AVERAGE D-DISTANCE BETWEEN VERTICES OF A GRAPH

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Abstract. The D-distance between vertices of a graph G is obtained by considering the path lengths and as well as the degrees of vertices present on the path. The average D-distance between the vertices of a connected graph is the average of the D-distances between all pairs of vertices of the graph. In this article we study the average D-distance between the vertices of a graph.

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1. Introduction

The concept of distance is one of the important concepts in study of graphs. It is used in isomorphism testing, graph operations, hamiltonicity problems, extremal problems on connectivity and diameter, convexity in graphs etc. Distance is the basis of many concepts of symmetry in graphs.

In addition to the usual distance, $d(u, v)$ between two vertices $u, v \in V(G)$ we have detour distance (introduced by Chartrand et al, see [2]), superior distance (introduced by Kathiresan and Marimuthu, see [6]), signal distance (introduced by Kathiresan and Sumathi, see [7]), degree distance etc.

In an earlier article [9], the authors introduced the concept of D-distance between vertices of a graph G by considering not only path length between vertices, but also the degrees of all vertices present in a path while defining the D-distance. In a natural way we can extend this concept to D-distance between edges.

The article is arranged as follows. In §2, we collect some definitions and results for easy reference. In §3, we study some properties of average D-distance and in §4, we calculate the average D-distance between vertices for some classes of graphs.

2. Preliminaries

Throughout this article, by a graph $G(V, E)$ or simply G, we mean a non-trivial, finite, undirected graph without multiple edges and loops. Further all graphs we consider are *connected.*

In this section we given some definitions and state some results for later use. We begin with D-distance in graphs.

Definition 1 If u, v are vertices of a connected graph G the D-length of a connected $u-v$ path s is defined as $l^D(s) = l(s) + deg(v) + deg(u) + \sum deg(w)$ where sum runs over all intermediate vertices w of s and $l(s)$ is the length of the path.

Definition 2 The D-distance, $d^D(u, v)$, between two vertices u, v of a connected graph G is defined as $d^D(u, v) = min\left\{l^D(s)\right\}$ where the minimum is taken over all $u - v$ paths s in G. In other words,

$$d^D(u, v) = min\left\{l(s) + deg(u) + deg(v) + \sum deg(w)\right\}$$

where the sum runs over all intermediate vertices w in s and minimum is taken over all $u - v$ paths s in G.

Definition 3 Let G be a connected graph of order n. The average distance of G denoted by $\mu(G)$, is defined as

$$\mu(G) = ({}^n c_2)^{-1} \sum d(u, v),$$

where $d(u, v)$ denotes the distance between the vertices u and v. See [3, 4, 5].

Similarly, we can define the average D-distance of a graph as follows:

Definition 4 Let G be a connected graph of order n. The average D-distance between vertices of G denoted by μ^D, is defined as

$$\mu^D(G) = ({}^n c_2)^{-1} \sum d^D(u, v),$$

where $d^D(u, v)$ denotes the D-distance between the vertices u and v.

Definition 5 Let G be a connected graph of order n having m edges with $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$. The D-distance matrix of G denoted as $D^D(G)$, is defined as

$$D^D(G) = \left[d_{i,j}^D\right]_{n\times n},$$

where $d_{i,j}^D = d^D(v_i, v_j)$ is the D-distance between the vertices v_i and v_j.

Obviously, $D^D(G)$ is an $n \times n$ symmetric matrix with all diagonal entries being zero.

Definition 6 Let G be a graph, then the average degree of G, denoted as $d(G)$, is given by

$$d(G) = \frac{1}{|V|}\sum d(v),$$

where $d(v)$ is the degree of the vertex v .

Definition 7 The total D-distance $[TDD]$ of graph G is the number given by

$$\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} d^D(v_i, v_j),$$

where n is the number of vertices.

3. Average D-distance

In this section, we prove some results on average D-distance between vertices. We begin with a theorem which connects the number of vertices and average D-distance. This leads to some more results.

Theorem 1 *Let G_1 and G_2 be two connected graphs having same number edges and same diameters. If the number of vertices in G_1 is more than the number of vertices in G_2 then average D-distance of G_1 is more than average D-distance of G_2.*

Proof. Since the diameters of these two graphs are the same, the largest entries in the D-distance matrix of these graphs are the same. The number of the pairs of vertices is more in G_1 and hence total D-distance value of G_1 is more. Since number of edges in G_1 and G_2 are same. This implies the average D-distance of G_1 is more than average D-distance of G_2. ∎

Theorem 2 *Let G_1 and G_2 be two connected graphs of same number of edges and $diam(G_1) < diam(G_2)$. Then $\mu^D(G_1) > \mu^D(G_2)$.*

Proof. Since G_1, G_2 have same number of edges and $diam(G_1) < diam(G_2)$, it is clear that $|V(G_1)| > |V(G_2)|$. Then by Theorem 1, we have $\mu^D(G_1) > \mu^D(G_2)$. ∎

Theorem 3 *Let G_1 and G_2 be two connected graphs having same number of edges and diameters. If $\delta(G_1) < \delta(G_2)$ then $\mu^D(G_1) > \mu^D(G_2)$.*

Proof. $\delta(G_1) < \delta(G_2)$ implies $|V(G_1)| > |V(G_2)|$. Then, by Theorem 1, $\mu^D(G_1) > \mu^D(G_2)$. ∎

Theorem 4 *Let G_1 and G_2 be two connected graphs having same number of edges and same diameters. If $\delta^1(G_1) < \delta^1(G_2)$ then $\mu^D(G_1) > \mu^D(G_2)$.*

Proof. Since $\delta^1(G_1) < \delta^1(G_2)$ we have $|V(G_1)| > |v(G_2)|$. Then, by Theorem 1, $\mu^D(G_1) > \mu^D(G_2)$. ∎

4. Results on some classes of graphs

Here we calculate the average D-distance for some classes of graphs.

Theorem 1 *The average D-distance of complete graph K_n is $\mu^D(K_n) = 2n - 1$.*

Proof. Every vertex taken from K_n has $n-1$ vertex neighbors. The D-distance between any vertex and its neighbors is $2n-1$. Thus the total D-distance $TDD = ({}^nc_2)(2n-1)$ and hence

$$\mu^D(K_n) = \frac{TDD}{{}^nc_2} = 2n - 1.$$

∎

Theorem 2 *The average D-distance of complete bipartite graph $K_{m,n}$ is*

$$\mu^D(K_{m,n}) = \frac{2mn(m+n+1) + n(n-1)(2m+n+2) + m(m-1)(2n+m+2)}{(m+n)(m+n-1)}.$$

Proof. Let $V(K_{m,n}) = A \cup B$, where $A = \{v_1, v_2, v_3, ..., v_m\}$, $B = \{w_1, w_2, w_3, ..., w_n\}$. Then $d^D(v_i, v_j) = 2m+n+2$, $d^D(w_i, w_j) = 2n+m+2$ and $d^D(v_i, w_j) = m+n+1$. Thus, the total D-distance

$$TDD = ({}^mc_2)\,(2m+n+2) + ({}^nC_2)\,(2n+m+2) + mn(m+n+1).$$

Hence the average D-distance

$$\begin{aligned}\mu^D(K_{m,n}) &= \frac{TDD}{{}^{(m+n)}c_2} \\ &= \frac{2mn(m+n+1) + n(n-1)(2m+n+2) + m(m-1)(2n+m+2)}{(m+n)(m+n-1)}\end{aligned}$$

∎

Theorem 3 *The average D-distance of star graph $St_{n,1}$ is given by*

$$\mu^D(St_{n,1}) = \frac{2(n+2)+(n-1)(n+4)}{n-1}\,.$$

Proof. In Star graph there are $n+1$ vertices. For the central vertex there are n neighbor vertices and all other vertices has only one neighbor vertex, namely the central vertex. For the central vertex there are no distinct vertices, where as all others have $n-1$ distinct vertices. The D-distance between any vertex and its distinct vertex is $(n+2)(n+4)$. Finally, $TDD = n(n+2) + ({}^nc_2)\,(n+4)$, then

$$\mu^D(St_{n,1}) = \frac{TDD}{{}^nc_2} = \frac{2(n+2)+(n-1)(n+4)}{n-1}\,.$$

■

Theorem 4 *The average D-distance of the path graph P_n is*

$$\mu^D(P_n) = \frac{2a_n}{n}\,,$$

where $a_n = a_{n-1} + n + 1$ with $a_1 = 0$.

Proof. For P_n, the D-distance matrix is the $n \times n$ symmetric matrix

$$\begin{bmatrix} 0 & 4 & 7 & 10 & \cdots & 3n-5 & 3n-2 \\ & 0 & 5 & 8 & \cdots & 3n-7 & 3n-4 \\ & & 0 & 5 & \cdots & 3n-12 & 3n-9 \\ & & & \cdots & \cdots & \cdots & \cdots \\ & & & & 0 & 5 & 7 \\ & & & & & 0 & 4 \\ & & & & & & 0 \end{bmatrix}$$

By adding all entries in the upper triangular or lower triangular matrix we get the total D-distance, which is $a_n(n-1)$ in this case, where $a_n (n \geq 2)$ is a constant given by $\{0, 3, 7, 12, 18, 25, \cdots\}$ or, recursively, $a_n = a_{n-1} + n + 1$ with $a_1 = 0$. Then,

$$\mu^D(P_n) = \frac{TDD}{({}^nC_2)} = \frac{2a_n}{n}\,.$$

■

Theorem 5 *The average D-distance of the cyclic graph C_n is*

$$\mu^D(C_{2n}) = \frac{a_n}{2n-1},$$

where $a_n = a_{n-1} + 6n + 1$ with $a_2 = 18$ and $\mu^D(C_{2n-1}) = \frac{3}{2}n + 2$, $\forall n \geq 2$.

Proof. Case (i). First, we consider the case of cyclic graphs of odd order, (C_{2n-1}) $(n \geq 2)$. As $\mu^D(C_{2n-1})$ is regular, the elements of any row in the DDM, except the diagonal element are $0, 5, 8, 11, ..., 3n-1, 3n-1, ..., 11, 8, 5$. Then, the total D-distance, $TDD = 2(5+8+11+\cdots+(3n-1) = \frac{1}{4}(3n+4)(2n-1)(2n-2)$. Thus the average D-distance, $\mu^D(C_{2n-1}) = \frac{3}{2}n+2$.

Case (ii). Here, we consider the case of even order cyclic graphs $(C_{2n})(n \geq 2)$. In this case, the elements of any row in DDM except the diagonal element, are $0, 5, 8, 11, ..., 3n+2, ..., 11, 8, 5$ Like above we can show that $\mu^D(C_{2n}) = \frac{a_n}{2n-1}$ where $a_n = a_{n-1} + 6n + 1$ with $a_2 = 18$. ∎

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