ON A SPECIAL CLASS OF FINITE p-GROUPS OF MAXIMAL CLASS

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Abstract. In this paper, we study the finite $p-$group G of maximal class in which every nonabelian subgroup H satisfies $C_G(H) = Z(H)$. We prove here that a finite $p-$group G of maximal class is metabelian if every nonabelian subgroup H satisfies $C_G(H) = Z(H)$, furthermore, if $p \neq 3$, $|G| \geq p^{2p}$, then there is an abelian subgroup of index p in G.

Keywords and phrases: a $CGZ-$group; a $p-$group of maximal class; an abelian $p-$group.

AMS Subject Classification: 20D15.

1. Introduction

Finite $p-$groups of maximal class is an important class of finite $p-$groups. It is well known that a finite $p-$group of maximal class can be determined by centralizers of some subgroups. For example, M. Suzuki [1, Proposition 1.8] proved that a finite $p-$group G is of maximal class if and only if there is a subgroup N of order p^2 such that $C_G(N) = N$; in [1, Proposition 10.17], it is showed that if G is a finite $p-$group, $B \leq G$ is a nonabelian subgroup of order p^3 such that $C_G(B) = Z(B)$, then G is of maximal class.

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In [1], it is posed the following problem: Classify every the finite $p-$ group G all of whose every nonabelian subgroup H satisfies $C_G(H) = Z(H)$. In this paper, we will study this problem. For convenience, we give the following definition.

Definition 1. A nonabelian $p-$group G is called a $CGZ-$group if and only if every nonabelian subgroup H of G satisfies $C_G(H) = Z(H)$.

In this paper, we mainly study the finite $CGZ-$groups of maximal class. We get that a finite $p-$group G of maximal class is metabelian if G is a $CGZ-$group. Furthermore, if $p \neq 3$ and $|G| \geq p^{2p}$, then there is an abelian subgroup of index p in G.

2. Main results and proofs

Lemma 2.1 *Let G be a nonabelian $p-$group and let H be a nonabelian subgroup of G. If G is a $CGZ-$group, then H is also a $CGZ-$group.*

Proof. Let H_1 be a nonabelian subgroup of H. Since $C_H(H_1) = H \cap C_G(H_1)$ and $C_G(H_1) \leq H_1$, we have $C_H(H_1) \leq H_1$. So H is also a $CGZ-$group. ∎

Proposition 2.2 *Let G be a nonabelian $p-$group with exp$(G) = p$,where $p \geq 3$. Then G is a $CGZ-$group if and only if G is a finite $p-$group of maximal class, and there is an abelian subgroup of index p.*

Proof. Since G is nonabelian, there is an element $a \in Z_2(G) - Z(G)$ and $b \in G$ such that $ab \neq ba$. We have that the subgroup $H = \langle a, b\rangle$ is a nonabelian subgroup of order p^3 with $|Z(H)| = p$. Since $Z(G) \leq C_G(H) = Z(H)$, then $Z(G)$ has order p. By [1, Proposition 10.17], G is a $p-$group of maximal class.

Let N be a normal subgroup of order p^2 and let $K = C_G(N)$. Then $|G : K| \leq p$. If K is not abelian, by Lemma 2.1, K is also a $CGZ-$group, similarly, we have $|Z(K)| = p$, which contradicts to $N \leq Z(K)$. So K is an abelian subgroup.

Conversely, if G is a finite $p-$group of maximal class, and G has a maximal abelian subgroup A with $|G : A| = p$. Then $|Z(G)| = p$. Let H be a nonabelian subgroup of G and let $H_1 = A \cap H$. Then $|H : H_1| = p$ and there is an element $x \in G - A$ such that $H = \langle x, H_1\rangle$, $G = \langle A, x\rangle$. It follows that $H_1 \trianglelefteq G$ and $Z(G) \leq H_1$. We see that

$$A \geq C_G(H) = C_G(x) = C_A(x) = Z(G).$$

Clearly $C_H(x) = Z(H) \leq Z(G) = C_G(x)$. Hence we have $C_G(H) = Z(H) = Z(G)$ and G is a $CGZ-$group. ∎

Noticing that a $p-$group G of maximal class with $|G| \geq p^{p+1}$ has an element g of order p^2, we have by Proposition 2.2, that if a nonabelian $p-$group G with $\exp(G) = p$ and G is a $CGZ-$group, then $|G| \leq p^p$.

Lemma 2.3 *Let G be a nonabelian $p-$group. If G is a $CGZ-$group and $\Omega_1(G) \cap Z(G)$ has an elementary abelian subgroup of order $\geq p^3$, then $\Omega_1(G) \leq Z(G)$ and $|\Omega_1(G)| = p^3$.*

Proof. Let $N = \Omega_1(G) \cap Z(G)$. If there is an element x of order p such that $x \notin Z(G)$, then there is an element $y \in Z_2(G) - Z(G)$ of order p. We can get an element $y_1 \in G$ such that $[y, y_1] \neq 1$. Let $H = \langle y, y_1 \rangle$. Then $|\Omega_1(H)| \leq p^3$ and $y \in \Omega_1(H)$. We have that the subgroup N is not contained in H, which contradicts to $N \leq C_G(H)$. It follows that $\Omega_1(G) \leq Z(G)$.

Let K be a minimal nonabelian subgroup of G. Then $|\Omega_1(K)| \leq p^3$. Since $\Omega_1(G) \leq C_G(K)$, we have $|\Omega_1(G)| = p^3$. ∎

For convenience, we give the following two lemmas.

Lemma 2.4 [3, lemma 14.14] *Let G be a finite $p-$group of maximal class. If $|G| \leq p^{p+1}$, then $exp(G^{'}) \leq p$.*

Lemma 2.5 [4] *Let G be a finite $p-$group of maximal class and order p^m, where $p > 2$, $m > p+1$. Then G is irregular and*

(a) *Let G_1 be a maximal subgroup of G. Then G_1 is either of maximal class or regular with $|G_1 : \mho_1(G_1)| = p^{p-1}$.*

(b) *If $N \trianglelefteq G$ is of order p^{p-1}, then $exp(N) = p$.*

(c) $[G_i, G_{j+1}] \leq G_{i+j+1}$.

Proposition 2.6 *Let G be a finite $p-$group of maximal class. If G is a CGZ-group, then $G^{'}$ is abelian.*

Proof. If $|G| \leq p^5$, it is easy to see that $G^{'}$ is abelian. If $p = 2$, by [5], G has a cyclic subgroup of index 2. By [2, Theorem 6], for a $3-$group G of maximal class, $G^{'}$ is also abelian. Hence we only need to consider the cases $|G| \geq p^6$ and $p \geq 5$.

Suppose that $|G| \leq p^{p+1}$. By Lemma 2.4, we have $\exp(G^{'}) = p$. Let N be a normal subgroup of order p^2. Then $|G : C_G(N)| \leq p$. Note that G is a $p-$group of maximal class, we have $N \leq G^{'}$ and $|Z(G^{'})| \geq p^2$. By Proposition 2.2 we get that $G^{'}$ is abelian.

Suppose that $|G| \geq p^{p+2}$. Let M be a normal subgroup of order p^3. Then $M = G_{n-3} \leq G^{'} = G_2$. By Lemma 2.5, $[G_2, G_{n-3}] \leq G_n = 1$, we have that $M \leq Z(G^{'})$. It follows from Lemma 2.5 that $|\Omega_1(G^{'})| = p^{p-1}$. By Lemma 2.3, we have that $G^{'}$ is abelian. ∎

By [3, Theorem 14.11], if G is a finite $p-$group of maximal class and of order p^n in which $G^{'}$ is abelian, then $C_G(G_i/G_{i+2}) = C_G(G_j/G_{j+2})$ for positive numbers $1 \leq i, j \leq n-2$. Hence we have the following corollary.

Corollary 2.7 *Let G be a finite $p-$group of maximal class of order p^n. If G is a $CGZ-$group, then $C_G(G_i/G_{i+2}) = C_G(G_j/G_{j+2})$ for every positive numbers $1 \leq i, j \leq n-2$.*

Proposition 2.8 *Let G be a finite $3-$group of maximal class. Then G is a $CGZ-$group.*

Proof. Suppose $|G| \leq 3^4$. Then G has an abelian subgroup of index 3 in G. It is easy to get that G is a $CGZ-$group.

Now, suppose $|G| \geq 3^5$. Let $H < G$ be a nonabelian subgroup. Then $H \leq M$, where M is a maximal subgroup of G. By Lemma 2.5, M is either of maximal class or regular with $|M : \mho_1(M)| = 3^2$. By [2, Theorem 6], G is metabelian, thus $G^{'}$ is abelian and $|M : G^{'}| = 3$. If M is regular, then M is metacyclic subgroup since $|M : \mho_1(M)| = 3^2$. It follows that M is a minimal nonabelian subgroup. Hence $H = M$, we have that $C_G(H) = Z(H)$. Suppose that M is a $3-$group of maximal class. Let $H_1 = H \cap G^{'}$. Then there is an element $x \in M$ such that $M = \langle x, G^{'}\rangle$ and $H = \langle x, H_1\rangle$, where $|H : H_1| = 3$. We have that

$$G^{'} \geq C_M(H) = C_M(x) = Z(H).$$

If $C_G(H) \neq Z(H)$, there is an element $y \in G - M$ such that $y \in C_G(H)$. Note that

$$G = \langle y, M\rangle = \langle x, y, G^{'}\rangle = \langle x, y\rangle,$$

we have that G is abelian, a contradiction. Hence $C_G(H) = Z(H)$ and G is a $CGZ-$group. ∎

Since the finite $2-$groups of maximal class are classified and every finite $2-$group of maximal class has a cyclic subgroup of index 2 (see [5]). From the proof of Proposition 2.2, we have that all finite $2-$groups of maximal class are $CGZ-$groups. Proposition 2.8 shows that all finite $3-$group of maximal class are $CGZ-$groups too. So we assume in following theorem that $p \geq 5$.

Theorem 2.9 *Let G be finite $p-$group of maximal class, where $p \geq 5$. If G is a $CGZ-$group of order $p^n > p^{2p}$, then G has an abelian subgroup of index p.*

Proof. Let M be a normal subgroup of order p^3. Note that $|G| = p^n \geq p^{2p}$ and $p \geq 5$. By Lemma 2.5, G has a normal subgroup T of order p^{p-1} and $\exp(T) = p$. It is easy to see that $M \leq T$ and then M is an elementary abelian subgroup. Suppose that

$$M = \langle x\rangle \times \langle y\rangle \times \langle z\rangle,$$

where $x \in Z_3(G) - Z_2(G), y \in Z_2(G) - Z(G), z \in Z(G)$. Then $Z(G) = G_{n-1} = \langle z\rangle$. Let $N = \langle y\rangle \times \langle z\rangle$. We have $N = Z_2(G) \trianglelefteq G$.

By Corollary 2.7, $C_G(G_i/G_{i+i}) = C_G(G_j/G_{j+2})$ for positive numbers $1 \leq i$, $j \leq n-2$. It follows that

$$C_G(M/Z(G)) = C_G(N).$$

Let $G_1 = C_G(N)$, then $|G : G_1| = p$. Hence there are two elements $a, b \in G$ such that $G = \langle a, b\rangle$ and $G_1 = \langle b, G^{'}\rangle$. Since $N \leq Z(G_1)$ is of order $\geq p^2$, then G_1 is not of maximal class. By Lemma 2.5, G_1 is a regular $p-$ subgroup.

Let $H = \langle x, b\rangle$. Notice that $G_1 = C_G(M/Z(G))$, we have $[x, b] \in Z(G)$. Suppose that $[x, b] = 1$. Since $|G_1 : G^{'}| = p$ and $G^{'}$ is abelian, one has that $x \in Z(G_1)$, it follows that $M \leq Z(G_1)$. By Lemma 2.3, G_1 is abelian.

Suppose that $[x, b] \neq 1$. Now $\mathrm{cl}(H) = 2$. It follows from $C_G(M/Z(G)) = C_G(N) = G_1$ that $[x, b] \in Z(G)$. We have that $H^{'} = Z(G)$ has order p. Hence H is a minimal nonabelian $p-$subgroup, and we have $Z(H) = \langle b^p, z\rangle$. Assume that $z \in \langle b^p\rangle$, then $Z(H) = \langle b^p\rangle$. But

$$N \leq Z(G_1) \leq C_{G_1}(H) = Z(H),$$

a contradiction. Hence $z \notin \langle b^p\rangle$, and then $Z(H) = \langle b^p\rangle \times \langle z\rangle$. Since $|G_1 : G^{'}| = p$ and $G^{'}$ is abelian. We have $b^p \in G^{'}$, and then $b^p \in Z(G_1)$. Similarly, $(b^g)^p = (b^p)^g \in Z(G_1)$ for every element $g \in G$. Since $Z(G_1) \leq C_{G_1}(H) \leq Z(H)$, it follows that $(b^p)^g \in Z(H)$. Assume that $|b^p| \geq p^2$. Then $(b^p)^g = (b^p)^{l_1} z^{l_2}$, where $(l_1, p) = 1$. It follows that $\langle b^{p^2}\rangle = \langle (b^{p^2})^g\rangle$, so $\langle b^{p^2}\rangle \trianglelefteq G$. By the hypothesis, G is a $p-$group of maximal class, surely $|Z(G)| = p$. Notice that we have proved that $\langle b^p\rangle \cap Z(G) = 1$, we get $b^{p^2} = 1$. Since $G_1 = \langle b\rangle G^{'}$ and $G^{'} = \langle [b, a]\rangle^G$, we have by Lemma 2.5, that G_1 is regular and $|\Omega_1(G_1)| \leq p^{p-1}$. Hence $\exp(G_1) \leq p^2$ and $|G_1 : \mho_1(G_1)| = |\Omega_1(G_1)|$, consequently $\mho_1(G_1) \leq \Omega_1(G_1)$. It follows that

$$|G_1| = |\mho_1(G_1)||\Omega_1(G_1)| \leq p^{2p-2}.$$

We have $|G| \leq p^{2p-1}$, which contradicts to the hypothesis that $|G| \geq p^{2p}$. ∎

Remark. There exists a finite $p-$group G of maximal class which is also a $CGZ-$group, but G has no abelian subgroup of index p in G and $|G| < p^{2p}$. For example,

$$G=\langle a, b, c, d | a^{p^2}=b^p=c^p=d^p=1, [b, a]=c, [c, a]=d, [c, b]=a^p, [d, b]=a^p, [d, a]=1\rangle,$$

where $p \geq 5$. We have G is a $p-$group of maximal class with $|G| = p^5$, and G is a $CGZ-$group without any abelian subgroup of index p in G.

Acknowledgement. This work is supported by National Natural Science Foundation of China (11471226, 11271301) and Natural Science Foundation of CQC-STC (cstc 2014 jcyj A 00010).

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Accepted: 10.06.2014