CONVERGENCE OF LAGRANGE-HERMITE INTERPOLATION

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Abstract. In this paper, we consider explicit representations and convergence of Lagrange–Hermite Interpolation on two disjoint set of nodes, which are obtained by projecting vertically the zeros of $\left(1-x^2\right)P_n^{(\alpha,\beta)}(x)$ and $\left(1-x^2\right)P_n^{(\alpha,\beta)\prime}(x)$ respectively on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomials.

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1. Introduction

In 1975, L.G. Pál [10] introduced a new type of interpolation on the zeros of two different polynomials. He considered two systems of real numbers $\{x_n\}_{k=0}^{n}$ and $\{x_n^*\}_{k=0}^{n-1}$, which are the zeros of $W_n(x)$ and $W_n'(x)$ respectively, then there exists a unique polynomials $P(x)$ of degree at most $2n-1$ satisfying the interpolatory properties:

$$P(x_k) = y_k, \quad k = 1\,(1)\,n,$$
$$P'(x_k^*) = y_k, \quad k = 1\,(1)\,n-1,$$

and gave the explicit formulae of this polynomial. In another paper, L.G. Pál [11] considered the $(0;0,1)$-Interpolation and obtained the convergence for the same. In 2003, H.P. Dikshit [7] also considered the Pál-type interpolation on non-uniformly distributed nodes on the unit circle. Later on P. Mathur [9] considered $(0,1;0)$ interpolation on infinite interval. P. Mathur and his associates [13] considered $(0;0,1)$ interpolation.

In a paper, S. Xie [15] considered the regularity of $(0,1,...,r-2,r)$ and $(0,1,...,r-2,r)^*$ −interpolation on the sets obtained by projecting vertically the zeros of $(1-x^2)\,P_n^{(\alpha,\beta)}(x)$ on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for the Jacobi polynomials. After that S. Bahadur [3, 5] considered $(0,1;0)$ and $(0;0,1)$ interpolation on the unit circle and establish the convergence theorem for the same. In another paper, the authors [6] have considered Hermite–Lagrange interpolation and established a convergence theorem on the unit circle.

In this paper, we consider Lagrange–Hermite interpolation on the unit circle. Here, we consider two pairwise disjoint sets $\{z_k\}_{k=0}^{2n+1}$ and $\{t_k\}_{k=0}^{2n-1}$, which are the vertically projected zeros of $(1-x^2)\,P_n^{(\alpha,\beta)}(x)$ and $(1-x^2)\,P_n^{(\alpha,\beta)\prime}(x)$ on the unit circle, respectively.

Let Z_n and T_n be two sets satisfying:

$$Z_n = \begin{cases} z_k : k = 0(1)2n+1 : \\ z_0 = 1,\ z_{2n+1} = -1 \\ z_k = \cos\theta_k + i\sin\theta_k,\ z_{n+k} = -z_k,\ k = 1\,(1)\,n \end{cases}$$

(1.1) and

$$T_n = \begin{cases} t_k : k = 0(1)2n-1 : \\ t_0 = 1,\ t_{2n-1} = -1 \\ t_k = \cos\phi_k + i\sin\phi_k,\ t_{n+k} = -t_k, k = 1\,(1)\,n-1 \end{cases},$$

which are the zeros of $(1-x^2)\,P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha,\beta)\prime}(x)$, respectively.

In Section 2, we give some preliminaries and in Section 3, we describe the problem and obtained the regularity of the same. In Section 4, we give the explicit formulae of the interpolatory polynomials. In Sections 5 and 6, estimation of interpolatory polynomials and convergence are given, respectively.

2. Preliminaries

In this section, we shall give some well known results, which we shall use.

The differential equation satisfied by $P_n^{(\alpha,\beta)}(x)$ is:

$$(2.1)\qquad \left(1-x^2\right) P_n^{(\alpha,\beta)\prime\prime}(x) + \left[\beta-\alpha-(\alpha+\beta+2)\,x\right] P_n^{(\alpha,\beta)\prime}(x) + n\,(n+\alpha+\beta+1)\,P_n^{(\alpha,\beta)}(x) = 0$$

$$(2.2)\qquad W(z) = \prod_{k=1}^{2n}(z-z_k) = K_n P_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right) z^n$$

$$(2.3)\qquad H(z) = \prod_{k=1}^{2n-2}(z-t_k) = K_n^* P_n^{(\alpha,\beta)\prime}\left(\frac{1+z^2}{2z}\right) z^{n-1}.$$

We shall require the fundamental polynomials of Lagrange interpolation based on Z_n and T_n

$$L_k\ (z) = \frac{R(z)}{R'(z_k)(z - z_k)},\ k = 0(1)2n+1, \tag{2.4}$$

where $R(z) = (z^2-1)\,W(z)$,

$$l_k\ (z) = \frac{H(z)}{H'(t_k)(z - t_k)},\ k\ = 1(1)2n-2. \tag{2.5}$$

We will also use the following results

$$(-1)^n W'(z_{n+k}) \ = W'(z_k) = -\frac{1}{2} K_n P_n^{(\alpha,\beta)\prime}(x_k)\left(1 - z_k^2\right) z_k^{n-2}, \quad k = 1\,(1)\,n. \tag{2.6}$$

We will also use the following well known inequalities (see [8])

$$(1-x^2)^{\frac{1}{2}}\, P_n^{(\alpha,\beta)}(x) = o\left(n^{\alpha-1}\right), \tag{2.7}$$

$$(1-x_k^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2} \tag{2.8}$$

$$\left|\ P_n^{(\alpha,\beta)\prime}(x_k)\right| \sim k^{-\alpha-\frac{3}{2}}\ n^{\alpha+2} \tag{2.9}$$

$$\left|P_n^{(\alpha,\beta)}(x)\right| = o\left(n^{\alpha}\right), \tag{2.10}$$

$$(1-x^2)\left|P_n^{(\alpha,\beta)\prime}(x)\right| \le cn^{\alpha+1}, \tag{2.11}$$

$$\left|\ P_n^{(\alpha,\beta)}(x_k)\right| \sim\ k^{-\alpha-\frac{1}{2}}\ n^{\alpha}, \tag{2.12}$$

$$\left|\ P_n^{(\alpha,\beta)\prime}(x)\right| \sim o\left(n^{\alpha+2}\right). \tag{2.13}$$

3. The problem

Let $\{z_k\}_{k=0}^{2n+1}$ and $\{t_k\}_{k=0}^{2n-1}$ be the two disjoint sets of nodes obtained by projecting vertically the zeros of $(1-x^2)\,P_n^{(\alpha,\beta)}(x)$ and $(1-x^2)P_n^{(\alpha,\beta)\prime}(x)$ on the unit circle respectively, we seek to determine the interpolatory polynomial $R_n(z)$ of degree $\le 6n-1$ satisfying the conditions:

$$\begin{cases} R_n(z_k) = \alpha_k, & k = 0\,(1)\,2n+1, \\ R_n(t_k) = \beta_k, & k = 1\,(1)\,2n-2, \\ R_n'(t_k) = \gamma_k, & k = 0\,(1)\,2n-1, \end{cases} \tag{3.1}$$

where α_k, β_k and γ_k are arbitrary complex numbers. We are also interested in establishing a convergence theorem for the same.

Regularity

Theorem 1. *The Lagrange–Hermite interpolation is regular on Z_n and T_n.*

Proof. It is sufficient, if we show the unique solution of (3.1) is $R_n(z) \equiv 0$, when all data $\alpha_k = \beta_k = \gamma_k = 0$. Clearly, in this case we have $R_n(z) = R(z)\, H(z)\, q(z)$, where $q(z)$ is a polynomial of degree $\leq 2n-1$.

Let $R_n(z) = R(z)\, H(z)\, q(z)$. Obviously, $R_n(z_k) = 0$ and $R_n(t_k) = 0$. Then, from $R_n'(t_k) \equiv 0$, we have $q(t_k) = 0$. Therefore, $q(z) = (az+b)\, H(z)$, where a and b are arbitrary constants.

As $q(\pm 1) = 0$, we get $a = b = 0$. It indicates

$$R_n(z) \equiv q(z) \equiv 0.$$

Hence the theorem follows.

4. Explicit representation of interpolatory polynomials

We shall write $R_n(z)$ satisfying (3.1) as

$$R_n(z) = \sum_{k=0}^{2n+1} \alpha_k A_k(z) + \sum_{k=1}^{2n-2} \beta_k B_k(z) + \sum_{k=0}^{2n-1} \gamma_k C_k(z), \tag{4.1}$$

where $A_k(z)$, $B_k(z)$ and $C_k(z)$ are unique polynomials, each of degree atmost $6n-1$ satisfying the following conditions:

For $k = 0\,(1)\,2n+1$

$$\begin{cases} A_k(z_j) = \delta_{jk}; & j = 0\,(1)\,2n+1, \\ A_k(t_j) = 0; & j = 1\,(1)\,2n-2, \\ A_k'(t_j) = 0; & j = 0\,(1)\,2n-1. \end{cases} \tag{4.2}$$

For $k = 1\,(1)\,2n-2$

$$\begin{cases} B_k(z_j) = 0; & j = 0\,(1)\,2n+1, \\ B_k(t_j) = \delta_{jk}; & j = 1\,(1)\,2n-2, \\ B_k'(t_j) = 0; & j = 0\,(1)\,2n-1. \end{cases} \tag{4.3}$$

For $k = 0\,(1)\,2n-1$

$$\begin{cases} C_k(z_j) = 0; & j = 0\,(1)\,2n+1, \\ C_k(t_j) = 0; & j = 1\,(1)\,2n-2 \\ C_k'(t_j) = \delta_{jk}; & j = 0\,(1)\,2n-1 \end{cases} \tag{4.4}$$

Theorem 2. *For* $k = 1(1)2n-2$, *we have,*

$$C_k(z) = \frac{(z^2-1)\,R(z)\,H(z)\,l_k(z)}{(t_k^2-1)\,R(t_k)\,H'(t_k)}. \tag{4.5}$$

For $k = 0, 2n-1$

$$C_k(z) = -\frac{(1+t_k z)\,H^2(z)\,R(z)}{2H^2(t_k)\,R'(t_k)}. \tag{4.6}$$

Theorem 3. *For* $k = 1(1)2n-2$, *we have,*

$$B_k(z) = \frac{(z^2-1)\,R(z)\,l_k^2(z)}{(t_k^2-1)\,R(t_k)} - \left\{\frac{2t_k}{(t_k^2-1)} + \frac{R'(t_k)}{R(t_k)} + \frac{H''(t_k)}{H'(t_k)}\right\} C_k(z), \tag{4.7}$$

where $C_k(z)$ *is given by* (4.5).

Theorem 4. *For* $k = 0(1)2n+1$, *we have,*

$$A_k(z) = \frac{(z^2-1)\,H^2(z)\,L_k(z)}{(z_k^2-1)\,H^2(z_k)}. \tag{4.8}$$

One can establish Theorems 2, 3 and 4 owing to conditions (4.2), (4.3) and (4.4), respectively.

5. Estimation of fundamental polynomials

Lemma 1. [2] *Let* $L_k(z)$ *be given by* (2.4). *Then*

$$\sum_{k=0}^{2n+1} |\,L_k(z)\,| \le c \sum_{k=0}^{2n+1} \frac{1}{k^{-\alpha+\frac{3}{2}}}, \tag{5.1}$$

where c *is a constant independent of* n *and* z.

Lemma 2. [6] *Let* $l_k(z)$ *be given by* (2.5). *Then*

$$\sum_{k=1}^{2n-2} |\,l_k(z)\,| \le c \sum_{k=1}^{2n-2} \frac{1}{k^{-\alpha-\frac{1}{2}}}, \tag{5.2}$$

where c *is a constant independent of* n *and* z.

Lemma 3. *Let* $C_k(z)$ *be given by* (4.5),*we have*

$$\sum_{k=0}^{2n-1} |C_k(z)| \le c \log n, \qquad -1 < \alpha \le -\frac{1}{2},\ |z| \le 1 \tag{5.3}$$

and c *is a constant independent of* n *and* z.

Proof. From (4.5) and (4.6) using (2.8) , (2.10) , (2.12) and Lemma 2, we get the required result.

Lemma 4. *Let* $B_k(z)$ *be given by* (4.7), *we have*

$$\sum_{k=1}^{2n-2} |B_k(z)| \leq cn \log n, \quad -1 < \alpha \leq -\frac{1}{2}, \ |z| \leq 1 \tag{5.4}$$

where c is a constant independent of n and z.

Proof. From (4.6), using (2.7) , (2.8) , (2.12) and Lemmas 2 and 3, we get the required result.

Lemma 5. *Let* $A_k(z)$ *be given by* (4.8), *we have*

$$\sum_{k=0}^{2n+1} |A_k(z)| \leq cn \log n, \ -1 < \alpha \leq -\frac{1}{2}, \ |z| \leq 1, \tag{5.7}$$

where c is a constant independent of n and z.

Proof. Proof is similar to Lemma 3.

6. Convergence

Let $f(z)$ be analytic for $|z| < 1$ and continuous for $|z| \leq 1$ and $\omega(f, \delta)$ be the modulus of continuity of $f(e^{ix})$.

Theorem 5. *Let* $f(z)$ *be continuous in* $|z| \leq 1$ *and analytic in* $|z| < 1$. *Let the arbitrary numbers* β_k*'s and* γ_k'*s be such that:*

$$\begin{cases} |\beta_k| = o\left(\omega_2(f, n^{-1})\right), & k = 1\,(1)\,2n-2, \\ |\gamma_k| \ = o\left(n\omega_2(f, n^{-1})\right), & k = 1\,(1)\,2n-2. \end{cases} \tag{6.1}$$

Then, $\{R_n\}$ *defined by*

$$R_n(z) = \sum_{k=0}^{2n+1} f(z_k) A_k(z) + \sum_{k=1}^{2n-2} \beta_k B_k(z) + \sum_{k=0}^{2n-1} \gamma_k C_k(z) \tag{6.2}$$

satisfies the relation

$$|R_n(z) - f(z)| = o\left(n\omega_2(f, n^{-1}) \log n\right) \ \textit{for} \ -1 < \alpha \leq -\frac{1}{2} \tag{6.3}$$

where $\omega_2(f, n^{-1})$ *is the modulus of continuity of* $f(z)$.

Remark. To prove Theorem 5, we shall need the following:

Let $f(z)$ be continuous in $|z| \leq 1$ and $f' \in Lip\,\nu$, $\nu > 0$. Then the sequence $\{R_n\}$ converges uniformly to $f\,(z)$ in $|z| \leq 1$, follows from (6.3) provided

$$\omega_2(f, n^{-1}) = o\left(n^{-1-\nu}\right). \tag{6.4}$$

Let $f(z)$ be continuous in $|z| \leq 1$ and analytic in $|z| < 1$. Then there exists a polynomial $F_n(z)$ of degree $6n-1$ satisfying Jackson's inequality

$$|F_n\,(z) - f(z)| \leq c\omega_2\left(f, n^{-1}\right), \quad z = e^{i\theta}\,(0 < \theta \leq 2\pi) \tag{6.5}$$

and also an inequality due to O. Kiš [8]

$$\left|F_n^{(m)}\,(z)\right| \leq cn^m\omega_2\left(f, n^{-1}\right), \quad \text{for } m \in I^+. \tag{6.6}$$

Proof. Since $R_n\,(z)$ be given by (6.2) is a uniquely determined polynomial of degree $\leq 6n-1$, the polynomial $F_n\,(z)$ satisfying (6.5) and (6.6) can be expressed as

$$R_n\,(z) = \sum_{k=0}^{2n+1} F_n(z_k)A_k\,(z) + \sum_{k=1}^{2n-2} F_n\,(t_k)\,B_k\,(z) + \sum_{k=0}^{2n-1} F_n'\,(t_k)\,C_k\,(z)\,.$$

Then,

$$\begin{aligned}|R_n\,(z) - f(z)| &\leq |R_n\,(z) - F_n(z)| + |F_n\,(z) - f(z)| \\ &\leq \sum_{k=0}^{2n+1} |f\,(z_k) - F_n(z_k)|\,|A_k(z)| + \sum_{k=1}^{2n-2} \{|\beta_k| + |F_n\,(t_k)|\}\,|B_k(z)| \\ &+ \sum_{k=0}^{2n-1} \{|\gamma_k| + |F_n'\,(t_k)|\}\,|C_k\,(z)| + |F_n\,(z) - f(z)|\,.\end{aligned}$$

Using $z = e^{i\theta}$ $(0 < \theta \leq 2\pi)$, (6.1), (6.4), (6.5), (6.6) and Lemmas 3, 4 and 5, we get (6.3).

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