_2 + 240k^3a_0a_2 - 1310k^3a_1a_2 + 80ka_1^2a_2 \\ &+ 1080k^3a_2^2 + 80ka_0a_2^2 - 100ka_1a_2^2 = 0,\end{aligned} \tag{3.14}$$

$$120k^5a_1 - 2400k^5a_2 + 550k^3a_1a_2 - 1340k^3a_2^2 + 100ka_1a_2^2 - 40ka_2^3 = 0, \tag{3.15}$$

$$720k^5a_2 + 540k^3a_2^2 + 40ka_2^3 = 0. \tag{3.16}$$

Solving the system of equations (3.9)-(3.15) by using the Maple or Mathematica, we obtain

Case 1.

$$\omega = -\frac{k^5}{16}, \quad a_0 = -\frac{k^2}{8}, \quad a_1 = \frac{3k^2}{2}, \quad a_2 = -a_1. \tag{3.17}$$

The solution of equation (3.1) corresponding to (3.16) is

$$u_1(x,t) = \frac{-k^2}{8} + \frac{3k^2}{8} \sec h^2 \left[\frac{k}{2}x - \frac{k^5}{32}t\right], \tag{3.18}$$

$$u_2(x,t) = \frac{-k^2}{8} - \frac{3k^2}{8} \csc h^2 \left[\frac{k}{2}x - \frac{k^5}{32}t\right], \tag{3.19}$$

Case 2.

$$\omega = -11k^5, \quad a_0 = -k^2, \quad a_1 = 12k^2, \quad a_2 = -a_1. \tag{3.20}$$

The solution of equation (3.1) corresponding to (3.19) is

$$u_3(x,t) = -k^2 + 3k^2 \sec h^2 \left[\frac{k}{2}x - \frac{11k^5}{2}t\right], \tag{3.21}$$

$$u_4(x,t) = -k^2 - 3k^2 \csc h^2 \left[\frac{k}{2}x - \frac{11k^5}{2}t\right]. \tag{3.22}$$

3.2. Example 2. The Ito equation

This equation is well known [4], [23], [24] and has the form:

$$u_t + 2u^2 u_x + 6u_x u_{xx} + 3uu_{3x} + u_{5x} = 0. \tag{3.23}$$

Let us solve equation (3.22) by using the homogeneous balance method. To this end, we use the wave transformation (2.2) to reduce equation (3.22) to the following ODE:

$$\omega u' + 2ku^2 u' + 6k^3\ u'u'' + 3k^3 uu^{(3)} + k^5 u^{(5)} = 0. \tag{3.24}$$

Balancing $u^{(5)}$ with u^2u' yields $N = 2$. Consequently, equation (3.23) has the formal solution (3.3). Substituting (3.3)-(3.8) into (3.23) and equating all the coefficients of powers of $Q(\xi)$ to zero, we obtain

$$-k^5 a_1 - \omega a_1 - 3k^3 a_0 a_1 - 2ka_0^2 a_1 = 0, \tag{3.25}$$

$$\begin{aligned} &31k^5 a_1 + \omega a_1 + 21k^3 a_0 a_1 + 2ka_0^2 a_1 - 9k^3 a_1^2 - 4ka_0 a_1^2 - 32k^5 a_2 \\ &\quad -2\omega a_2 - 24k^3 a_0 a_2 - 4ka_0^2 a_2 = 0, \end{aligned} \tag{3.26}$$

$$\begin{aligned} &-180k^5 a_1 - 36k^3 a_0 a_1 + 45k^3 a_1^2 + 4ka_0 a_1^2 - 2ka_1^3 + 422k^5 a_2 \\ &\quad +2\omega a_2 + 114k^3 a_0 a_2 + 4ka_0^2 a_2 - 63k^3 a_1 a_2 - 12ka_0 a_1 a_2 = 0, \end{aligned} \tag{3.27}$$

$$\begin{aligned} &390k^5 a_1 + 18k^3 a_0 a_1 - 66k^3 a_1^2 + 2ka_1^3 - 1710k^5 a_2 - 162k^3 a_0 a_2 \\ &\quad +267k^3 a_1 a_2 + 12ka_0 a_1 a_2 - 8ka_1^2 a_2 - 72k^3 a_2^2 - 8ka_0 a_2^2 = 0, \end{aligned} \tag{3.28}$$

$$\begin{aligned} &-360k^5 a_1 + 30k^3 a_1^2 + 3000k^5 a_2 + 72k^3 a_0 a_2 - 354k^3 a_1 a_2 \\ &\quad +8ka_1^2 a_2 + 282k^3 a_2^2 + 8ka_0 a_2^2 - 10ka_1 a_2^2 = 0, \end{aligned} \tag{3.29}$$

$$120k^5 a_1 - 2400k^5 a_2 + 150k^3 a_1 a_2 - 354k^3 a_2^2 + 10ka_1 a_2^2 - 4ka_2^3 = 0, \tag{3.30}$$

$$720k^5 a_2 + 144k^3 a_2^2 + 4ka_2^3 = 0. \tag{3.31}$$

Solving the system of equations (3.24)-(3.30), using the Maple or Mathematica we obtain

$$\omega = -6k^5, \quad a_0 = -\frac{5k^2}{2}, \quad a_1 = 30k^2, \quad a_2 = -a_1. \tag{3.32}$$

The solution of equation (3.22) corresponding to (3.31) is

$$u_1(x,t) = \frac{-5k^2}{2} + \frac{15k^2}{2} \sec h^2 \left[\frac{k}{2}x - \frac{6k^5}{2}t\right], \tag{3.33}$$

$$u_2(x,t) = \frac{-5k^2}{2} - \frac{15k^2}{2} \csc h^2 \left[\frac{k}{2}x - \frac{6k^5}{2}t\right]. \tag{3.34}$$

3.3. Example 3. The Caudrey-Dodd-Gibbon equation (CDG)

This equation is well known [2], [23], [24] and has the form:

$$u_t + 180u^2u_x + 30u_xu_{xx} + 30uu_{3x} + u_{5x} = 0. \tag{3.35}$$

Let us solve equation (3.34) by using the homogeneous balance method. To this end. we use the wave transformation (2.2) to reduce equation (3.34) to the following ODE :

$$\omega u' + 180ku^2u' + 30k^3\ u'u'' + 30\ k^3uu^{(3)}\ + k^5u^{(5)} = 0. \tag{3.36}$$

Balancing $u^{(5)}$ with u^2u' yields $N = 2$. Consequently, equation (3.35) has the formal solution (3.3). Substituting (3.3)-(3.8) into (3.35) and equating all the coefficients of powers of $Q(\xi)$ to zero, we obtain

$$-k^5a_1 - \omega a_1 - 30k^3a_0a_1 - 180ka_0^2a_1 = 0, \tag{3.37}$$

$$\begin{aligned} &31k^5a_1 + \omega a_1 + 210k^3a_0a_1 + 180ka_0^2a_1 - 60k^3a_1^2 \\ &\quad -360ka_0a_1^2 - 32k^5a_2 - 2\omega a_2 - 240k^3a_0a_2 - 360ka_0^2a_2 = 0, \end{aligned} \tag{3.38}$$

$$\begin{aligned} &-180k^5a_1 - 360k^3a_0a_1 + 330k^3a_1^2 + 360ka_0a_1^2 - 180ka_1^3 \\ &\quad +422k^5a_2 + 2\omega a_2 + 1140k^3a_0a_2 + 360ka_0^2a_2 - 450k^3a_1a_2 \\ &\quad -1080ka_0a_1a_2 = 0, \end{aligned} \tag{3.39}$$

$$\begin{aligned} &390k^5a_1 + 180k^3a_0a_1 - 510k^3a_1^2 + 180ka_1^3 - 1710k^5a_2 \\ &\quad -1620k^3a_0a_2 + 2010k^3a_1a_2 + 1080ka_0a_1a_2 - 720ka_1^2a_2 \\ &\quad -480k^3a_2^2 - 720ka_0a_2^2 = 0, \end{aligned}$$

$$\begin{aligned} &-360k^5a_1 + 240k^3a_1^2 + 3000k^5a_2 + 720k^3a_0a_2 - 2760k^3a_1a_2 \\ &\quad +720ka_1^2a_2 + 1980k^3a_2^2 + 720ka_0a_2^2 - 900ka_1a_2^2 = 0, \end{aligned} \tag{3.40}$$

$$\begin{aligned} &120k^5a_1 - 2400k^5a_2 + 1200k^3a_1a_2 - 2580k^3a_2^2 \\ &\quad +900ka_1a_2^2 - 360ka_2^3 = 0, \end{aligned} \tag{3.41}$$

$$720k^5a_2 + 1080k^3a_2^2 + 360ka_2^3 = 0. \tag{3.42}$$

Solving the system of equations (3.36)-(3.42) using the Maple or Mathematica, we obtain

Case 1.

$$\omega = -k^5 - 30k^3a_0 - 180ka_0{}^2, \quad a_1 = k^2, \quad a_2 = -a_1. \tag{3.43}$$

The solution of equation (3.34) corresponding to (3.43) is

$$u_1(x,t) \ = \ a_0 + \frac{k^2}{4}\sec h^2\left[\frac{k}{2}\ x - \frac{(k^5 + 30k^3a_0 + 180ka_0{}^2)}{2}\ t\right], \tag{3.44}$$

$$u_2(x,t) \ = \ a_0 - \frac{k^2}{4}\csc h^2\left[\frac{k}{2}\ x - \frac{(k^5 + 30k^3a_0 + 180ka_0{}^2)}{2}\ t\right], \tag{3.45}$$

where a_0 is an arbitrary constant.

Case 2.

$$\omega = -k^5, \quad a_0 = -\frac{k^2}{6}, \quad a_1 = 2k^2, \quad a_2 = -a_1. \tag{3.46}$$

The solution of equation (3.34) corresponding to (3.46) is

$$u_1(x,t) = \frac{-k^2}{6} + \frac{k^2}{2}\sec h^2\left[\frac{k}{2}\,x - \frac{k^5}{2}\,t\right]. \tag{3.47}$$

$$u_2(x,t) = \frac{-k^2}{6} - \frac{k^2}{2}\csc h^2\left[\frac{k}{2}\,x - \frac{k^5}{2}\,t\right]. \tag{3.48}$$

3.4. Example 4. The Lax equation

This equation is well known [12], [23], [24] and has the form:

$$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{3x} + u_{5x} = 0. \tag{3.49}$$

Let us now solve equation (3.49) by using the homogeneous balance method. To this end, we use the wave transformation (2.2) to reduce equation (3.49) to the following ODE:

$$\omega u' + 30ku^2u' + 20k^3\; u'u'' + 10k^3uu^{(3)} + k^5u^{(5)} = 0. \tag{3.50}$$

Balancing $u^{(5)}$ with u^2u' yields $N = 2$. Consequently, equation (3.50) has the formal solution (3.3). Substituting (3.3)-(3.8) into (3.50) and equating all the coefficients of powers of $Q(\xi)$ to zero, we obtain

$$-k^5a_1 - \omega a_1 - 10k^3a_0a_1 - 30ka_0^2a_1 = 0, \tag{3.51}$$

$$\begin{aligned} &31k^5a_1 + \omega a_1 + 70k^3a_0a_1 + 30ka_0^2a_1 - 30k^3a_1^2 - 60ka_0a_1^2 \\ &\quad -32k^5a_2 - 2\omega a_2 - 80k^3a_0a_2 - 60ka_0^2a_2 = 0, \end{aligned} \tag{3.52}$$

$$\begin{aligned} &-180k^5a_1 - 120k^3a_0a_1 + 150k^3a_1^2 + 60ka_0a_1^2 - 30ka_1^3 \\ &\quad +422k^5a_2 + 2\omega a_2 + 380k^3a_0a_2 + 60ka_0^2a_2 \\ &\quad -210k^3a_1a_2 - 180ka_0a_1a_2 = 0, \end{aligned} \tag{3.53}$$

$$\begin{aligned} &390k^5a_1 + 60k^3a_0a_1 - 220k^3a_1^2 + 30ka_1^3 - 1710k^5a_2 \\ &\quad -540k^3a_0a_2 + 890k^3a_1a_2 + 180ka_0a_1a_2 \end{aligned} \tag{3.54}$$

$$-120ka_1^2a_2 - 240k^3a_2^2 - 120ka_0a_2^2 = 0, \tag{3.55}$$

$$\begin{aligned} &-360k^5a_1 + 100k^3a_1^2 + 3000k^5a_2 + 240k^3a_0a_2 - 1180k^3a_1a_2 \\ &\quad +120ka_1^2a_2 + 940k^3a_2^2 + 120ka_0a_2^2 - 150ka_1a_2^2 = 0, \end{aligned} \tag{3.56}$$

$$\begin{aligned} &120k^5a_1 - 2400k^5a_2 + 500k^3a_1a_2 - 1180k^3a_2^2 \\ &\quad +150ka_1a_2^2 - 60ka_2^3 = 0, \end{aligned} \tag{3.57}$$

$$720k^5a_2 + 480k^3a_2^2 + 60ka_2^3 = 0. \tag{3.58}$$

Solving the system of equations (3.51)-(3.57)using the Maple or Mathematica, we obtain

Case 1.

$$\omega = -k^5 - 10k^3 a_0 - 30ka_0^2, \quad a_1 = 2k^2, \quad a_2 = -a_1. \tag{3.59}$$

The solution of equation (3.49) corresponding (3.58) is

$$u_1(x,t) = a_0 + \frac{k^2}{2}\sec h^2\left[\frac{k}{2}\, x - \frac{(k^5 + 10k^3 a_0 + 30ka_0^2)}{2}\, t\right], \tag{3.60}$$

$$u_2(x,t) = a_0 - \frac{k^2}{2}\csc h^2\left[\frac{k}{2}\, x - \frac{(k^5 + 10k^3 a_0 + 30ka_0^2)}{2}\, t\right], \tag{3.61}$$

where a_0 is an arbitrary constant.

Case 2.

$$\omega = -\frac{7k^5}{2}, \quad a_0 = -\frac{k^2}{2}, \quad a_1 = 6k^2, \quad a_2 = -a_1. \tag{3.62}$$

The solution of equation (3.49) corresponding (3.61) is

$$u_3(x,t) = \frac{-k^2}{2} + \frac{3k^2}{2}\sec h^2\left[\frac{k}{2}x - \frac{7k^5}{4}t\right], \tag{3.63}$$

$$u_4(x,t) = \frac{-k^2}{2} - \frac{3k^2}{2}\csc h^2\left[\frac{k}{2}x - \frac{7k^5}{4}t\right]. \tag{3.64}$$

3.5. Example 5. The Sawada-Kotera (SK) equation

This equation is well known [19], [23], [24] and has the form:

$$u_t + 5u^2 u_x + 5u_x u_{xx} + 5uu_{3x} + u_{5x} = 0. \tag{3.65}$$

Let us solve equation (3.64)using the homogeneous balance method. To this end, we use the wave transformation (2.2) to reduce equation (3.64) to the following ODE:

$$\omega u' + 5ku^2 u' + 5k^3\ u'u'' + 5k^3 uu^{(3)} + k^5 u^{(5)} = 0. \tag{3.66}$$

Balancing $u^{(5)}$ with u^2u' yields $N = 2$. Consequently, equation (3.65) has the formal solution (3.3). Substituting (3.3)-(3.8) into (3.65) and equating all the coefficients of powers of $Q(\xi)$ to zero, we obtain

$$-k^5a_1 - \omega a_1 - 5k^3a_0a_1 - 5ka_0^2a_1 = 0, \tag{3.67}$$

$$31k^5a_1 + \omega a_1 + 35k^3a_0a_1 + 5ka_0^2a_1 - 10k^3a_1^2 - 10ka_0a_1^2 - 32k^5a_2 = 0$$
$$-2\omega a_2 - 40k^3a_0a_2 - 10ka_0^2a_2 = 0, \tag{3.68}$$

$$\begin{aligned}&-180k^5a_1 - 60k^3a_0a_1 + 55k^3a_1^2 + 10ka_0a_1^2 - 5ka_1^3\\ &+422k^5a_2 + 2\omega a_2 + 190k^3a_0a_2 + 10ka_0^2a_2\\ &-75k^3a_1a_2 - 30ka_0a_1a_2 = 0,\end{aligned} \tag{3.69}$$

$$\begin{aligned}&390k^5a_1 + 30k^3a_0a_1 - 85k^3a_1^2 + 5ka_1^3 - 1710k^5a_2\\ &-270k^3a_0a_2 + 335k^3a_1a_2 + 30ka_0a_1a_2\end{aligned} \tag{3.70}$$
$$-20ka_1^2a_2 - 80k^3a_2^2 - 20ka_0a_2^2 = 0, \tag{3.71}$$

$$\begin{aligned}&-360k^5a_1 + 40k^3a_1^2 + 3000k^5a_2 + 120k^3a_0a_2 - 460k^3a_1a_2\\ &+20ka_1^2a_2 + 330k^3a_2^2 + 20ka_0a_2^2 - 25ka_1a_2^2 = 0,\end{aligned} \tag{3.72}$$

$$\begin{aligned}&120k^5a_1 - 2400k^5a_2 + 200k^3a_1a_2 - 430k^3a_2^2\\ &+25ka_1a_2^2 - 10ka_2^3 = 0,\end{aligned} \tag{3.73}$$

$$720k^5a_2 + 180k^3a_2^2 + 10ka_2^3 = 0. \tag{3.74}$$

Solving the system of equations (3.66)-(3.72) using the Maple or Mathematica, we obtain

Case 1.

$$\omega = -k^5 - 5k^3a_0 - 5ka_0^2, \quad a_1 = 6k^2, \quad a_2 = -a_1. \tag{3.75}$$

The solution of equation (3.64) corresponding (3.73) is

$$u_1(x,t) = a_0 + \frac{3k^2}{2}\sec h^2\left[\frac{k}{2}\,x - \frac{(k^5 + 5k^3a_0 + 5ka_0^2)}{2}\,t\right], \tag{3.76}$$

$$u_2(x,t) = a_0 - \frac{3k^2}{2}\csc h^2\left[\frac{k}{2}\,x - \frac{(k^5 + 5k^3a_0 + 5ka_0^2)}{2}\,t\right], \tag{3.77}$$

where a_0 is an arbitrary constant.

Case 2.

$$\omega = -k^5, \quad a_0 = -k^2, \quad a_1 = 12k^2, \quad a_2 = -a_1. \tag{3.78}$$

The solution of equation (3.64) corresponding (3.76) is

$$u_3(x,t) = -k^2 + 3k^2\sec h^2\left[\frac{k}{2}\,x - \frac{k^5}{2}\,t\right], \tag{3.79}$$

$$u_4(x,t) = -k^2 - 3k^2\csc h^2\left[\frac{k}{2}\,x - \frac{k^5}{2}\,t\right]. \tag{3.80}$$

4. Physical explanations of our obtained solutions

Solitary bell-type waves have been obtained. In this section we have presented some graphs of these solutions by taking suitable values of involved unknown parameters to visualize the underlying mechanism of the original equations. Using mathematical software Maple or Mathematica, the plots of some obtained solutions of equations (3.1), (3.22) and (3.34) have been shown in Figs. 1-3.

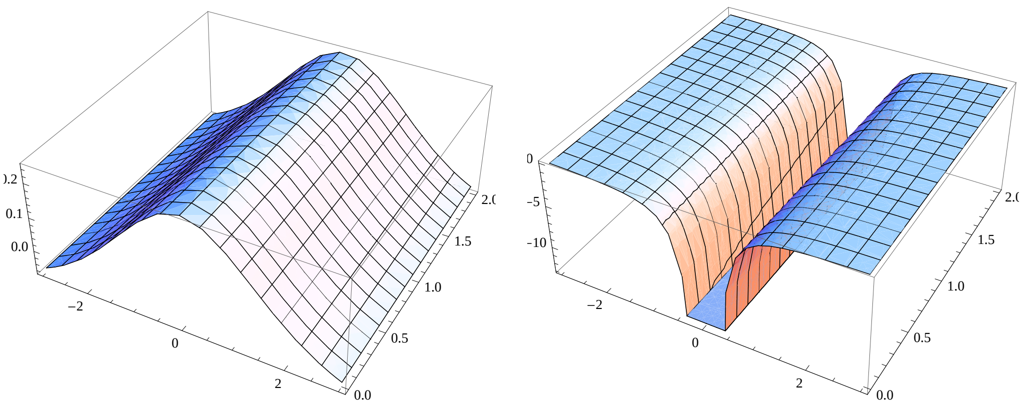

Figure 1: The plot of the solutions (3.17) and (3.18), when $k = 1$

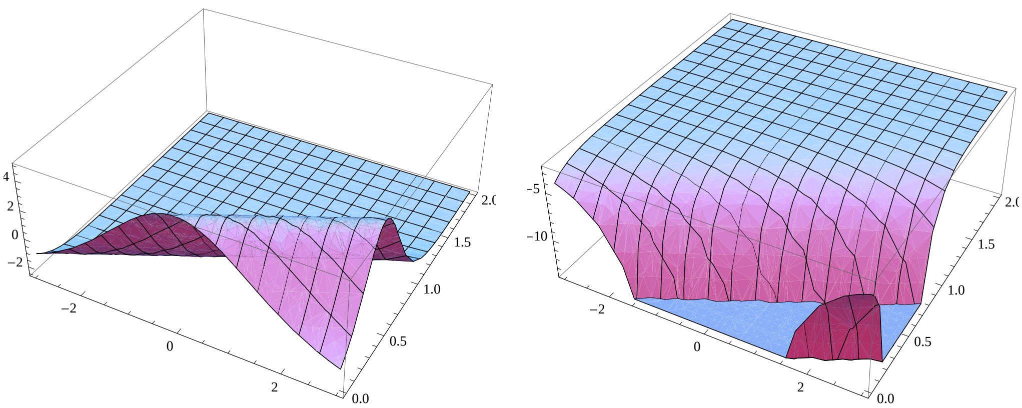

Figure 2: The plot of the solutions (3.32) and (3.33), when $k = 1$

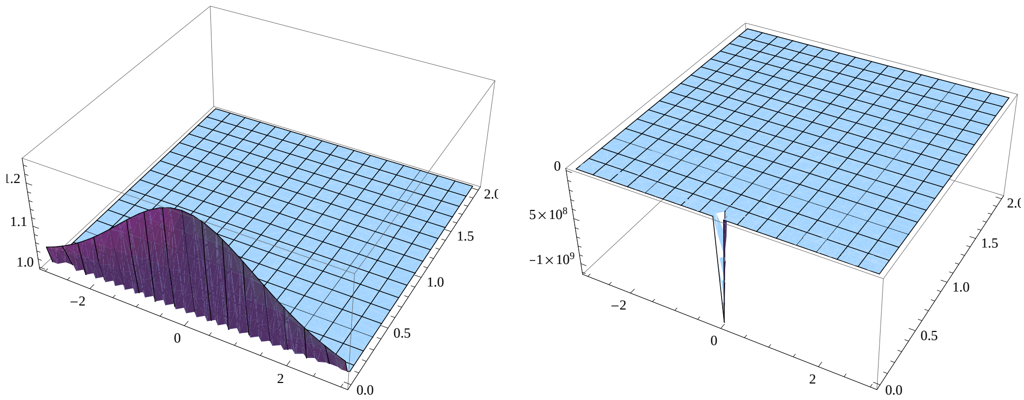

Figure 3: The plot of the solutions (3.44) and (3.45), when $a_0 = 1$, $k = 1$

5. Conclusions

The homogeneous balance method presented in this article has been applied to the nonlinear Kaup-Kupershmidt equation, the nonlinear Ito equation, the nonlinear Caudrey-Dodd-Gibbon equation, the nonlinear Lax equation and the Sawada-Kotera equation for finding the exact solutions of these equations which attract the attention of many authors. On comparing this method with the other methods, we see that the homogeneous balance method is much more simpler than these methods. Also we deduce that the homogeneous balance method is direct, effective and can be applied to many other nonlinear evolution equations.

References

[1] Ablowitz, M.J., Clarkson, P.A., *Solitons, nonlinear evolution equation and inverse scattering*, Cambridge University Press, New York, 1991.

[2] Caudrey, P.J., Dodd, R.K., Gibbon, J.D., *A new hierarchy of Korteweg-de Vries equations*, Proc. Roy. Soc. Lond., A 351 (1976), 407422.

[3] Hirota, R., *Exact solutions of KdV equation for multiple collisions of solitons*, Phys. Rev. Lett., 27 (1971), 1192-1194.

[4] Ito, M., *An extension of nonlinear evolution equations of the KdV (mKdV) type to higher orders*, J. Phys. Soc. Japan, 49 (2) (1980), 771778.

[5] Jawad, A.J.M., Petkovic, M.D., Biswas, A., *Modified simple equation method for nonlinear evolution equations*, Appl. Math. Comput., 217 (2010), 869-877.

[6] Kaup, D.J., *On the inverse scattering problem for cubic eigenvalue problems of the class* $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$, Stud. Appl. Math., 62 (1980), 189216.

[7] Kudryashov, N.A., *Exact solutions of generalized Kuramoto. Sivashinsky equation,* Phys. Lett. A, 147 (1990), 287-291.

[8] Kudryashov, N.A., *On one of methods for finding exact solutions of nonlinear differential equations*, arXiv:1108.3288v1[nlin.SI] 16 Aug 2011.

[9] Kudryashov, N.A., *On types of nonlinear non-integrable equations with exact solutions*, Phys. Lett. A, 155 (1991), 269-275.

[10] Kudryashov, N.A., Loguinova, N.B., *Extended simplest equation method for nonlinear differential equations*, Appl. Math. Comput., 205 (2008), 396-402.

[11] Kuperschmidt, B.A., *A super KdV equation: an integrable system*, Phys. Lett., A 102, (1984), 213215.

[12] Lax, P.D., *Integrals of nonlinear equations of evolution and solitary waves*, Commun. Pure Appl. Math., 21 (1968), 467490.

[13] Liu, S., Fu, Z., Liu, S., Zhao, Q., *Jacobi elliptic function expansion method and periodic wave solutions of non linear wave equations*, Phys. Lett., A 289 (2001), 69-74.

[14] Lu, D., Shi, Q., *New Jacobi elliptic functions solutions for the combined KdV-mKdV equation*, Int. J. Nonlinear Sci., 10 (2010), 320-325.

[15] Miura, M.R., *Backlund transformation*, Berlin, Springer, 1978.

[16] Ryabov, P.N., *Exact solutions of the KudryashovSinelshchikov equation*, Appl. Math. Comput., 217 (2010), 3585-3590.

[17] Ryabov, P.N., Sinelshchikov, D.I., Kochanov, M.B., *Application of the Kudryashov method for finding exact solutions of the high order nonlinear evolution equations*, Appl. Math. Comput., 218 (2011), 3965-3972.

[18] Parkes, E.J., Duffy, B.R., *An automated tansh-function method for finding solitary wave solutions to nonlinear evolution equations*, Comput. Phys. Comm., 698 (2005), 288-300.

[19] Sawada, K., Kotera, T., *A method for finding N-soliton solutions of the KdV equation and the KdV-like equation*, Progr. Theoret. Phys., 51 (1974), 13551367.

[20] EL-Wakil, S.A., Madkour, M.A., Abdou, M.A., *Application of exp-function method for nonlinear evolution equations with variable coefficients*, Phys. Lett., A 369 (2007), 62-69.

[21] Wang, Y.P., *Solving the* (3+1)*-dimensional potential-YTSF equation with Exp-function method 2007 ISDN*, J. Phys. Conf. Ser. 2008, 96:012186.

[22] Wang, M., Li, X., Zhang, J., *The* (G'/G)*-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics*, Phys. Lett., A 372 (2008), 417-423.

[23] Wazwaz, A.M., *Partial Differential Equations and Solitary Waves Theory, Springer-Verlag*, Heidelberg, 2009.

[24] Wazwaz, A.M., *The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations*, Appl. Math. Comput., 184 (2007), 1002-1014.

[25] Zayed, E.M.E., *A note on the modified simple equation method applied to Sharma-Tasso-Olver equation*, Appl. Math. Comput., 218 (2011), 3962-3964.

[26] Zayed, E.M.E., Arnous, A.H., *DNA dynamics studied using the homogeneous balance method*, Chin. Phys. Lett., 29 (2012), 80-203.

Accepted: 09.07.2014