CONJUGACY CLASS SIZES OF SUBGROUPS AND THE STRUCTURE OF FINITE GROUPS

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Abstract. The authors investigate the influences of conjugacy class sizes of subgroups of a finite groups G on the structure of G. Some sufficient conditions for a finite group to be p-nilpotent, p- solvable and supersolvable are obtained.

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1. Introduction

One of the questions that were studied extensively is what can be said about the structure of the group G if some information is known about the arithmetical structure of $Con(G)$, the set of the conjugacy classes of G. Answers in many cases were given. On the other hand, few studies about the conjugacy classes of subgroups of a group G were done. In [12], the author proved that a finite group G is p-nilpotent for some prime p if and only if $(p, |G : N_G(Q)|) = 1$ for any Sylow subgroups Q of G. In the same paper, the author showed also that if $|G : N_G(Q)|$ is square-free for any Sylow subgroups Q of G, then either G is supersolvable or $G = HK$, where H is normal in G and $H = PSL(2, p)$ or $SL(2, p)$ for some prime $p = 8k + 5$, K is a supersolvable subgroup of G. Guo Wenbin proved in [6] that if $|G : N_G(Q)|$ is prime power numbers for any Sylow subgroups Q of G, then G is solvable. Further, if $|G : N_G(Q)|$ is prime power numbers or odd numbers, then G is a solvable group. Recently, in [2], Berkovich and Kazarin showed that G is solvable with $nl(G) \leq 2$ if $|G : N_G(H)|$ is a power of a prime for all primary subgroup $H \leq G$. In this paper, we consider the conjugacy class sizes of subgroups of a finite groups G and investigate the influences of conjugacy class sizes of subgroups of G on the structure of G.

In what follows, G is a finite group of order $|G|$; $\pi(G)$ denotes the set of all prime divisors of $|G|$; $nl(G)$ denotes the nilpotent length of $|G|$ and $cl(G)$ denotes the nilpotent class of $|G|$. The p-length of $|G|$ is denoted by $l_p(G)$. All further unexplained notation and terminologies are standard can be found in [5].

2. Preliminaries

In this section, we give some lemmas which are useful in the sequel.

Lemma 2.1 *Let $N \trianglelefteq G$, and $H \leq G$. Then*

(1) $|N : N_N(H)|$ *and* $|\overline{G} : N_{\overline{G}}(\overline{H})|$ *divide* $|G : N_G(H)|$*, if* N *is contained in* H*, where* $\overline{G} = G/N$.

(2) $|G : N_G(NH)|$ *divides* $|G : N_G(H)|$.

Proof. (1) Clearly,

$$|N : N_N(H)| = |NN_G(H) : N_G(H)|,$$

which divides $|G : N_G(H)|$.

Also

$$|\overline{G} : N_{\overline{G}}(\overline{H})|||\overline{G} : \overline{N_G(H)N}| = |G : N_G(H)N|.$$

On the other hand,

$$|G : N_G(H)| = |G : N_G(H)N||N_G(H)N : N_G(H)|.$$

Hence

$$|\overline{G} : N_{\overline{G}}(\overline{H})|||G : N_G(H)|.$$

(2) follows by the fact that $N_G(H) \leq N_G(NH)$. ∎

Lemma 2.2 *Suppose that $\pi \subseteq \pi(G)$ and $x \in H$, where H is a π-Hall subgroup of group G. If $|G : N_G(\langle x\rangle)|$ is a π-number. Then $\langle x\rangle \leq O_\pi(G)$.*

Proof. Indeed, $G = N_G(\langle x\rangle)H$. So $\langle x\rangle^G = \langle x\rangle^{N_G(\langle x\rangle)H} = \langle x\rangle^H \leq H$. ∎

Lemma 2.3 *Let G be a group and p be the smallest odd prime divisor of $|G|$. Suppose that there exits an element x of order p such that $|G : N_G(\langle x\rangle)|$ is a power of two. Then G is not a non-abelian simple group.*

Proof. Assume that G is a non-abelian simple group, $H = N_G(\langle x\rangle)$ is a proper subgroup of G with index of 2^α, where α is a natural number. Then G is one of the groups list in theorem 1 of [7].

If $G = A_{2^n}$, then $H = A_{2^n-1}$. As $2^n > 6$, we obtain that A_{2^n-1} is also a simple group. This is impossible.

If $G = PSL(n, q)$, then $|G : N_G(\langle x\rangle)| = \frac{q^n-1}{q-1}$. It is easy to check that $n = 2$ and $q = 2^\alpha - 1$ is a prime in this case. Hence, $|N_G(\langle x\rangle)| = q(q-1)$. Suppose that $o(x) = q$, then $q - 1 = 2^\alpha - 2$ is a divisor of $|G|$. Hence $2^\alpha - 2 = 2^\beta$ by the choice of x, where α, β are both natural numbers. Therefore, $\alpha = 2$, and $q = 3$, a contradiction. Hence, we may assume that $o(x)|q-1$. Since $N_G(\langle x\rangle)/C_G(\langle x\rangle)$ is isomorphic to some subgroups of $Aut(\langle x\rangle)$, we have $|N_G(\langle x\rangle)| = q(q-1)$ is a divisor of $|C_G(\langle x\rangle)||Aut(\langle x\rangle)|$. On the other hand, by the structure of $PSL(2, q)$ we know that $q \not| |C_G(\langle x\rangle)|$, which implies that $(|C_G(\langle x\rangle)||Aut(\langle x\rangle)|, q) = 1$. Thus, we obtain a contradiction.

Assume that $G = PSL(2, 11)$. Then, $N_G(\langle x\rangle) = H = A_5$. Hence, we obtain $\langle x\rangle \trianglelefteq A_5$, a contradiction.

If $G = M_{23}$, then $H = M_{22}$. Since in this case $|G : N_G(\langle x\rangle)| = 23 \neq 2^\alpha$, we get a contradiction. Similarly, we have that $G \neq M_{11}$ and $PSU(4, 2)$, a final contradiction. The proof is complete. ■

Lemma 2.4 [4] *Every group of odd order is solvable.*

Lemma 2.5 [3, Theorem 1] *If the subgroup H of the group G is quasinormal in G, then H/H_G is nilpotent.*

Lemma 2.6 [1, Theorem 3] *Let the group $G = HK$ be the m-permutable product of the subgroups H and K. Assume that H is supersolvable and K is nilpotent. If K permutes with every Sylow subgroup of H, then G is supersolvable.*

3. Main results

We first prove the following:

Theorem 3.1 *Let p be the minimal odd divisor of $|G|$. Suppose that $|G : N_G(\langle x\rangle)|$ is a power of two for any element $x \in G$ of order p. Then G is p-solvable.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. Then we have:

(1) Any nontrivial normal subgroup of G is p-solvable and $O_{p'}(G) = 1$.

This follows by Lemma 2.1 and the choice of G.

(2) $O_p(G) > 1$.

It follows by Lemma 2.3 that there exists a nontrivial minimal normal subgroup N of G. By Lemma 2.1, N is p-solvable. Since $O_{p'}(N) \leq O_{p'}(G) = 1$, we obtain $O_{p'}(N) = 1$. Hence $1 < O_p(N) \leq O_p(G)$.

(3) Every subgroup of order p is contained in $O_p(G)$.

Suppose that there is a subgroup $\langle x\rangle$ of order p such that $\langle x\rangle$ is not contained in $O_p(G)$. It follows that $|G : N_G(\langle x\rangle)|$ is a power of two by hypothesis. Hence $N_G(\langle x\rangle)$ contains a Sylow p-subgroup of G, giving $O_p(G) \leq N_G(\langle x\rangle)$. Let $C = C_G(O_p(G))$. Then $\langle x\rangle \leq C \trianglelefteq G$, and $O_p(C) = O_p(G)$ and hence $O_{pp'}(C) = O_p(C) \times H$, where, H is a nontrivial p'-group. Therefore, $1 < H < O_{p'}(G) = 1$, a contradiction. Hence $C = G$.

Let $\overline{G} = G/O_p(G)$, and $\overline{x} = xO_p(G)$ be an element of order p. Then

$$|\overline{G} : N_{\overline{G}}(\langle\overline{x}\rangle)|\,|\,|G : N_G(\langle x\rangle)|$$

is a power of two. Again by Lemma 2.3, $\overline{G}$ is not a non-abelian simple group. Hence there exists a normal subgroup N of G such that $O_p(G) < N < G$. Now, $O_p(N) = O_p(G) \leq Z(G)$. This implies that $O_{p'}(G) > 1$, which contradicts to (1).

(4) G/M is a non-abelian simple group and $p||G/M|$, where M be a maximal normal subgroup of G which is p-solvable.

This follows immediately by (1).

(5) The final contradiction.

Let S_0 be a Sylow 2-subgroup of M and $H = N_G(S_0)$. Then, by Frattini argument, we obtain that $G = MH$. It follows by (4) that $H/H \cap M \cong G/M$ is a non-abelian simple group and $p||G/H/H \cap M|$. Let $D = C_H(O_p(H))$. Then D is p-nilpotent by Ito' theorem. Since $(H \cap M)D/H \cap M$ is normal in $H/H \cap M$, we have $D \leq H \cap M$.

By Lemma 2.4, there is a Sylow 2-subgroup S of G such that $S_0 < S$ and $S \leq H$. Let S_1/S_0 be a subgroup of order two of $Z(S/S_0)$. Then $S_1 = S_0\langle u\rangle$, where $u^2 \in S_0$ and $S_0 \trianglelefteq S_1$. Let $K = O_p(H)S_1$. If S_1 is normal in K, then $S_0 \leq C_H(O_p(H)) = D \leq H \cap M$, which contradicts to the fact that S_1 be a Sylow 2-subgroup of M. Therefore, K has no normal Sylow 2-subgroup, of course K is not nilpotent. Suppose that W is a minimal non-nilpotent group of K. Then $W = X\langle v\rangle$, where $o(v) = 2^\alpha$, $\Phi(\langle v\rangle) = \langle v^2\rangle \leq Z(W)$, the center of W, and X is a normal p-group of W of exponent p. Now, $\langle v\rangle$ acts irreducible on $X/\Phi(X)$ and v induces an automorphism of order two of $X/\Phi(X)$. Hence, $|X/\Phi(X)| = p$ and $X = \langle x\rangle$ is a cyclic group of order p. Further, $S_0 \leq C_G(\langle x\rangle) < N_G(\langle x\rangle)$. So $N_G(\langle x\rangle)$ contains $S_0\langle v\rangle$, a Sylow 2-subgroup of K. Without lose of generality, we may assume $S_1 = S_0\langle v\rangle$. Now, $|G : N_G(\langle x\rangle)|$ is a power of two by assumption. Thus $G = N_G(\langle x\rangle)S$. If $G = N_G(\langle x\rangle)$, then $C_G(\langle x\rangle) \trianglelefteq G$. By (1), $C_G(\langle x\rangle)$ is p-solvable, of course we have G is p-solvable, a contradiction. If $G > N_G(\langle x\rangle)$, we have

$$S_1^G = S_1^{SN_G(\langle x\rangle)} = S_1^{N_G(\langle x\rangle)} \leq N_G(\langle x\rangle)$$

since $S_1/S_0 \leq Z(S/S_0)$. Hence S_1^G is p-solvable, and we have $S_1^G \leq M$. Therefore, $|M|_2 = |S_0| < |S_1| \leq |M|_2$, the final contradiction. This completes our proof. ∎

Theorem 3.2 *Let G be a group. If $|G : N_G(\langle x\rangle)|$ is a power of a prime for all $x \in G$ of prime power order. Then G is solvable with $nl(G) \leq 2$ and $l_p(G) \leq 1$ for any prime divisor p of $|G|$.*

Proof. First of all, G is not a non-abelian simple group. Suppose G is simple. Then $|\pi(G)| \geq 3$. Let $P \in Syl_p(G)$, and $x \in Z(P)$, the center of P, where $p \in \pi(G)$. Then, by hypothesis $|G : N_G(\langle x \rangle)|$ is power of a prime r, which is different from p. Let R be a Sylow r-subgroup of G and let $y \in Z(R)$. Then $|G : N_G(\langle y \rangle)|$ is also a power of a prime. It is well known that G must be the simple group $PSL(2, 7)$ (see [7, p.304, Note]). Obviously, $PSL(2, 7)$ can not satisfy the hypothesis of our Theorem. This is a contradiction. Hence G is not a non-abelian simple group and there exists a proper normal subgroup N in G. By Lemma 2.1, G/N and N are all solvable groups, therefor G is solvable.

In order to prove that $l_p(G) \leq 1$, we suppose G is a counterexample of minimal order. Then by [8, Lemma 6.9, VI], we know $\Phi(G) = 1$ and $F(G) = O_p(G)$ is the unique minimal normal subgroup of G. Hence there exists a proper subgroup M of G such that $G = F(G) \rtimes M$, the semiproduct of $F(G)$ and M. Let $M_p \in Syl_p(M)$ and $x \in Z(M_p)$. Then $F(G)$ is not contained in $N_G(\langle x \rangle)$. Otherwise we have that $x \in C_G(F(G))$, a contradiction since $C_G(F(G)) \leq F(G)$. Thus $|G : N_G(\langle x \rangle)|$ is a power of p. By Lemma 2.2, $x \in O_p(G) = F(G)$, a contradiction. Then we obtain that $F(G)$ must be a Sylow subgroup of G. Therefore, $l_p(G) = 1$, and G is not a counterexample, a contradiction too.

Now, we will show that $nl(G) \leq 2$. Assume that G is a counterexample of minimal order. Since $F(G/\Phi(G)) = F(G)/\Phi(G)$, we have $\Phi(G) = 1$ by induction. Moreover, $F(G) = O_p(G)$ is the unique minimal normal subgroup of G. Hence, there exists a proper subgroup M of G such that $G = F(G) \rtimes M$. By using the same argument as the above, we get M is a p'-group and $|G : N_G(\langle x \rangle)|$ is a power of p for any $x \in M$ of prime power order. It follows by Lemma 2.1 that $|G : N_G(F(G)\langle x \rangle)|$ divides $|G : N_G(\langle x \rangle)|$. However, $|G : N_G(F(G)\langle x \rangle)|$ is coprime to p since $F(G)$ is the Sylow p-subgroup of G. Hence $F(G)\langle x \rangle$ is normal in G, which implies that all cyclic subgroup of $G/F(G)$ is normal in $G/F(G)$. Hence $G/F(G)$ is a Dedekind group, and $nl(G) \leq 2$, which contradicts to the choice of G. The proof is complete. ∎

Moreover, for a group G, let $Norm(G) = \cap\{N_G(\langle a \rangle) | \forall a \in G\}$. Then, we have $Norm(G) \leq Z_2(G)$ by [11].

The following theorem gives a sufficient condition for a group to be p-nilpotent.

Theorem 3.3 *Let G be a solvable group and p a prime divisor of $|G|$ such that q does not divide $p - 1$ for any prime divisor q of $|G|$. Suppose that $|G : N_G(\langle x \rangle)|$ is not divided by p^2 for any $x \in G$ of prime power order. Then G is a p-nilpotent group. Furthermore, if P is a Sylow p-subgroup of G, then $cl(P) \leq 3$.*

Proof. It follows by Lemma 2.1 that the conclusion holds for proper quotient groups of G. Hence we may assume that G has a unique minimal normal subgroup N, since the class of p-nilpotent groups forms a saturated formation. Clearly, we may assume also that N is an elementary abelian group of order r^n for a prime r and a natural number n. Obviously $r = p$. Let M be a maximal normal subgroup of G. Then $|G/M| = q$ is a prime. By Lemma 2.1, M satisfies the assumptions of the theorem and therefore it is p-nilpotent. If $q = p$, then it follows that the normal p-complement of M is also the normal p-complement of G, a contradiction. So we have $q \neq p$. Since G/N is p-nilpotent and it has no quotient group of order p, we have G/N is a p'-group. If $M \neq N$, then $O_{p'}(M) \neq 1$ since M is p-nilpotent. Hence $N \leq O_{p'}(M)$ by the uniqueness of N, a contradiction. Therefore $M = N$ and $|G/N| = q$. If $n = 1$, then we have $q|p-1$, since $|G/N| = |N_G(N)/C_G(N)|$ divides $|Aut(N)|$. This is a contradiction. Hence $n \geq 2$. By the Schur-Zassenhaus Theorem, $G = N\langle x\rangle$, where $o(x) = q$. If there is an element $u(\neq 1) \in N$ such that $u \in N_G(\langle x\rangle)$, then $u \in Z(G)$ since N is abelian. Thus, $N = \langle u\rangle \leq Z(G)$, a contradiction. Hence, we obtain $N_G(\langle x\rangle) = \langle x\rangle$. This implies that $p^2||G : N_G(\langle x\rangle)|$, a contradiction too.

It remains to prove that $cl(P) \leq 3$. It follows by the first part of the proof that $P \cong G/O_{p'}(G)$. Thus P satisfies the assumptions of the theorem. Let $x \in P$, by the hypothesis, $|P : N_P(\langle x\rangle)| \leq p$, and so $\Phi(P) \leq N_P(\langle x\rangle)$ for all $x \in P$, giving $\Phi(P) \leq Norm(P) \leq Z_2(P)$. Hence $cl(P) \leq 3$. Our proof is complete now. ∎

The following two theorems give some sufficient conditions for a group to be supersolvable.

Theorem 3.4 *Let G be a solvable group. Suppose that $|G : N_G(\langle x\rangle)|$ is a square-free number for all $x \in G$ of prime power order. Then G is supersolvable.*

Proof. Assume that the result is false and G be a counterexample of minimal order. Since G is solvable, we have that G has a minimal normal subgroup N of order p^n, where p is a prime and n is a natural number. Since the class of supersolvable groups forms a saturated formation, we may suppose that N is a unique minimal normal subgroup of G and $\Phi(G) = 1$. If $n = 1$, then G is supersolvable since G/N is supersolvable, which contradicts to the choice of G. Hence $n > 1$. Since $N \not\leq \Phi(G)$, there exists a maximal subgroup M of G such that $G = MN$, $M \cap N = 1$, and $M \cong G/N$ is supersolvable. Let Q be a minimal normal subgroup of M. Then $Q = \langle x\rangle$, and $N_G(\langle x\rangle) \geq M$, where $x \in M$ is of prime order. Assume that $N_G(\langle x\rangle) \cap N \neq 1$, then $N_G(\langle x\rangle) = G$. Hence $Q \trianglelefteq G$. This implies that G is supersolvable since G/Q is supersolvable, a contradiction. Now,

$$p < p^n||NN_G(\langle x\rangle) : N_G(\langle x\rangle)| = |G : N_G(\langle x\rangle)|,$$

contrary to the hypothesis. The proof is hence completed. ∎

Recall that a subgroup K of a group G is said to be quasi-normal in G if $KH = HK$ for any subgroup H of G.

Theorem 3.5 *Let A and B be quasi-normal subgroups of a solvable group G such that $G = AB$. Suppose that $|G : N_G(\langle x\rangle)|$ is a square-free number for every $x \in A \cup B$ of prime power order. Then G is supersolvable.*

Proof. Assume that the theorem is not true and G a counterexample of minimal order. Because supersolvable groups form a saturated formation, we may suppose that G has a unique minimal normal subgroup N and $\Phi(G) = 1$. Let $|N| = p^n$ for a prime p and a natural number n, then $n > 1$. Obviously, $F(G) = N = C_G(N)$. If either $A_G = 1$ or $B_G = 1$, then either A or B is nilpotent by Lemma 2.5. Therefore, by Lemma 2.6 we obtain G is supersolvable, a contradiction. Now, we have $N \leq A_G$ and $N \leq B_G$ by the uniqueness of N. Since A and B are quasi-normal in G, $F(A)$ and $F(B)$ are contained in $F(G) = N$. Therefore, $F(A) = F(B) = N$.

Let q be the largest prime divisor of $|A|$, and $S_q \in Syl_q(A)$. Then $S_q \trianglelefteq A$ since A is supersolvable and $S_q \leq N$. This implies that $p = q$ and N is the Sylow p- subgroup of A. By the same reason, we know that p is the largest prime divisor of $|B|$ and N is a Sylow subgroup of B. Therefore, N is a Sylow subgroup of G and p is the largest prime divisor of $|G|$. Let K/N be a minimal normal subgroup of G/N. By a result of [9], we may assume that $K \leq A$ or $K \leq B$. Since G/N is supersolvable, we have $|K/N| = q$ is a prime, which is not equal to p. By the Schur-Zassenhaus Theorem, $K = N\langle v\rangle$, where $o(v) = q$. If there is an element $u(\neq 1) \in N$ such that $u \in N_K(\langle v\rangle)$, then $u \in Z(K)$ since N is abelian. Thus, we have either $Z(K) = K$ or $Z(K) < N$, a contradiction. Hence we obtain $N_K(\langle v\rangle) = \langle v\rangle$. This implies that $|N| = |K : N_K(\langle v\rangle)|$, that is, $p^2 \| K : N_K(\langle v\rangle)|$, contrary to the hypothesis. The proof is complete. ∎

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