A REFINEMENT ON THE GROWTH FACTOR IN GAUSSIAN ELIMINATION FOR ACCRETIVE-DISSIPATIVE MATRICES

Junjian Yang

School of Mathematics and Statistics
Hainan Normal University
Haikou, 571158
P.R. China
e-mail: junjianyang1981@163.com

Abstract. In this note, we give a refinement of the growth factor in Gaussian elimination for accretive-dissipative matrix A which is due to Lin [Calcolo, 2013, DOI 10.1007/s10092-013-0089-1].

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1. Introduction

Let $M_n(\mathbb{C})$ be the set of $n \times n$ complex matrices and A be a non-singular matrix in $M_n(\mathbb{C})$. Consider the linear system

$$Ax = b \tag{1.1}$$

and let $A^{(k)} = (a_{ij}^{(k)})$ be the matrix resulted from applying the first $k(1 \le k \le n-1)$ steps of Gaussian elimination to A; in particular, $A^{(n-1)}$ is the upper triangular matrix obtained from the LU factorization of A.

The quantity

$$\rho_n(A) = \frac{\max\limits_{i,j,k} |a_{ij}^{(k)}|}{\max\limits_{i,j} |a_{ij}|} \tag{1.2}$$

is called the growth factor (in Gaussian elimination) of A.

For any $A = (a_{ij}) \in M_n(\mathbb{C})$, A^* stands for the conjugate transpose of A. Similarly, x^* means the conjugate transpose of x for any $x \in \mathbb{C}$. $A \in M_n(\mathbb{C})$ is accretive-dissipative if it can be written as

$$A = B + iC, \tag{1.3}$$

where B and C are both (Hermitian) positive definite.

If B, C are real symmetric positive definite in (1.3), then A is called a Higham matrix.

For nonsingular matrix A, its condition number is denoted by

$$\kappa(A) := \sqrt{\frac{\lambda_{\max}(A^*A)}{\lambda_{\min}(A^*A)}}$$

which is the ratio of largest and smallest singular values of A.

It is conjectured in [1] that

$$\rho_n(A) \le 2 \tag{1.4}$$

for any Higham matrix A.

It is proved in [1] that if A in (1.1) is a Higham matrix, then no pivoting is needed in Gaussian elimination.

George et al. obtained the following result in [2]:

Theorem 1 *Let $A \in \mathrm{M}_n(\mathbb{C})$ be accretive-dissipative. Then*

$$\rho_n(A) < 3\sqrt{2}. \tag{1.5}$$

Furthermore, if A is a Higham matrix, then

$$\rho_n(A) < 3. \tag{1.6}$$

They proved Theorem 1 by the Theorem 2 in [2] below:

Theorem 2 *Let $A \in \mathrm{M}_n(\mathbb{C})$ be accretive-dissipative. Then*

$$\frac{|a_{ij}^{(k)}|}{|a_{ij}|} < 3, \qquad j = 1, \dots, n; \quad k = 1, \dots, n-1. \tag{1.7}$$

Lin [3] got a stronger result as follows:

Theorem 3 *Let $A \in \mathrm{M}_n(\mathbb{C})$ be accretive-dissipative. Then*

$$\frac{|a_{ij}^{(k)}|}{|a_{ij}|} < 2\sqrt{2}, \qquad j = 1, \dots, n; \quad k = 1, \dots, n-1. \tag{1.8}$$

Consequently,

$$\rho_n(A) < 4. \tag{1.9}$$

If A is a Higham matrix, then

$$\rho_n(A) < 2\sqrt{2}. \tag{1.10}$$

2. The main theorem

In this paper, we show a refinement of (1.8) which is a main result. Moreover, we get the refinements of (1.9) and (1.10):

Theorem 4 *Let $A \in \mathrm{M}_n(\mathbb{C})$ be accretive-dissipative. Then*

$$\frac{|a_{ij}^{(k)}|}{|a_{ij}|} < \left[1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right] \sqrt{2}, \ j = 1, ..., n; \ k = 1, ..., n-1, \tag{2.1}$$

Consequently, $\rho_n(A) < 2\left[1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right]$. If A is a Higham matrix, then

$$\rho_n(A) < \left[1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right] \sqrt{2},$$

where $\kappa \in [1, +\infty)$ is the maximum of the condition numbers of B and C.

Proof. We fix numbers $k \in 1, 2, ..., n-1$ and j, where $j \geq k+1$. Denote by A_k, B_k and C_k, respectively, the leading principal order k submatrices in A, B and C. Consider the $(k+1) \times (k+1)$ matrix

$$A_{kj} = \begin{pmatrix} A_k & u \\ v^T & a_{jj} \end{pmatrix}$$

where

$$u^T = (a_{1j}, a_{2j}, ..., a_{kj})$$

and

$$v^T = (a_{j1}, a_{j2}, ..., a_{jk}).$$

Note that A_{kj} is a principal order $k+1$ submatrix in A.

Defining the vectors

$$b^T = (b_{1j}, b_{2j}, ..., b_{kj})$$

and

$$c^T = (c_{1j}, c_{2j}, ..., c_{kj}),$$

we can rewrite A_{kj} as

$$A_{kj} = \begin{pmatrix} B_k + iC_k & b + ic \\ b^* + ic^* & b_{jj} + ic_{jj} \end{pmatrix}$$

It is easy to see that $a_{jj}^{(k)}$ can be obtained by performing block Gaussian eliminations in A_{kj}; namely,

$$a_{jj}^{(k)} = a_{jj} - v^T A_k^{-1} u = b_{jj} + ic_{jj} - (b^* + ic^*)(B_k + iC_k)^{-1}(b + ic).$$

Setting

$$a_{jj}^{(k)} = \beta + i\gamma, \quad \beta, \gamma \in \mathbf{R},$$

we have by [2, Theorem 2.1]

$$\beta = b_{jj} - b^* X_k b + c^* X_k c - b^* Y_k c - c^* Y_k b$$

and

$$\gamma = c_{jj} + b^* Y_k b - c^* Y_k c - b^* X_k c - c^* X_k b,$$

where

$$X_k = (B_k + C_k B_k^{-1} C_k)^{-1} \tag{2.2}$$

$$Y_k = (C_k + B_k C_k^{-1} B_k)^{-1} \tag{2.3}$$

with

$$\begin{pmatrix} B_k & b \\ b^* & b_{jj} \end{pmatrix} \text{ and } \begin{pmatrix} C_k & c \\ c^* & c_{jj} \end{pmatrix} \tag{2.4}$$

positive definite. It is known that $\beta, \gamma > 0$.

By simple computation, we have

$$\pm(b^* Y_k c + c^* Y_k b) \leq b^* Y_k b + c^* Y_k c; \tag{2.5}$$

$$\pm(b^* X_k c + c^* X_k b) \leq b^* X_k b + c^* X_k c. \tag{2.6}$$

From (2.2) and (2.3) we have [2, Lemma 2.3]

$$X_k \leq \frac{1}{2} C_k^{-1} \text{ and } Y_k \leq \frac{1}{2} B_k^{-1}. \tag{2.7}$$

From (2.4) and [4, (6)], we get

$$\left(\frac{\lambda_1 - \lambda_n}{\lambda_1 + \lambda_n}\right)^2 b_{jj} \geq b^* B_k^{-1} b \text{ and } \left(\frac{\lambda_1' - \lambda_n'}{\lambda_1' + \lambda_n'}\right)^2 c_{jj} \geq c^* C_k^{-1} c \tag{2.8}$$

In (2.8), λ_1 and λ_n (λ_1' and λ_n') are the largest and the smallest eigenvalues of B (C), respectively.

Note that $f(x) = (\frac{x-1}{x+1})^m (m \geq 1)$ is increasing for $x \in [1, \infty)$. Then we have

$$
\begin{aligned}
|a_{jj}^{(k)}| &= |\beta + i\gamma| \\
&\leq \beta + \gamma \\
&= b_{jj} - b^* X_k b + c^* X_k c - b^* Y_k c - c^* Y_k b \\
&\quad + c_{jj} + b^* Y_k b - c^* Y_k c - b^* X_k c - c^* X_k b \\
&\leq b_{jj} - b^* X_k b + c^* X_k c + (b^* Y_k b + c^* Y_k c) \qquad \text{(by (2.5))} \\
&+ \; c_{jj} + b^* Y_k b - c^* Y_k c + (b^* X_k b + c^* X_k c) \qquad \text{(by (2.6))} \\
&= b_{jj} + 2b^* Y_k b + c_{jj} + 2c^* X_k c \\
&\leq b_{jj} + b^* B_k^{-1} b + c_{jj} + c^* C_k^{-1} c \qquad \text{(by (2.7))} \\
&\leq b_{jj} + \left(\frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}\right)^2 b_{jj} + c_{jj} + \left(\frac{\lambda_n^{'} - \lambda_1^{'}}{\lambda_n^{'} + \lambda_1^{'}}\right)^2 c_{jj} \qquad \text{(by (2.8))} \\
&= \left[1 + \left(\frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1}\right)^2\right] b_{jj} + \left[1 + \left(\frac{\lambda_n^{'} - \lambda_1^{'}}{\lambda_n^{'} + \lambda_1^{'}}\right)^2\right] c_{jj} \\
&\leq \left[1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right] (b_{jj} + c_{jj}) \\
&\leq \left[1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right] \sqrt{2} |b_{jj} + i c_{jj}| \\
&= \left[1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right] \sqrt{2} |a_{jj}|.
\end{aligned}
$$

where $\kappa = \max(\frac{\lambda_1}{\lambda_n}, \frac{\lambda_1^{'}}{\lambda_n^{'}}) \geq 1$, i.e., the maximum of the condition numbers of B and C. This completes the proofs of (2.1).

It is easy to know that $\left[1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right] \sqrt{2} < 2\sqrt{2}$ for $\kappa \in [1, +\infty)$. So (2.1) is a refinement of (1.8).

To show the remaining claims, we need the following facts:

Fact1. [2, Corollary 2.3] The property of being an accretive-dissipative matrix is hereditary under Gaussian elimination.

Fact2. [2, Lemma 2.1, 2.2] If $A = (a_{ij}) \in M_n(\mathbb{C})$ is accretive-dissipative, then $\sqrt{2} \max_{l} |a_{ll}| \geq \max_{l \neq j} |a_{lj}|$. If A is a Higham matrix, then $\max_{l} |a_{ll}| \geq \max_{l,j} |a_{lj}|$.

Suppose $\max_{j,k} |a_{jj}^{(k)}| = |a_{j_0 j_0}^{(k_0)}|$ for some j_0, k_0, then by (2.1) the result below holds:

$$
\rho_n(A) = \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}|} \leq \frac{\sqrt{2} \max_{j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}|} \leq \frac{\sqrt{2} |a_{j_0 j_0}^{(k_0)}|}{|a_{j_0 j_0}|} < 2 \left[1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right].
$$

Similarly, if A is a Higham matrix, then $\rho_n(A) < \left[1 + \left(\frac{1-\kappa}{1+\kappa}\right)^2\right] \sqrt{2}$. The proof is thus complete. ∎

3. Conclusion

Our results in Theorem 4 are refinements of the results in Lin [3, Theorem 3]. Although it is a minor improvement, the result is much closer to the final solution of Higham's conjecture.

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