ON AN INERTIA FACTOR GROUP OF $2^8{:}O_8^+(2)$

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Abstract. The group $\overline{G} = 2^6{:}A_8$ is an inertia factor group of $2^8{:}O_8^+(2)$. As an inertia factor group, our group $\overline{G}$ plays an essential role in the construction of the character table of $2^8{:}O_8^+(2)$. In this paper we look at two ways of constructing $\overline{G}$. In the first method, we use combinatorics and the natural action of A_8 on $W \cong 2^6$. In the second method, we use a computational method to construct $\overline{G} = N{:}A_8$ inside $O_8^+(2)$. We show that A_8 acts irreducibly on both W and N and we prove that the two groups are indeed isomorphic.

Keywords: split extension, maximal subgroup, inertia factor groups, orthogonal group
2010 Mathematics Subject Classification: 20D06, 20C15, 20E22.

1. Introduction

The group $\overline{G} = 2^6{:}A_8$ is an inertia factor group of $2^8{:}O_8^+(2)$. This group is also a maximal subgroup of $O_8^+(2)$ of index 135 and order 1290240. As an inertia factor it plays an essential role in the construction of the character table of $2^8{:}O_8^+(2)$ as there is a block of irreducible characters in this table that corresponds to $\overline{G}$. In the construction of $\overline{G}$, A_8 acts on the elementary abelian group 2^6. The action on 2^6 is multiplication on the right of the six dimensional row vector space $N = 2^6$. This requires A_8 to be represented by 6×6 matrices. It then becomes necessary to reconstruct A_8 from a 8×8 representation to a 6×6 representation. In this paper we look at two ways to do this.

[1]The first author was supported by research grants from NRF (SA). The second author was supported by a postgraduate bursary from the NRF (SA). Supports from NWU (MaSIM, Mafikeng) are also acknowledged.

Although it is much simpler and natural to consider the embedding of $2^6{:}A_8$ into $O_8^+(2)$ (see Section 3), but it is interesting to construct this group combinatorially and this is our main reason for discussing the first method. In our first method (Section 2), we first take an 8-dimensional module V on which S_8 acts naturally by permuting its basis elements. We then obtain two submodules of V, namely M_1 and M_2 of dimensions 1 and 7 respectively. Let $W = M_2/M_1$, then $dim(W) = 6$ and W is a G-invariant where $G = S_6$ or A_8 (see Theorem 2.2). Let α and β be two permutation cycles of orders 7 and 3 respectively where $A_8 =< \alpha, \beta >$. Then, by the action of α and β on the generators of W, we get a matrix representation of both α and β. These are 6×6 matrix representations. We are then able to represent A_8 by 6×6 matrices. We show that (in Section 2) A_8 acts irreducibly on W and this action produces three orbits of lengths $1, 28$ and 35 respectively. These have corresponding point stabilizers which we obtain from the ATLAS [4] . We are then able to construct $\overline{G}$.

For the second method (Section 3) we use GAP [6]. We first construct $O_8^+(2)$ from the general orthogonal group $GO_8^+(2)$. We then construct, $\overline{G} = 2^6{:}A_8$, inside $O_8^+(2)$. This has only one proper normal subgroup, namely $N \cong 2^6$, which we can always obtain from $\overline{G}$. We then obtain the 6 generators of N which are 8×8 matrices. From the generators of $\overline{G}$, we are able to get two, 8×8 matrix generators of A_8 namely, a and b each of order 4. We then let a and b act on the generators of N by conjugation. Since $N \trianglelefteq \overline{G}$ the result of these actions are elements of N. We get a 6×6 matrix representation of both. This leads us to a 6×6 representation of A_8. We then let this A_8, using GAP, to act on N and in Theorem 3.1 we show that N is irreducible under this action. The two groups constructed in Sections 2 and 3 are indeed isomorphic (see Corollary 3.2).

This work is taken from the dissertation of the second author [18], for more of his work, one can also go to [14], [15], [16]. For further references one can also read [1], [8], [9], [12], [17], [19]. For the general theory of ordinary characters of finite groups, readers are referred to [7].

Our notation will be consistent with that used in the ATLAS [4].

2. The Combinatorics Method

The combinatorics method can also be found in [1] and [17] and is used extensively in [13] and [20]. The group S_8 acts naturally on a module of dimension 8 by permuting the basis elements which generate the module. Let V be the 8-dimensional natural module of A_8 over $GF(2)$, where $V = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8 \rangle$, and $e_i^2 = 1$ for $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. We regard V as a multiplicative elementary abelian 2-group of order 2^8.

Theorem 2.1 *Let V be the natural module of S_8 over $GF(2)$. Then there exist S_8 submodules M_1 and M_2 of V such that $V \supset M_2 \supset M_1 \supset 0$ and that*

$$dim(M_2) = 7 \quad and \quad dim(M_1) = 1.$$

Proof. Let $V = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\rangle$, and $e_i^2 = 1$ for $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$. Then S_8 acts naturally on V and this natural action results in the following orbits:

1. $O_0 = \{1_V\}$ and $|O_0| = 1$.
2. $O_1 = \{e_i | 1 \leq i \leq 8\}$ and $|O_1| = 8$.
3. $O_2 = \{e_ie_j | 1 \leq i, j \leq 8, i \neq j\}$ and $|O_2| = \begin{pmatrix} 8 \\ 2 \end{pmatrix} = 28$.
4. $O_3 = \{e_ie_je_k | 1 \leq i, j, k \leq 8,$ distinct $i, j, k\}$ and $|O_3| = \begin{pmatrix} 8 \\ 3 \end{pmatrix} = 56$.
5. $O_4 = \{e_ie_je_ke_l | 1 \leq i, j, k, l \leq 8,$ distinct $i, j, k, l\}$ and $|O_4| = \begin{pmatrix} 8 \\ 4 \end{pmatrix} = 70$.
6. $O_5 = \{e_ie_je_ke_le_m | 1 \leq i, j, k, l, m \leq 8,$ distinct $i, j, k, l, m\}$ and $|O_5| = \begin{pmatrix} 8 \\ 5 \end{pmatrix} = 56$.
7. $O_6 = \{e_ie_je_ke_le_me_n | 1 \leq i, j, k, l, m, n \leq 8,$ distinct $i, j, k, l, m, n\}$ and $|O_6| = \begin{pmatrix} 8 \\ 6 \end{pmatrix} = 28$.
8. $O_7 = \{e_ie_je_ke_le_me_ne_o | 1 \leq i, j, k, l, m, n, o \leq 8,$ distinct $i, j, k, l, m, n, o\}$ and $|O_7| = \begin{pmatrix} 8 \\ 7 \end{pmatrix} = 8$.
9. $O_8 = \{e_ie_je_ke_le_me_ne_oe_p | 1 \leq i, j, k, l, m, n, o, p \leq 8,$ distinct $i, j, k, l, m, n, o, p\}$ and $|O_8| = \begin{pmatrix} 8 \\ 8 \end{pmatrix} = 1$.

Thus S_8 produces 9 orbits on V.

Set $M_1 = \langle e_1e_2e_3e_4e_5e_6e_7e_8\rangle$. Then M_1 is an S_8-invariant submodule of V with $dim(M_1) = 1$.

Now, set $M_2 = O_0 \cup O_2 \cup O_4 \cup O_6 \cup O_8$. Then $|M_2| = 128$, so we have $dim(M_2) = 7$.

Since $M_1 = O_0 \cup O_8$, we obtain that $V \supset M_2 \supset M_1 \supset 0$. This implies that M_2 is a reducible S_8-invariant submodule of V. ∎

Since S_8 is 8-transitive, A_8 is 6-transitive on $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$. It is clear that $O_0, O_1, O_2, O_4, O_5, O_6$ are also orbits under the action of A_8. Now since A_8 does not have a proper subgroup of index less than 8, O_7 remains as an orbit of length 8. Obviously O_8 also remains as an orbit of length 1.

Theorem 2.2 *Let $W = M_2/M_1$, then $dim(W) = 6$. Also W is a G-invariant module where $G = S_8$ or A_8.*

Proof. It is clear that $dim(W) = 6$, since $dim(M_1) = 1$ and $dim(M_2) = 7$. If $g \in G$ and $\alpha \in M_2$, then since M_2 is G invariant, $g(\alpha M_1) = g(\alpha)M_1 \in M_2/M_1$ $\forall\ g \in G$ and $\alpha \in M_2$. So W is S_8 (A_8) invariant. ∎

Let

$$W = \langle e_1e_2M_1, e_1e_3M_1, e_1e_4M_1, e_1e_5M_1, e_1e_6M_1, e_1e_7M_1\rangle .$$

The set $B = \{e_1e_2, e_1e_3, e_1e_4, e_1e_5, e_1e_6, e_1e_7\}$ is a linearly independent set.

Let

$$\begin{aligned} \gamma_1 &= e_1e_2M_1, & \gamma_2 &= e_1e_3M_1, \\ \gamma_3 &= e_1e_4M_1, & \gamma_4 &= e_1e_5M_1, \\ \gamma_5 &= e_1e_6M_1, & \gamma_6 &= e_1e_7M_1. \end{aligned}$$

Also, if $\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7)$ and $\beta = (6\ 7\ 8)$, then

$$A_8 = \langle \alpha, \beta\rangle .$$

We obtain

$$\alpha : \gamma_1 \to \gamma_1\gamma_2,\ \gamma_2 \to \gamma_1\gamma_3,\ \gamma_3 \to \gamma_1\gamma_4,\ \gamma_4 \to \gamma_1\gamma_5,\ \gamma_5 \to \gamma_1\gamma_6,\ \text{and} \quad \gamma_6 \to \gamma_1.$$

We give two examples for the action of α. Under the action of α we have

$$\gamma_2 = e_1e_3M_1 \to e_2e_4M_1 = e_1e_2e_1e_4M_1 = \gamma_1\gamma_3$$

That is $\alpha(\gamma_2) = \gamma_1\gamma_3$. Also

$$\gamma_6 = e_1e_7M_1 \to e_2e_1M_1 = \gamma_1.$$

That is $\alpha(\gamma_6) = \gamma_1$. Hence α can be represented by the following matrix

$$\alpha = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with $o(\alpha) = 7$.

Similarly, for β we have

$$\beta : \gamma_1 \to \gamma_1,\ \gamma_2 \to \gamma_2,\ \gamma_3 \to \gamma_3,\ \gamma_4 \to \gamma_4,\ \gamma_5 \to \gamma_6,\ \gamma_6 \to \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6$$

As an example, we see that

$$\gamma_6 = e_1e_7M_1 \to e_1e_8M_1 = e_2e_3e_4e_5e_6e_7M_1 = \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6.$$

That is $\beta(\gamma_6) = \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6$. Here we obtain

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

with $o(\beta) = 3$. We are now able to write all the elements of A_8 as 6×6 matrices.

By acting A_8 directly on W, using the orbits of A_8 on M_2 and the fact that $M_1 = \{1, e_1e_2e_3e_4e_5e_6e_7e_8\}$, we can see that A_8 has 3 orbits namely

$$\begin{aligned} \Delta_0 &= \{O_0M_1\} = \{O_8M_1\} = \{M_1\}, \\ \Delta_1 &= \{O_2M_1\} = \{O_6M_1\} = \{e_ie_jM_1 \mid \text{distinct } \ e_i, e_j\}, \\ \Delta_2 &= \{O_4M_1\} = \{e_ie_je_ke_lM_1 | \text{distinct } \ e_i, e_j, e_k, e_l\}. \end{aligned}$$

Clearly, $|\Delta_0| = 1,\ |\Delta_1| = 28,\ \ |\Delta_2| = \frac{70}{2} = 35$ and $W = \Delta_0 \cup \Delta_1 \cup \Delta_2$.

Theorem 2.3 *A_8 acts irreducibly on W.*

Proof. Let $U \leq W$ be such that $U \neq 0$ and U is A_8 invariant. Since $U \neq 0$ $\exists\ x \in U$ such that $x \neq 0$. Since $x \neq 0$ and $U \leq W$ we have two cases.

Case 1. Suppose $x \in \Delta_1$, $x = e_ie_jM_1$ for distinct i, j. Hence $g(x) \in U \ \ \forall \ \ g \in A_8$. However

$$\{g(x)|g \in A_8\} = \Delta_1 \quad \Rightarrow \Delta_1 \subseteq U \quad \Rightarrow e_ie_jM_1 \in U \ \ \forall\ i, j.$$

Hence we have $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6 \in U$.

Case 2. Suppose $x \in \Delta_2$, then $x = e_ie_je_ke_lM_1$ for some distinct e_i, e_j, e_k, e_l. Hence $g(x) \in U \quad \forall \ \ g \in A_8$. Now

$$\{g(x)|g \in A_8\} = \Delta_2 \quad \Rightarrow \Delta_2 \subseteq U.$$

Since $\Delta_2 \subseteq U$, $e_ke_le_me_rM_1$ and $e_ke_le_me_iM_1$ are in U for distinct k, l, m, r, i. Since U is closed we get

$$(e_ke_le_me_rM_1)(e_ke_le_me_iM_1) = e_ie_r \in U \ \ \forall \ \ \text{distinct} \ \ \ k, l, m, r, i.$$

This shows that $U \subseteq W$. So similar to case 1, we have $U = W$.

Hence W is a unique 6-dimensional $GF(2)$ module that A_8 acts irreducibly on. ■

Our group $\bar{G}$ can now be realized by the split extension $W{:}A_8$, where $W \cong 2^6$. In here the multiplication is defined by

$$(w_1, g_1).(w_2, g_2) = (w_1(w_2)^{g_1}, g_1g_2)$$

for $w_1, w_2 \in W$ and $g_1, g_2 \in A_8$.

Remark 2.4 In V, the set of vectors of even weight, that is our M_2, forms a 7-dimensional S_8-submodule (this is true in general for S_n, these vectors form an $(n-1)$-dimensional S_n-submodule). Our module M_2 under the standard dot product on V becomes a symplectic space.

We can define an invariant quadratic form Q associated with this symplectic form as follows:

$$Q(\alpha) = \begin{cases} 0, & \alpha \in O_0 \cup O_4 \cup O_8 \\ \\ 1, & \alpha \in O_2 \cup O_6 \ . \end{cases}$$

The quotient space $W = M_2/M_1$ still admits an invariant quadratic form $\overline{Q}$ given by

$$\overline{Q}(\alpha) = \begin{cases} 0, & \alpha \in \Delta_0 \cup \Delta_2 \\ \\ 1, & \alpha \in \Delta_1 \ , \end{cases}$$

where $W = \Delta_0 \cup \Delta_1 \cup \Delta_2$. Hence $S_8 \leq GO_6^{\epsilon}(2)$ and since $|GO_6^-(2)| = 2^6 \times 3^4 \times 5$, we can easily deduce that $S_8 = GO_6^+(2)$ since these two groups have same order. Therefore

$$A_8 = O_6^+(2) \cong GL(4, 2).$$

So our group $\overline{G} = 2^6{:}A_8$ can be identified as the extension of $O_6^+(2)$ by its natural module.

Since our group $\overline{G} = 2^6{:}A_8$ is an inertia factor group of $2^8{:}O_8^+(2)$, in the next section we aim to construct $\overline{G}$ as a subgroup of $O_8^+(2)$.

3. The Computational Method

In this section for all our computations we use GAP. We first construct $O_8^+(2)$ inside the general orthogonal group $GO_8^+(2)$. This we do by getting the maximal normal subgroup of $GO_8^+(2)$ and this is a group of 8×8 matrices of size 174182400 over $GF(2)$. We then construct $\overline{G} = 2^6{:}A_8$ inside $O_8^+(2)$ by first constructing an 8-dimensional row vector space U, over $GF(2)$. We then let $O_8^+(2)$ to act on U and we get three orbits of lengths $1, 120$ (non-isotropic points) and 135 (isotropic points). Using the ATLAS [4] and Programme C [18], the maximal subgroup of index 135 is $2^6{:}A_8$ which corresponds to the third orbit. We then get the stabilizer of a representative of this orbit in $O_8^+(2)$, which gives us a group of 8×8 matrices of size 1290240 which is our $2^6{:}A_8$. We are now ready to construct 2^6 and A_8 inside our $\overline{G}$. We use GAP for our computations.

The group $N = 2^6$ is the only proper normal subgroup of $\overline{G}$ and we use GAP to obtain this normal subgroup. We then obtain its generators, which are given below.

$$\gamma_1 = \begin{bmatrix} 1&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 1&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&1\\ 0&0&0&0&0&1&0&1\\ 1&1&0&0&1&1&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix},$$

$$\gamma_3 = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&1\\ 1&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}, \quad \gamma_4 = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&1\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&1\\ 0&0&0&0&0&1&0&0\\ 1&0&0&1&0&1&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix},$$

$$\gamma_5 = \begin{bmatrix} 1&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&1\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&1\\ 0&0&0&0&0&1&0&0\\ 0&1&0&1&0&1&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}, \quad \gamma_6 = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&1\\ 0&0&0&1&0&0&0&1\\ 0&0&0&0&1&0&0&1\\ 0&0&0&0&0&1&0&1\\ 0&0&1&1&1&1&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}.$$

We then turn our attention to A_8. We first obtain the generators of $\overline{G}$. We use GAP [6] to get two generators, a and b, of A_8 . We give a, b and their inverses below. Note that $o(a) = o(b) = 4$.

$$a = \begin{bmatrix} 1&0&0&0&0&1&0&0\\ 0&1&0&1&0&1&0&1\\ 1&0&1&0&0&1&0&0\\ 0&0&0&1&0&0&0&0\\ 1&1&0&0&1&1&0&0\\ 0&0&0&0&0&1&0&0\\ 1&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}, \quad b = \begin{bmatrix} 0&0&0&0&0&1&0&0\\ 0&0&1&1&1&1&1&0\\ 1&0&0&1&0&0&0&1\\ 1&0&1&0&0&0&0&1\\ 1&1&0&0&0&1&1&1\\ 1&1&1&1&0&1&0&1\\ 0&1&0&0&0&0&1&0\\ 1&1&1&1&0&0&0&0 \end{bmatrix}$$

$$a^{-1} = \begin{bmatrix} 1&0&0&0&0&1&0&0\\ 0&1&0&1&0&1&0&1\\ 1&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&1&0&1&1&1&0&1\\ 0&0&0&0&0&1&0&0\\ 1&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}, \quad b^{-1} = \begin{bmatrix} 0&0&0&0&1&1&1&1\\ 0&0&1&1&1&1&1&0\\ 1&0&0&1&1&0&1&0\\ 1&0&1&0&1&0&1&0\\ 1&1&0&0&1&1&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&1&1&1&1&0&0\\ 1&0&0&0&0&1&0&1 \end{bmatrix}.$$

Noting that the generators of 2^6 and A_8 are both 8×8 matrices, thus we have $\bar{G} = \langle \gamma_i, a, b : 1 \leq i \leq 6 \rangle$, as a maximal subgroup of $O_8^+(2)$.

Computing the conjugate of each γ_i with respect to a, that is $a\gamma_i a^{-1}$ and noting that 2^6 is normal in $2^6{:}A_8$, we get that

$$a\gamma_i a^{-1} = \gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_k},$$

where $\gamma_{j_r} = \gamma_j$ or 1 for some $j_r = 1, ..., 6$. We denote this as

$$\gamma_i \longrightarrow \gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_k}.$$

We then get

$$\gamma_1 \longrightarrow \gamma_1,\ \gamma_2 \longrightarrow \gamma_2\gamma_5,\ \gamma_3 \longrightarrow \gamma_2\gamma_4\gamma_5\gamma_6,\ \gamma_4 \longrightarrow \gamma_4,\ \gamma_5 \longrightarrow \gamma_1\gamma_4\gamma_5,\ \gamma_6 \longrightarrow \gamma_1\gamma_2\gamma_3\gamma_4.$$

Similarly with b we get

$$\gamma_1 \longrightarrow \gamma_1\gamma_2\gamma_3\gamma_4\gamma_6,\ \gamma_2 \longrightarrow \gamma_2\gamma_3\gamma_4\gamma_5\gamma_6,\ \gamma_3 \longrightarrow \gamma_1\gamma_2\gamma_3\gamma_5\gamma_6,\ \gamma_4 \longrightarrow \gamma_1\gamma_3\gamma_4,$$

$$\gamma_5 \longrightarrow \gamma_4,\ \gamma_6 \longrightarrow \gamma_2\gamma_3\gamma_4.$$

Representing this information in matrix form, where the i-th row will correspond to the i-th conjugate, we get a 6×6 matrix representation of $G = A_8$. Hence we have $A_8 = \langle a', b' \rangle$, where

$$a' = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&1&0\\ 0&1&0&1&1&1\\ 0&0&0&1&0&0\\ 1&0&0&1&1&0\\ 1&1&1&1&0&0 \end{bmatrix}, \quad b' = \begin{bmatrix} 1&1&1&1&0&1\\ 0&1&1&1&1&1\\ 1&1&1&0&1&1\\ 1&0&1&1&0&0\\ 0&0&0&1&0&0\\ 0&1&1&1&0&0 \end{bmatrix}.$$

By methods of coset analysis which can also be found in [10], [11], when $G = A_8$ acts on N we obtain three orbits of lengths $1, 28$ and 35 respectively. These have corresponding point stabilizers K_1, K_2 and K_3 of indices $1, 28$ and 35 respectively. One can immediately see that $K_1 = G$ and K_2, K_3 must each sit in a maximal subgroup of G. However any maximal subgroup of G which contains

K_i must have an order divisible by $|K_i|$ and its index in G must divide 28 and 35 respectively. From the ATLAS [4], we get that up to isomorphism and conjugacy there is only one maximal subgroup of G, in each case, that would contain K_2 and the other K_3 and these are the symmetric group S_6 and the group $2^4{:}(S_3 \times S_3)$ respectively. However, since $|K_2| = |S_6|$ we have $K_2 \cong S_6$. Similarly we have $K_3 \cong 2^4{:}(S_3 \times S_3)$. For each $g \in G$, the number of fixed points $g \in G$ in N is equal to $k = |C_N(g)|$. Since the zero vector of N is fixed by every $g \in G$ we have

$$k = 1 + \chi(G|K_2)(g) + \chi(G|K_3)(g) = 1 + (\chi(G|K_2) + \chi(G|K_3))(g).$$

From this, we determine $\chi = \chi(A_8|2^6)$, the permutation character of A_8 on 2^6. We have

$$\chi = 1a + I_{S_6}^{A_8} + I_{2^4:(S_3\times S_3)}^{A_8} = 3 \times 1a + 7a + 14a + 2 \times 20a,$$

where $I_{S_6}^{A_8} = 1a + 7a + 20a$ and $I_{2^4:(S_3\times S_3)}^{A_8} = 1a + 14a + 20a$ are the characters of A_8 induced from the identity characters of S_6 and $2^4{:}(S_3 \times S_3)$ respectively. Since $C_N(g) \leq N$, we must have $k = 2^n$ where $n \in \{1, 2, 3, 4, 5, 6\}$. Hence we obtain the values of the $k's$ in Table 1.

Table 1

$[g]_{A_8}$	1a	2a	2b	3a	3b	4a	4b	5a	6a	6b	7a	7b	15a	15b
$\chi(A_8\|K_2)$	28	4	8	10	1	0	2	3	1	2	0	0	0	0
$\chi(A_8\|K_3)$	35	11	7	5	2	3	1	0	2	1	0	0	0	0
k	64	16	16	16	4	4	4	4	4	4	1	1	1	1

Theorem 3.1 *A_8 acts irreducibly on N.*

Proof. As we seen above, the action of A_8 on N produces three orbits of lengths 1, 28 and 35. If H is an A_8-invariant subgroup of N, then H is a union of these orbits and hence $|H| = 1+28a+35b$, where $a, b \in \{0, 1\}$. Now since $H = 2^i$, where $0 \leq i \leq 6$, we must have $a = b = 0$ or $a = b = 1$. This implies that $H = \{1_N\}$ or $H = N$. ∎

Corollary 3.2 *The two groups constructed in Sections 2 and 3 are isomorphic.*

Proof. Since by [2] A_8 has a unique (up to isomorphism) modular representation of degree 2 over $GF(2)$, proof follows from Theorems 2.3 and 3.1. ∎

Since the two $2^6{:}A_8$ constructed are isomorphic, from now on, we use one of them in the rest of our discussion. So we consider $\bar{G} = N{:}G$, where $N \cong 2^6$ and $G \cong A_8$ as in this section (Section 3).

4. Actions of A_8 on N and $Irr(N)$

We determined the action of A_8 on N in Section 3. We notice that A_8 produces three orbits of lengths 1, 28 and 35 on N. Hence it produces three orbits on $Irr(N)$. Here we use Programme C [18] to act G on $Irr(N)$. To be able to do this we need to rewrite N as a row vector space V of dimension 6 over $GF(2)$, that is V:= FullRowSpace(GF(2),6). We have two procedures at our disposal. First we can act G on V from right and this action gives us the orbits of G acting as a permutation group on the conjugacy classes of N. Secondly, we act G^T, that is the set consists of transpose of elements of G, on V from right. This action is equivalent to multiplying the column vectors of V on the left by G. This action gives the orbits of G acting as a permutation group on the irreducible characters of N.

From the above, the action of G on $Irr(N)$ produces three orbits of lengths $1, 28$ and 35 respectively. We then take representatives of the orbits of lengths 28 and 35. For each of the orbit representative we find its stabilizer in G. For the representative of the orbit of length 28, the corresponding stabilizer, that is H_2, is a group of 6×6 matrices of size 720 isomorphic to S_6. For the orbit of length 35 the corresponding stabilizer, that is H_3, is a group of 6×6 matrices of size 576 isomorphic to $2^4 : (S_3 \times S_3)$.

Remark 4.1 Since $A_8 = O_6^+(2)$ acts naturally on $N = 2^6$, the three orbits mentioned above are the zero vector, non-isotropic points and isotropic points respectively.

5. The Conjugacy Classes of $2^6{:}A_8$

We first give the representatives of the conjugacy classes of A_8, in terms of 6×6 matrices in Table 2.

From the methods of coset analysis, which can also be found in [1],[10], [8], [11], [12], [17], [19] and by Programmes A and B [18], we are able to compute the conjugacy classes of $2^6{:}A_8$ which are given in Table 3. We give a very brief summary of coset analysis. We look at the action of $\overline{G}$ on $N\overline{g}$, for the split extension it is suffices to look at the coset $Ng, g \in G$. First N acts on Ng and we get k orbits. Then we act $C_G(g)$ on these orbits and f_j of these orbits, fuse to form one orbit with $\sum f_j = k$, and d_j a representative of these fused orbits. For this we use Programme A [18].

Table 2. Conjugacy classes of $H_1 = G = A_8$

$[g]_G$	6×6 matrix	$\lvert[g]_G\rvert$	$[g]_G$	6×6 matrix	$\lvert[g]_G\rvert$
1A	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	1	2A	$\begin{pmatrix} 0&0&0&1&0&0\\ 1&0&1&0&0&1\\ 0&1&0&1&0&1\\ 1&0&0&0&0&0\\ 0&1&1&1&1&1\\ 0&0&0&0&0&1 \end{pmatrix}$	105
2B	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&1\\ 1&0&1&0&1&0 \end{pmatrix}$	210	3A	$\begin{pmatrix} 0&1&1&1&1&0\\ 0&1&0&0&0&0\\ 1&1&0&0&0&0\\ 1&1&1&1&0&0\\ 0&0&0&1&0&0\\ 0&0&0&1&1&1 \end{pmatrix}$	112
3B	$\begin{pmatrix} 1&1&1&0&0&0\\ 1&0&1&1&0&1\\ 1&0&1&0&0&1\\ 0&1&1&1&0&0\\ 0&1&0&0&0&0\\ 0&0&1&1&1&1 \end{pmatrix}$	1 120	4B	$\begin{pmatrix} 0&0&1&1&1&1\\ 0&0&1&1&0&0\\ 0&0&0&0&1&0\\ 1&0&1&0&1&1\\ 0&1&0&1&0&1\\ 1&1&1&0&0&0 \end{pmatrix}$	1 260
4A	$\begin{pmatrix} 0&0&1&1&1&0\\ 0&0&1&1&0&0\\ 0&0&0&1&0&0\\ 1&0&1&1&0&1\\ 0&1&0&1&0&1\\ 1&1&1&1&1&1 \end{pmatrix}$	2 520	5a	$\begin{pmatrix} 0&1&0&1&0&1\\ 1&0&1&0&0&1\\ 0&0&0&1&0&0\\ 0&0&0&1&1&1\\ 0&0&0&0&0&1\\ 0&1&1&1&1&0 \end{pmatrix}$	1 344
6A	$\begin{pmatrix} 0&0&0&1&0&0\\ 1&0&1&0&0&1\\ 1&0&1&0&1&1\\ 0&1&1&1&1&0\\ 1&0&0&0&0&0\\ 1&1&1&1&1&1 \end{pmatrix}$	1 680	6B	$\begin{pmatrix} 0&0&1&1&1&1\\ 0&0&1&0&0&0\\ 1&0&1&1&0&1\\ 1&0&1&0&1&1\\ 0&1&1&1&1&0\\ 1&0&0&0&0&0 \end{pmatrix}$	3 360
7A	$\begin{pmatrix} 0&0&0&0&1&0\\ 0&0&1&1&0&0\\ 1&0&0&1&0&1\\ 1&0&1&1&0&1\\ 0&1&1&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}$	2 880	7B	$\begin{pmatrix} 1&0&1&0&0&1\\ 1&0&0&1&0&1\\ 1&1&0&0&0&0\\ 1&1&1&1&1&1\\ 0&0&0&1&1&1\\ 0&1&1&0&0&0 \end{pmatrix}$	2 880
15A	$\begin{pmatrix} 1&0&1&0&1&1\\ 1&0&1&0&0&1\\ 1&1&1&1&0&0\\ 1&1&1&1&1&1\\ 0&0&0&1&1&1\\ 0&1&1&0&0&0 \end{pmatrix}$	1 344	15B	$\begin{pmatrix} 0&0&1&1&1&0\\ 0&0&1&1&1&1\\ 1&1&1&1&1&1\\ 1&0&0&0&0&0\\ 0&1&1&0&0&0\\ 1&0&1&1&0&1 \end{pmatrix}$	1 344

Table 3. Conjugacy classes of $\bar{G} = 2^6{:}A_8$

$g \in A_8$	k	f_j	d_j	w	$[x]_{2^6:A_8}$	$\lvert C_{2^6:A_8}(x)\rvert$
$1A$	2^6	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$1A$	1 290 240
		28	$(0,0,0,0,0,1)$	$(0,0,0,0,0,1)$	$2A$	46 080
		35	$(0,0,0,1,0,1)$	$(0,0,0,1,0,1)$	$2B$	36 864
$2A$	2^4	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$2C$	3 072
		1	$(0,0,1,0,0,1)$	$(0,0,0,0,0,0)$	2D	3 072
		1	$(0,0,1,1,1,1)$	$(0,0,0,0,0,0)$	2E	3 072
		1	$(0,0,1,1,1,1)$	$(0,0,0,0,0,0)$	2F	3 072
		12	$(0,0,0,0,0,1)$	$(1,1,1,0,0,1)$	4A	256
$2B$	2^4	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$2G$	1 536
		1	$(0,0,1,1,1,1)$	$(0,0,1,0,0,0)$	4B	1 536
		3	$(1,0,1,0,1,0)$	$(0,0,0,0,0,0)$	2H	512
		3	$(0,0,1,1,0,1)$	$(0,0,1,0,1,0)$	4C	512
		8	$(0,0,0,0,0,1)$	$(0,0,0,1,1,0)$	4D	192
$3A$	2^4	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$3A$	2 880
		5	$(1,1,0,0,1,0)$	$(0,0,1,1,1,1)$	6A	576
		10	$(0,1,0,0,1,1)$	$(0,0,1,1,0,0)$	6B	288
$3B$	2^2	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$3B$	72
		1	$(1,1,0,1,0,0)$	$(1,0,0,0,0,1)$	$6B$	72
		1	$(1,0,1,0,1,0)$	$(1,0,1,0,1,0)$	$6C$	72
		1	$(0,0,0,1,0,0)$	$(1,1,1,1,1,1)$	$6D$	72
$4A$	2^2	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$4E$	64
		1	$(1,1,0,1,0,0)$	$(0,0,0,0,0,0)$	$4F$	64
		1	$(1,0,1,0,1,0)$	$(0,0,0,0,0,0)$	$4G$	64
		1	$(0,0,0,1,0,0)$	$(0,0,0,0,0,0)$	$4H$	64
$4B$	2^2	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$4I$	32
		1	$(1,1,0,1,0,0)$	$(0,0,1,1,1,1)$	$8A$	32
		1	$(1,0,1,0,1,0)$	$(0,0,0,0,0,0)$	$4J$	32
		1	$(0,0,0,1,0,0)$	$(1,1,1,0,1,0)$	$8B$	32
$5A$	2^2	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$5A$	60
		3	$(0,1,0,1,0,0)$	$(0,0,0,1,0,1)$	$10A$	20
$6A$	2^2	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6F$	48
		1	$(1,0,1,0,1,0)$	$(0,1,1,0,1,0)$	$12A$	48
		2	$(1,0,1,0,1,0)$	$(1,0,1,0,1,0)$	$12B$	24
$6B$	2^2	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6G$	24
		1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,1)$	$6H$	24
		1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6I$	24
		1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$6J$	24
$7A$	1	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$7A$	7
7B	1	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$7B$	7
15A	1	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$15A$	15
15B	1	1	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$15B$	15

Let $o(dg) = k$ and $o(g) = m$. If $w = (dg)^m$, then if $w = (0,0,0,0,0,0)$, $k = m$. On the other hand if $w \neq (0,0,0,0,0,0)$, then $k = 2m$. To get w we use Programme B [18]. We obtain that $2^6{:}A_8$ has altogether 41 conjugacy classes which are given in Table 3 above.

Remark 5.1 The group $O_8^+(2)$ has three conjugacy classes of maximal subgroups isomorphic to $2^6{:}A_8$. Our group $\overline{G}$, which is the stabilizer of a non-isotropic point in $O_8^+(2)$, is first of these groups as listed in the ATLAS. Table 3 shows that $N = [1A] \cup [2A]_{\overline{G}} \cup [2B]_{\overline{G}}$, with $|[2A]_{\overline{G}}| = 28$ and $|[2B]_{\overline{G}}| = 35$. The fusion of $\overline{G}$ into $O_8^+(2)$ implies $[2A]_{\overline{G}} \to [2B]_{O_8^+(2)}$ and $[2B]_{\overline{G}} \to [2A]_{O_8^+(2)}$. This confirms that $\overline{G} = N_{O_8^+(2)}(2A_{35}B_{28})$ as it is stated in the ATLAS.

Acknowledgements. The second author would like to thank the University of North West for giving him time to work with Prof. J. Moori and Dr. Z.E. Mpono who introduced him to Fischer-Clifford theory. Special mention is also given about Dr F. Ali and Dr T. Breuer who were most helpful with GAP.

References

[1] Ali, F., *Fischer-Clifford Theory and Character Tables of Group Extensions*, PhD Thesis, University of Natal, 2001.

[2] sc Jansen, C., Lux, K., Parker, R., Wilson, R.A., *An atlas of Brauer characters*, London Mathematical Society Monographs. New Series, 11. Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1995.

[3] Darafsheh, M.R., Iranmanesh, A., *Computation of the character table of affine groups using Fischer matrices*, London Mathematical Society Lecture Note Series, 211, vol. 1, C.M. Campbell et al., Cambridge University Press, 1995, 131-137.

[4] Conway, J.H., et al., *Atlas of Finite Groups*, Oxford University Press, Oxford, 1985.

[5] Fischer, B., *Clifford-Matrices*, Progress in Mathematics, 95, Michler, G.O. and Ringel, C.M. (eds.), Birkhauser, Basel, 1991, 1-16.

[6] The GAP Group, *GAP-Groups, Algorithms and Programming, Version 4.4*, Aachen, St Andrews, 2008, (`http://www-gap.dcs.st-and.ac.uk/~gap`).

[7] Isaacs, I.M., *Character Theory of Finite Groups*, Academic Press, San Diego, 1976.

[8] Ali, F., Moori, J., *Fischer-Clifford matrices of the non-split group extension* $2^6U_4(2)$, Quaest. Math., 31 (1) (2008), 27-36.

[9] Ali, F., Moori, J., *Fischer-Clifford matrices and character table of a maximal subgroup of Fi'_{24}*, Representation Theory, 7 (2003), 300-321.

[10] Moori, J., *On the Groups G^+ and G of the forms $2^{10}{:}M_{22}$ and $2^{10}{:}\overline{M}_{22}$*, PhD thesis, University of Birmingham, 1975.

[11] Moori, J., *On certain groups associated with the smallest Fischer group*, J. London Maths Soc., 2 (1981), 61-67.

[12] Moori, J., Mpono, Z.E., *The Fischer-Clfford matrices of the group $2^6{:}SP(6,2)$*, Quaest. Math., 22 (1999), 257-298.

[13] Moori, J., Zimba, K., *Permutation actions of the symmetric group S_n on the groups Z_m^n and $\overline{Z_m^n}$*, Quaest. Math., 28 (2) (2005), 179-193.

[14] Moori, J., Mpono, Z.E., Seretlo, T.T., *A group $2^7{:}S_8$ in $\overline{Fi}_{22}$*, South East Asian Bulletin of Mathematics, 37 (2013), 111-121.

[15] Moori, J., Seretlo, T.T., *On the Fischer-Clifford matrices of a maximal subgroup of the Lyons Group Ly*, Bulletin of the Iranian Maths Soc., to appear.

[16] Moori, J., Seretlo, T.T., *On two non-split extension groups associated with HS and HS:2*, Turkish Journal of Maths, to appear.

[17] Mpono, Z.E., *Fischer-Clifford Theory and Character Tables of Group Extensions*, PhD Thesis, University of Natal, 1998.

[18] Seretlo, T.T., *Fischer Clifford Matrices and Character Tables of Certain Groups Associated with Simple Groups $O_{10}^+(2)$, HS and Ly*, PhD thesis, University of KwaZulu-Natal, Pietermaritzburg, 2012.

[19] Whitley, N.S., *Fischer Matrices and Character Tables of Group Extensions*, MSc Thesis, University of Natal, 1994.

[20] Zimba, K., *Fischer-Clifford Matrices of the Generalized Symmetric Group and some Associated Groups*, PhD Thesis, University of KwaZulu Natal, 2005.

Accepted: 10.05.2014