ASYMMETRIC CLOPEN SETS IN THE BITOPOLOGICAL SPACES

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Abstract. In the paper the behavior of clopen sets in bitopological spaces and some properties of generalized objects (e.g., (i, j)-quasi components and (i, j)-clopen compact subsets) are investigated. By using asymmetric clopen sets we introduce new classes of (i, j)-clopen irresolute and (i, j)-weakly clopen-continuous maps. Also, some their relations to p-ultra-Hausdorff bitopological structures are established. Characterizations and a preserving theorem of pairwise connected spaces are obtained.

Keywords: bitopological space, clopen, zero dimension, quasi-component.

2010 Mathematics Subject Classification: Primary 54E55.

1. Introduction

The clopen sets (i.e., sets that are both closed and open) play an important role in characterizations of the objects which define fundamental constructions of classical topology (see, e.g., [4], [6], [7], [10] etc.). It is well known that such sets are actually used in mathematical analysis, logics and theoretical computer sciences. In bitopological spaces, considerations of so called $(1, 2)$-clopen and $(2, 1)$-clopen sets seem to be not applied widely, although there are few interesting articles in this direction (see, e.g., [1], [12]). Motivated by this gap in the bitopological case we try to develop some theoretical constructions for asymmetric quasi-components, ultra-Hausdorff separation and continuous-like mappings by using asymmetric clopens. We obtain new characterizations and a preservation theorem of pairwise connected spaces due to Pervin [11].

Throughout the paper, for a bitopological space (X, τ_1, τ_2) we use the following notations: the interior and the closure of a subset A of X with respect to the topology τ_i are denoted by $\tau_i intA$ and $\tau_i clA$, respectively, where $i \in \{1, 2\}$. If O is open in τ_i, then we write $O \in \tau_i$, while, for the τ_i-closed set F, we use the notation $F \in co\tau_i$ (in this case, for brevity, O and F are meant also as an i-open and an i-closed set, respectively). We denote by $\tau_i^A = \{A \cap U | U \in \tau_i\}$ the topology induced on the set A from the τ_i. Next, in several results, we apply few important notions on bitopological structures, which are completely concerned in [5], but for classical topological ones see, e.g., [7]. The family of all τ_i-open neighborhoods of a subset M of X is denoted by $\sum_i^X(M)$. The bitopological space (X, τ_1, τ_2) is briefly denoted by BS (X, τ_1, τ_2).

2. (i, j)-Clopen sets

Definition 2.1. A subset A of a BS (X, τ_1, τ_2) is called an (i, j)-clopen set if $A \in \tau_i \cap co\tau_j$, where $i, j \in \{1, 2\}, i \neq j$.

Below the class of all (i, j)-clopen subsets of (X, τ_1, τ_2) will be denoted by $(i, j) - Clp(X)$. If $i = j$, we get the well known notion of general topology –the clopen set. Therefore, the class of $i - Clp(X)$ will denote the collection of all τ_i-clopen subsets of (X, τ_1, τ_2).

The following three propositions might be easily verified and we omit the proofs.

Proposition 2.1. *Let A and B be subsets of a BS (X, τ_1, τ_2).*

(1) *$A \in (i, j) - Clp(X)$ if and only if $X \setminus A \in (j, i) - Clp(X)$.*

(2) *If $A \in (i, j) - Clp(X)$ and $B \in (j, i) - Clp(X)$, then $A \setminus B \in (i, j) - Clp(X)$.*

(3) *The following equation holds:*

$$(1, 2) - Clp(X) \cap (2, 1) - Clp(X) = 1 - Clp(X) \cap 2 - Clp(X).$$

Proposition 2.2. *Let $A_\alpha \in (i, j) - Clp(X)$ for each $\alpha \in \Lambda$, $A = \bigcap_{\alpha \in \Lambda} A_\alpha$ and $B = \bigcup_{\alpha \in \Lambda} A_\alpha$. Then the following hold:*

(1) *$A \in co\tau_j$ and $B \in \tau_i$,*

(2) *$A, B \in (i, j) - Clp(X)$ if Λ is finite,*

(3) *$A \in (i, j) - Clp(X)$ (resp. $B \in (i, j) - Clp(X)$) if (X, τ_i) (resp. (X, τ_j)) is an Alexandorff space.*

Proposition 2.3. *If A is a subset of a BS (X, τ_1, τ_2) and $B \in (i,j) - Clp(X)$, then $A \cap B$ is (i,j)-clopen in the subspace (A, τ_1^A, τ_2^A).*

In [3], Dochviri introduced the notion of p-open (resp. p-closed) sets to obtain various characterizations of bitopological objects. Recall that a nonempty set A of a BS (X, τ_1, τ_2) is said to be p-open (resp. p-closed) if there exist $G_1 \in \tau_1$ and $G_2 \in \tau_2$ (resp. $F_1 \in co\tau_1$ and $F_2 \in co\tau_2$) such that $A = G_1 \cap G_2$ (resp. $A = F_1 \cup F_2$). The classes of p-open and p-closed sets of a given BS (X, τ_1, τ_2) are denoted by $p - O(X)$ and $p - C(X)$, respectively. The conjugate classes of sets were introduced in [2]. According to [2], a set A of a BS (X, τ_1, τ_2) is said to be p-quasi-open (resp. p-quasi-closed) if there exist $G_1 \in \tau_1$ and $G_2 \in \tau_2$ (resp. $F_1 \in co\tau_1$ and $F_2 \in co\tau_2$) such that $A = G_1 \cup G_2$ (resp. $A = F_1 \cap F_2$). The classes of p-quasi-open and p-quasi closed sets are denoted by $p - qO(X)$ and $p - qC(X)$, respectively. It is obvious that the complement of a p-open (resp. p-quasi-open) set is p-closed (resp. p-quasi-closed), and vice versa. By applying the above mentioned classes of sets we conclude: if $A \in (i,j) - Clp(X)$ and $B \in (j,i) - Clp(X)$ then $A \cap B \in p - O(X) \cap p - qC(X)$ and $A \cup B \in p - C(X) \cap p - qO(X)$.

Definition 2.2. A map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is said to be

(1) i-open (resp. i-continuous) if $f : (X, \tau_i) \to (Y, \gamma_i)$ is an open (resp. continuous) map.

(2) j-closed if $f : (X, \tau_j) \to (Y, \gamma_j)$ is a closed map.

(3) p-continuous if both $f : (X, \tau_1) \to (Y, \gamma_1)$ and $f : (X, \tau_2) \to (Y, \gamma_2)$ are continuous [8].

(4) p-homeomorphism if f is bijective and both f and f^{-1} are p-continuous, where f^{-1} denotes the inverse to f.

The proof of the following proposition is obvious.

Proposition 2.4. *If a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is i-open and j-closed and $A \in (i,j) - Clp(X)$, then $f(A) \in (i,j) - Clp(Y)$.*

Below, we obtain another conditions under which (i,j)-clopen sets are preserved.

Definition 2.3. A BS (X, τ_1, τ_2) is said to be (i,j)-stable [Ko] if any $A \in co\tau_i$ implies j-compactenss of A.

Note that if (X, τ_1, τ_2) is (j,i)-stable and i-Hausdorff then $(i,j) - Clp(X) \subset i - Clp(X)$.

Definition 2.4. A map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is said to be $(i,j) - \Delta$ continuous if $f : (X, \tau_i) \to (Y, \gamma_j)$ is continuous.

Proposition 2.5. *Let (X, τ_1, τ_2) be a (j,i)-stable BS and a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ be both i-open and $(i,j)-\Delta$-continuous. If a BS (Y, γ_1, γ_2) is j-T_2, then $f(A) \in (i,j) - Clp(Y)$ for each $A \in (i,j) - Clp(X)$.*

Proof. Note that, in the (j,i)-stable BS (X, τ_1, τ_2), for each $A \in (i,j) - Clp(X)$ A is a τ_i-compact subset. By $(i,j) - \Delta$-continuity of f, $f(A)$ is a τ_j-compact subset of (Y, γ_1, γ_2). Hence $f(A) \in co\gamma_j$ and combining this fact with the i-openness of f the proof is done. ∎

3. p-Connected spaces

Definition 3.1. A BS (X, τ_1, τ_2) is said to be pairwise connected (briefly p-connected) if X could not be represented as the union of the disjoint sets $A \in \tau_1 \setminus \{\emptyset\}$ and $B \in \tau_2 \setminus \{\emptyset\}$ [11].

In another case (X, τ_1, τ_2) is called a p-disconnected BS.

Theorem 3.1. *For a BS (X, τ_1, τ_2), the following properties are equivalent:*

(1) *(X, τ_1, τ_2) is p-connected;*

(2) *X cannot be represented as the union of nonempty disjoint $A \in (i,j) - Clp(X)$ and $B \in (j,i) - Clp(X)$;*

(3) *There exists no nonempty proper (i,j)-clopen set.*

Proof. (1)⇒(2): Suppose that there exist $A \in (i,j) - Clp(X)$, $B \in (j,i) - Clp(X)$ such that $\emptyset \neq A$, $\emptyset \neq B$, $A \cap B = \emptyset$ and $A \cup B = X$. Then $\emptyset \neq A \in \tau_i$, $\emptyset \neq B \in \tau_j$, $A \cap B = \emptyset$ and $A \cup B = X$. This shows that (X, τ_1, τ_2) is not p-connected.

(2)⇒(3): Suppose that there exists $A \in (i,j) - Clp(X)$ such that $\emptyset \neq A \subset X$ and $A \neq X$. Then, by Proposition 2.1 $X \setminus A \in (j,i) - Clp(X)$. Therefore, X is represented as the union of nonempty disjoint sets $A \in (i,j) - Clp(X)$ and $X \setminus A \in (j,i) - Clp(X)$. This contradicts (2).

(3)⇒(1): Suppose that X is not p-connected. Then there exist $A \in \tau_1$ and $B \in \tau_2$ such that $\emptyset \neq A$, $\emptyset \neq B$, $A \cap B = \emptyset$ and $A \cup B = X$. Therefore, there exists a nonempty proper set A such that $A \in \tau_1 \cap co\tau_2$. This contradicts (3). ∎

Definition 3.2. A map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is said to be (i,j)-clopen-irresolute if $f^{-1}(V) \in (i,j) - Clp(X)$ for each $V \in (i,j) - Clp(Y)$, where $i \neq j$, $i, j \in \{1, 2\}$.

If a map is both $(1,2)$-clopen-irresolute and $(2,1)$-clopen-irresolute then it is called to be p-clopen-irresolute. Every p-continuous map is p-clopen-irresolute but the converse is not always true as shown by the following example.

Example 3.1. Let us consider a set $X = \{m, n, p, q, k\}$, together with topologies $\tau_1 = \{\emptyset, X\} \cup \{\{m\}, \{n, p\}, \{m, n, p\}\}$ and $\tau_2 = \{\emptyset, X\} \cup \{\{q, k\}\}$. Then, we observe that $(1,2) - Clp(X) = \{\emptyset, X, \{m, n, p\}\}$ and $(2,1) - Clp(X) = \tau_2$. Moreover, let $Y = \{a, b, c, d\}$ be endowed with the following topologies $\gamma_1 = \{\emptyset, Y\} \cup \{\{a, b\}, \{c\}, \{a, b, c\}\}$ and $\gamma_2 = \{\emptyset, Y\} \cup \{\{c, d\}\}$, then $(1,2) - Clp(Y) = \{\emptyset, Y, \{a, b\}\}$ and $(2,1) - Clp(Y) = \gamma_2$. If we define a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ via the equations:$f(m) = f(n) = a, f(p) = b, f(q) = c, f(k) = d$, then it is p-clopen-irresolute. But $f : (X, \tau_1) \to (Y, \gamma_1)$ is not continuous and f is not p-continuous.

It is known that the p-connectedness is preserved under p-continuous surjections [11]. The following proposition is an improvement of this result.

Proposition 3.1. *The p-connectedness is preserved by (i, j)-clopen-irresolute surjections.*

Proof. Let $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ be an (i, j)-clopen-irresolute surjection and (X, τ_1, τ_2) be p-connected. Suppose that (Y, γ_1, γ_2) is not p-connected. By Theorem 3.1, there exists a nonempty proper (i, j)-clopen set V of Y. Since f is an (i, j)-clopen-irresolute surjection, then $f^{-1}(V)$ is a nonempty proper (i, j)-clopen subset of X. By Theorem 3.1, (X, τ_1, τ_2) is not p-connected. ∎

Recall that a BS (X, τ_1, τ_2) is said to be (i, j)-zero dimensional if a basis $\mathbf{B}(\tau_i)$ for the topology τ_i is formed with $co\tau_j$, i.e. $\mathbf{B}(\tau_i) = co\tau_j$ [12]. It is obvious that a BS (X, τ_1, τ_2) is (i, j)-zero dimensional if and only if $\mathbf{B}(\tau_i) = (i, j) - Clp(X)$.

Theorem 3.2. *If (X, τ_1, τ_2) is an (i, j)-zero dimensional and i-T_1 BS, then $card(A) \leq 1$ for every p-connected subspace (A, τ_1^A, τ_2^A).*

Proof. Assume there exists a p-connected subset A of X with $card(A) \geq 2$. Then, there exists $U \in (i, j) - Clp(X)$ such that $a \in U \subset X \setminus \{b\}$ for any pair of distinct points $a, b \in A$. Note that $A \setminus U \in \tau_j^A \setminus \{\emptyset\}$, $A \cap U \in \tau_i^A \setminus \{\emptyset\}$ and A is the disjoint union of $(A \setminus U)$ and $(A \cap U)$. Hence we get a contradiction to our assumption. ∎

Definition 3.3. The subset $\bigcap\{U(x) | x \in U(x) \in (i, j) - Clp(X)\}$ of a BS (X, τ_1, τ_2) is called the (i, j)-quasi-component of a point $x \in X$ and is denoted by $(i, j) - Q_x$.

Proposition 3.2. *Let x be a point in a BS (X, τ_1, τ_2). Then the following hold:*

(1) $(i, j) - Q_x \in co\tau_j \setminus \{\emptyset\}$.

(2) *If $y \in (i, j) - Q_x$, then $(i, j) - Q_y \subset (i, j) - Q_x$.*

Proof. (1) This follows immediately from Definition 3.3.

(2) Suppose that $a \notin (i, j) - Q_x$. Then there exists $V_a \in (i, j) - Clp(X)$ such that $x \in V_a$ and $a \notin V_a$. Since $y \in (i, j) - Q_x, y \in U(x)$ for every

$U(x) \in (i,j)-Clp(X)$ containing x. Therefore, $y \in V_a \in (i,j)-Clp(X)$ and $a \notin V_a$. Hence $a \notin (i,j)-Q_y$. Consequently, $(i,j)-Q_y \subset (i,j)-Q_x$. ∎

Proposition 3.3. *If $f:(X,\tau_1,\tau_2)\to(Y,\gamma_1,\gamma_2)$ is (i,j)-clopen-irresolute, then $f((i,j)-Q_x)\subset(i,j)-Q_{f(x)}$ for each $x \in X$.*

Proof. Since $(i,j)-Q_{f(x)} = \cap\{V|f(x)\in V\in(i,j)-Clp(Y)\}$, we have

$$\begin{aligned} f^{-1}((i,j)-Q_{f(x)}) &= f^{-1}(\cap\{V|f(x)\in V\in(i,j)-Clp(Y)\}) \\ &= \cap f^{-1}(\{V|f(x)\in V\in(i,j)-Clp(Y)\}) \\ &\quad \supset \cap\{U|x\in U\in(i,j)-Clp(X)\} \\ &= (i,j)-Q_x. \end{aligned}$$

Therefore, $f((i,j)-Q_x)\subset f(f^{-1}((i,j)-Q_{f(x)}))\subset(i,j)-Q_{f(x)}$. ∎

Corollary 3.1. *Let $f:(X,\tau_1,\tau_2)\to(Y,\gamma_1,\gamma_2)$ be a map.*

(1) *If f is p-continuous, then $f((i,j)-Q_x)\subset(i,j)-Q_{f(x)}$.*

(2) *If f is a p-homeomorphism, then $f((i,j)-Q_x)=(i,j)-Q_{f(x)}$.*

Proof. (1) Since every p-continuous function is (i,j)-clopen-irresolute, the proof follows immediately from Proposition 3.3.

(2) Since a p-homeomorphism is a bijection such that f and f^{-1} are p-continuous, the proof follows immediately from (1). ∎

The greatest p-connected subset containing a point $x \in X$ is called the p-component of x in a BS (X,τ_1,τ_2) and is denoted by $p-C_x$.

Theorem 3.3. *In a BS (X,τ_1,τ_2), the following implication holds:*

$$p-C_x \subset (1,2)-Q_x \cap (2,1)-Q_x.$$

Proof. Consider a set $G_x \in (1,2)-Clp(X)$ containing a point $x\in X$. Then from $G_x\cap(X\setminus G_x)=\emptyset$ it follows that $(p-C_x\cap G_x)\cap(p-C_x\setminus G_x)=\emptyset$. Moreover, we have $p-C_x\cap G_x\neq\emptyset$. Hence $p-C_x\setminus G_x=\emptyset$, or equivalently $p-C_x\subset G_x$. It is obvious that $p-C_x\subset\bigcap G_x=(1,2)-Q_x$. Similarly we obtain that $p-C_x\subset(2,1)-Q_x$. Thereby, the implication $p-C_x\subset(1,2)-Q_x\cap(2,1)-Q_x$ is valid. ∎

4. p-ultra-Hausdorff spaces

Definition 4.1. A BS (X,τ_1,τ_2) is said to be

(1) p-Urysohn if for any pair of distinct points $x_1,\ x_2\in X$ there exist the neighborhoods $U\in\sum_i^X(x_1)$ and $V\in\sum_j^X(x_2)$ such that $\tau_j clU\cap\tau_i clV=\emptyset$ (see [5]).

(2) p-ultra-Hausdorff if for any pair of distinct points x_1,$x_2 \in X$ there exist $U_{x_1} \in (i,j) - Clp(X)$ and $V_{x_2} \in (j,i) - Clp(X)$ such that $x_1 \in U_{x_1}$, $x_2 \in V_{x_2}$ and $U_{x_1} \cap V_{x_2} = \emptyset$.

It should be especially noticed that if $i = j$ then the notion of p-ultra-Hausdorff coincides with the notion of ultra-Hausdorff for topological spaces, given in [13].

Example 4.1. Let X be a set with $card(X) \geq \aleph_0$, τ_d-discrete and $\tau_{cof.}$-cofinite (i.e., all finite subsets of X are closed, and vice versa) topologies on X, respectively. Then it is obvious that $(X, \tau_d, \tau_{cof.})$ is p-ultra-Hausdorff BS.

Remark 4.1. It should be mentioned that p-ultra-Hausdorff implies p-Urysohn but the converse does not hold. Find example which show that the BS Space is p-Urysohn but not p-ultra-Hausdorff. Under which conditions the converse is true?

Proposition 4.1. *A BS (X, τ_1, τ_2) is p-ultra-Hausdorff if and only if for any distinct points x_1, x_2, there exist $U \in (i,j) - Clp(X)$ such that $x_1 \in U$, $x_2 \notin U$ and $V \in (i,j) - Clp(X)$ such that $x_2 \in V, x_1 \notin V$.*

Definition 4.2. A subset K of a BS (X, τ_1, τ_2) is said to be (i,j)-clopen-compact relative to X if every cover of K by (i,j)-clopen sets of X has a finite subcover.

It should be noticed that every i-compact subset of (X, τ_1, τ_2) is (i,j)-clopen-compact relative to X.

Proposition 4.2. *If a BS (X, τ_1, τ_2) is (i,j)-zero dimensional and a subset K of X is (i,j)-clopen-compact relative to X, then K is a τ_i-compact subset of X.*

Proof. Consider a family $\mathcal{U} = \{U_\alpha | U_\alpha \in \tau_i\}_{\alpha \in \Lambda}$ such that $K \subset \bigcup\limits_{\alpha \in \Lambda} U_\alpha$. Then, since (X, τ_1, τ_2) is (i,j)-zero dimensional, we can write $U_\alpha = \bigcup\limits_{\alpha_\beta \in \Omega_\alpha} V_{\alpha_\beta}$ for each $\alpha \in \Lambda$, where $V_{\alpha_\beta} \in (i,j) - Clp(X)$. Denote $\Omega \equiv \bigcup\limits_{\alpha \in \Lambda} \Omega_\alpha$, then the family $\mathcal{V} = \{V_\beta\}_{\beta \in \Omega}$ is a cover of K by (i,j)-clopen sets of X. Therefore, one can extract from the family $\mathcal{V}$ a finite collection $\{V_{\beta_1}, V_{\beta_2}, ..., V_{\beta_n}\}$ such that $K \subset \bigcup\limits_{p=1}^{n} V_{\beta_p}$. Now we can choose finite collection $\{U_{\alpha(\beta_1)}, U_{\alpha(\beta_2)}, ..., U_{\alpha(\beta_n)}\}$ from the covering $\mathcal{U}$ such that $K \subset \bigcup\limits_{p=1}^{n} U_{\alpha(\beta_p)}$. ■

Theorem 4.1. *For a BS (X, τ_1, τ_2), the following are equivalent:*

(1) *X is p-ultra-Hausdorff;*

(2) *For each set K of X which is (i,j)-clopen compact relative to X, $K = \cap\{V | K \subset V \in (i,j) - Clp(X)\}$;*

(3) *For each $x \in X$, $x = \cap\{V | x \in V \in (i,j) - Clp(X)\}$.*

Proof. (1) $\Rightarrow$ (2): Let K be any set of X which is (i,j)-clopen-compact relative to X. It is obvious that $K \subset \cap\{V_\alpha | K \subset V_\alpha \in (i,j) - Clp(X)\}$. Suppose that $x \notin K$. Then, by Proposition 4.1, for each $k \in K$ there exists $V_k \in (i,j) - Clp(X)$ such that $x \notin V_k$ and $k \in V_k$. Since $\{V_k | k \in K\}$ is a cover of K by (i,j)-clopen sets of X, there exist a finite points $k_1, k_2, ..., k_n \in K$ such that $K \subset \bigcup\limits_{m=1}^{n} V_{k_m}$. Now, set $V = \bigcup\limits_{m=1}^{n} V_{k_m}$. Then $K \subset V \in (i,j) - Clp(X)$ and $x \notin V$. Therefore, we have $x \notin \cap\{V_\alpha | K \subset V_\alpha \in (i,j) - Clp(X)\}$. Hence $K \supset \cap\{V_\alpha | K \subset V_\alpha \in (i,j) - Clp(X)\}$. Consequently, we obtain the assertion (2).

(2) $\Rightarrow$ (3): This is obvious since every singleton is (i,j)-clopen-compact relative to X.

(3) $\Rightarrow$ (1): For any distinct points $x_1, x_2 \in X$, by (3) we have $x_1 \notin \cap\{V_\alpha \mid x_2 \in V_\alpha \in (i,j) - Clp(X)\}$. Therefore, there exists $V \in (i,j) - Clp(X)$ such that $x_2 \in V$ and $x_1 \notin V$. Similarly, we have $x_2 \notin \cap\{V_\alpha | x_1 \in V_\alpha \in (i,j) - Clp(X)\}$. Therefore, there exists $U \in (i,j) - Clp(X)$ such that $x_1 \in U$ and $x_2 \notin U$. By Proposition 4.1, X is p-ultra-Hausdorff. ∎

Corollary 4.1. *If a BS (X, τ_1, τ_2) is p-ultra-Hausdorff and K is (i,j)-clopen-compact relative to X, then $K \in co\tau_j$.*

Proof. This follows from Proposition 2.2 and Theorem 4.1. ∎

Definition 4.3. A map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is said to be (i,j)-weakly clopen-continuous (resp. (i,j)-clopen-continuous) if, for each $x \in X$ and each $V \in \sum_i^Y (f(x))$, there exists a set $U \in (i,j) - Clp(X)$ containing x such that $f(U) \subset \gamma_j cl(V)$ (resp. $f(U) \subset V$), where $i \neq j$, $i,j \in \{1,2\}$.

Example 4.2. Let us consider the set $X = \{m, n, p, q, k\}$, together with topologies $\tau_1 = \{\emptyset, X\} \cup \{\{m\}, \{n,p\}, \{m,n,p\}\}$ and $\tau_2 = \{\emptyset, X\} \cup \{\{q,k\}\}$. Then, we observe that $(1,2) - Clp(X) = \{\emptyset, X, \{m,n,p\}\}$ and $(2,1) - Clp(X) = \{\emptyset, X, \{q,k\}\}$. Moreover, let $Y = \{a,b,c\}$ be endowed with the following topologies $\gamma_1 = \{\emptyset, Y\} \cup \{\{a\}\}$ and $\gamma_2 = \{\emptyset, Y\} \cup \{\{c\}\}$. If we define a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ via the equations: $f(m) = f(n) = a, f(p) = b, f(q) = f(k) = c$, then f is $(1,2)$-weakly clopen-continuous. But it is not $(1,2)$-clopen-continuous.

Theorem 4.2. *If $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is (i,j)-weakly clopen-continuous (resp. (i,j)-clopen-continuous) and A is a subset of X, then $f|_A : (A, \tau_1^A, \tau_2^A) \to (Y, \gamma_1, \gamma_2)$ is (i,j)-weakly clopen-continuous (resp. (i,j)-clopen-continuous).*

Proof. We prove only the case of (i,j)-weakly clopen-continuous. Let $x \in A$ and $V \in \sum_i^Y (f(x))$. Since f is (i,j)-weakly clopen-continuous, then there exists an (i,j)-clopen set U in X containing x such that $f(U) \subset W = \gamma_j cl(V)$. Because $U \subset f^{-1}(W)$, we have the implication $U \cap A \subset f^{-1}(W) \cap A = (f|_A)^{-1}(W)$ and $x \in U \cap A \in (i,j) - Clp(A)$. Consequently, $f|_A$ is (i,j)-weakly clopen-continuous. ∎

Theorem 4.3. *If $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is a p-weakly clopen-continuous injective map and a BS (Y, γ_1, γ_2) is p-Urysohn, then (X, τ_1, τ_2) is p-ultra-Hausdorff.*

Proof. For any pair of distinct points $x, y \in X$, there exist $U \in \sum_i^Y(f(x))$ and $V \in \sum_j^Y(f(y))$ such that $\gamma_j cl(U) \cap \gamma_i cl(V) = \emptyset$. Since f is p-weakly clopen-continuous, there exist $G_x \in (i,j) - Clp(X)$ and $H_y \in (j,i) - Clp(X)$ containing x and y, respectively, such that $f(G_x) \subset \gamma_j cl(U)$ and $f(H_y) \subset \gamma_i cl(V)$. Since $\gamma_j cl(U) \cap \gamma_i cl(V) = \emptyset$, then $G_x \cap H_y = \emptyset$. This shows that (X, τ_1, τ_2) is p-ultra-Hausdorff. ∎

Theorem 4.4. *If a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is (i,j)-weakly clopen-continuous and (X, τ_j) is Alexandorff, then f is (i,j)-clopen-irresolute.*

Proof. Let V be any (i,j)-clopen set of Y and $x \in f^{-1}(V)$. Then $V \in \sum_i^Y(f(x))$. Since f is (i,j)-weakly clopen-continuous, there exists $U_x \in (i,j) - Clp(X)$ such that $x \in U_x$ and $f(U_x) \subset \gamma_j cl(V) = V$. For each $x \in f^{-1}(V)$, we have $x \in U_x \subset f^{-1}(V)$ and hence $\bigcup_{x \in f^{-1}(V)} U_x = f^{-1}(V)$. Since (X, τ_j) is Alexandorff, by Proposition 2.2 $f^{-1}(V) \in (i,j) - Clp(X)$. Therefore, f is (i,j)-clopen-irresolute. ∎

Theorem 4.5. *If a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is (i,j)-clopen irresolute and (Y, γ_1, γ_2) is (i,j)-zero dimensional, then f is (i,j)-clopen-continuous.*

Proof. Let $x \in X$ and $V \in \sum_i^Y(f(x))$. Since Y is (i,j)-zero dimensional, there exists $W \in (i,j) - Clp(Y)$ containing $f(x)$ such that $W \subset V$. Since f is (i,j)-clopen irresolute, we have $f^{-1}(W) \in (i,j) - Clp(X)$. Set $U = f^{-1}(W)$, then $x \in U$ and $f(U) \subset W \subset V$. This shows f is (i,j)-clopen-continuous. ∎

Corollary 4.6. *Let (X, τ_j) be an Alexandroff topological space and (Y, γ_1, γ_2) is an (i,j)-zero dimensional BS. Then a map $f : (X, \tau_1, \tau_2) \to (Y, \gamma_1, \gamma_2)$ is (i,j)-clopen-continuous if and only if it is (i,j)-clopen irresolute.*
Proof. This follows immediately from Theorems 4.4 and 4.5. ∎

Acknowledgement. We are grateful to Professor S. Jafari for his suggestions.

References

[1] Bezhanishvili, G., Bezhanishvili, D. Gabelaia, D., Kurz, A., *Bitopological duality for distributive lattices and Heyting algebras*, Math. Struct. Comput. Sci., 20 (3) (2010), 359–393.

[2] Datta, M.C., *Projective bitopological spaces*, J. Aust. Math. Soc., 13 (1972), 327–334.

[3] Dochviri, I., *Characterization of some bitopological properties via pairwise sets*, Proc. A. Razmadze Math. Inst., 140 (2006), 75–82.

[4] Dochviri, I., Noiri, T., *Some remarks on mildly compact spaces in bitopologies*, Quest. Answers Gen. Topology, 32 (1) (2014), 69-72.

[5] Dvalishvili, B.P., *Bitopological spaces: theory, relations with generalized algebraic structures and applications*, Math. Studies, 199, North-Holland, Elsevier, 2005.

[6] Ekici E., *Generalization of perfectly continuous, regular set-connected and clopen functions*, Acta. Math. Hungar., 107 (3) (2005), 193–206.

[7] Engelking, R., *General topology.* (Russian Edit.) Moscow, Publ. House Mir, 1986.

[8] Kelly, J.C., *Bitopological spaces*, Proc. London Math Soc., 9 (13) (1963), 71–89.

[9] Kopperman, R.D., *Assymetry and duality in topology*, Topol. Appl., 66 (1995), 1–39.

[10] Noiri, T., *Super-continuity and some strong forms of continuity*, Indian J. Pure Appl. Math., 15 (3) (1984), 241–250.

[11] Pervin, W.J., *Connectedness in bitopological spaces*, Indag. Math., 29 (1967), 369–372.

[12] Reilly, I., *Zero-dimensional bitopological spaces*, Indag. Math., 35 (1973), 127–131.

[13] Staum, R., *The algebra of bounded continuous functions into a non-archimedean field*, Pacific J. Math., 50 (1974) 169–185.

Accepted: 24.05.2014