STRONG CONVERGENCE THEOREMS FOR FIXED POINT PROBLEMS AND EQUILIBRIUM PROBLEMS WITH APPLICATIONS

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Abstract. In this paper, we present a new iterative algorithm for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem, and we prove the strong convergence theorems in Hilbert spaces. We also apply our results to the convex minimization and variational inequality problems. Our results extend and improve some recent results of Cai, Tang and Liu [Cai, Y., Tang, Y. and Liu, L.: *Iterative algorithms for minimum-norm fixed point of nonexpansive mapping in Hilbert space*. Fixed Point Theory Appl., 2012:49 (2012)] and others.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The set of fixed points of T is denoted by $\mathrm{Fix}(T)$. It is well known that $\mathrm{Fix}(T)$ is closed and convex. Halpern [1] introduced an iterative method for approximation of fixed points of a nonexpansive mapping as follows: $u \in C,\ x_1 \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n,$$

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where $\{\alpha_n\}$ is a sequence in $[0,1]$. Wittmann [2] proved that $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(T)}u$ if $\{\alpha_n\}$ satisfies the conditions:

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

where $P_{\mathrm{Fix}(T)}$ is the metric projection onto $\mathrm{Fix}(T)$. Especially, letting $0 = u \in C$, we see that the sequence $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(T)}0$, i.e., the minimum-norm fixed point of T. Cai, Tang and Liu [3] presented an iterative method for finding the minimum-norm fixed point of T.

Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for f is to find $\overline{x} \in C$ such that $f(\overline{x}, y) \geq 0$ for all $y \in C$. The set of such solutions is denoted by $EP(f)$. Numerous problems in physics, optimization and economics can be reduced to find a solution of the equilibrium problem (for instance, see [4]). For solving equilibrium problem, we assume that the bifunction f satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;

(A3) For every $x, y, z \in C$, $\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x,y)$;

(A4) $f(x,\cdot)$ is convex and lower semicontinuous for each $x \in C$.

Some methods have been proposed to solve the equilibrium problem (see [4]–[12], [15]).

The methods for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem have received much attention in recent years. Tada and Takahashi [5] introduced an iterative scheme as follows.

$$\begin{cases} x_0 = x \in H, \\ u_n \in C, \\ \text{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \text{ for all } y \in C, \\ \omega_n = (1-\alpha_n)x_n + \alpha_n T u_n, \\ C_n = \{z \in H : \|\omega_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n}(x_0), \end{cases}$$

where $\{\alpha_n\} \subset [a,1]$ for some $a \in (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. They proved that the sequence $\{x_n\}$ converges to $P_{\mathrm{Fix}(T)\cap EP(f)}x$. Takahashi and Takahashi [6] obtained the following iterative scheme

$$\begin{cases} x_1 \in H, \\ u_n \in C, \\ \text{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \text{ for all } y \in C, \\ x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n)Tu_n, \end{cases}$$

where g is a contraction on H. Under appropriate conditions they showed that the sequence $\{x_n\}$ converges strongly to an element of the set $\mathrm{Fix}(T) \cap EP(f)$.

Motivated by the above results, in this paper, we present a new iterative algorithm for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem, and prove the strong convergence theorems in Hilbert spaces. Finally, we give the applications to the convex minimization and variational inequality problems. Our results extend and improve some recent results of Cai, Tang and Liu [3] and others.

2. Preliminaries

Throughout this paper, let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let C be a nonempty closed convex subset of H. We write $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Similarly, the notation $x_n \rightharpoonup x$ means weak convergence. It is well known that H satisfies Opial's condition, that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \text{ for all } y \neq x. \tag{2.1}$$

For any $x \in H$, there exists a unique point $P_C x \in C$ such that

$$\|x - P_C x\| \leq \|x - y\|, \text{ for all } y \in C.$$

P_C is called the metric projection of H onto C. Note that P_C is a nonexpansive mapping of H onto C. For $x \in H$ and $z \in C$, we have

$$z = P_C x \Longleftrightarrow \langle z - y, x - z\rangle \geq 0 \text{ for every } y \in C. \tag{2.2}$$

We need the following lemmas.

Lemma 2.1. [4] *Let C be a nonempty closed convex subset of H, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1) − (A4). If $r > 0$ and $x \in H$, then there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \textit{ for all } y \in C.$$

Lemma 2.2. [11] *Let C be a nonempty closed convex subset of H, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1) – (A4)*. For $r > 0$, define a mapping $T_r : H \to 2^C$ as follows:*

$$T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C\}.$$

Then the following hold:

(i) *T_r is single-valued;*

(ii) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\langle x - y, T_r x - T_r y\rangle \geq \|T_r x - T_r y\|^2;$$

(iii) $\mathrm{Fix}(T_r) = EP(f)$*;*

(iv) *$EP(f)$ is closed and convex.*

Lemma 2.3. [12] *Suppose that (A1) – (A4) hold. If $x, y \in H$ and $r_1, r_2 > 0$, then*

$$\|T_{r_2} y - T_{r_1} x\| \leq \|y - x\| + \frac{|r_2 - r_1|}{r_2}\|T_{r_2} y - y\|.$$

Lemma 2.4. [13] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose $x_{n+1} = (1 - \beta_n)x_n + \beta_n y_n$ for all integers $n \geq 1$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

The following lemma is an immediate consequence of the inner product on H.

Lemma 2.5. *For all $x, y \in H$, the inequality $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ holds.*

Lemma 2.6. [14] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\beta_n$, where*

(i) $\{\alpha_n\} \subset (0, 1)$, $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Strong convergence theorems

In this section, we introduce a new iterative algorithm for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem, and we prove the strong convergence theorems in Hilbert spaces.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1) – (A4). *Suppose that $T : C \to C$ is a nonexpansive mapping such that* $\mathrm{Fix}(T) \cap EP(f) \neq \emptyset$. *For $\lambda \in (0,1)$ and $\omega \in C$, let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ u_n \in C, \\ \text{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \text{ for all } y \in C, \\ x_{n+1} = (1-\alpha_n)[\lambda T u_n + (1-\lambda)x_n] + \alpha_n \omega, \end{cases} \tag{3.1}$$

where the sequences $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy the following conditions:

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(2) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(T)\cap EP(f)}\omega$.

Proof. Note that $\mathrm{Fix}(T) \cap EP(f)$ is a closed convex subset of H since $\mathrm{Fix}(T)$ and $EP(f)$ are closed and convex. For simplicity, we write $\Omega := \mathrm{Fix}(T) \cap EP(f)$.

From Lemmas 2.1 and 2.2, we have $u_n = T_{r_n} x_n$, and for any $z \in \Omega$,

$$\|u_n - z\| = \|T_{r_n} x_n - T_{r_n} z\| \leq \|x_n - z\|. \tag{3.2}$$

It follows that

$$\begin{aligned} \|x_{n+1} - z\| &= \|(1-\alpha_n)[\lambda T u_n + (1-\lambda)x_n] + \alpha_n \omega - z\| \\ &\leq (1-\alpha_n)\|\lambda T u_n + (1-\lambda)x_n - z\| + \alpha_n\|\omega - z\| \\ &\leq (1-\alpha_n)[\lambda\|T u_n - z\| + (1-\lambda)\|x_n - z\|] + \alpha_n\|\omega - z\| \\ &\leq (1-\alpha_n)[\lambda\|u_n - z\| + (1-\lambda)\|x_n - z\|] + \alpha_n\|\omega - z\| \\ &\leq (1-\alpha_n)\|x_n - z\| + \alpha_n\|\omega - z\| \\ &\leq \max\{\|x_n - z\|, \|\omega - z\|\}. \end{aligned} \tag{3.3}$$

From a simple inductive process, we get

$$\|x_{n+1} - z\| \leq \max\{\|x_1 - z\|, \|\omega - z\|\},$$

which yields that $\{x_n\}$ is bounded. So is the sequence $\{u_n\}$.

Setting

$$y_n = \frac{(1-\alpha_n)\lambda T u_n + \alpha_n \omega}{\alpha_n + (1-\alpha_n)\lambda},$$

one has

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&\quad = \Big\| \frac{(1-\alpha_{n+1})\lambda T u_{n+1} + \alpha_{n+1}\omega}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} - \frac{(1-\alpha_n)\lambda T u_n + \alpha_n \omega}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} \\
&\quad + \frac{(1-\alpha_n)\lambda T u_n + \alpha_n \omega}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} - \frac{(1-\alpha_n)\lambda T u_n + \alpha_n \omega}{\alpha_n + (1-\alpha_n)\lambda} \Big\| \\
&\quad \le \Big\| \frac{(\alpha_{n+1} - \alpha_n)\omega + (1-\alpha_{n+1})\lambda T u_{n+1} - (1-\alpha_n)\lambda T u_n}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} \Big\| \\
&\quad + \Big| \frac{1}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} - \frac{1}{\alpha_n + (1-\alpha_n)\lambda} \Big| \cdot \big\| (1-\alpha_n)\lambda T u_n + \alpha_n \omega \big\| \\
&\quad \le \frac{1}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} \big[\big\| (\alpha_{n+1} - \alpha_n)\omega \big\| \\
&\qquad + \big\| (1-\alpha_{n+1})\lambda T u_{n+1} - (1-\alpha_n)\lambda T u_n \big\| \big] \\
&\quad + \Big| \frac{1}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} - \frac{1}{\alpha_n + (1-\alpha_n)\lambda} \Big| \cdot \big\| (1-\alpha_n)\lambda T u_n + \alpha_n \omega \big\| \\
&\quad \le \frac{1}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} \big[\big\| (\alpha_{n+1} - \alpha_n)\omega \big\| \\
&\qquad + \big\| (1-\alpha_{n+1})\lambda T u_{n+1} - (1-\alpha_{n+1})\lambda T u_n \big\| \\
&\qquad + \big\| (1-\alpha_{n+1})\lambda T u_n - (1-\alpha_n)\lambda T u_n \big\| \big] \\
&\qquad + \Big| \frac{1}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} - \frac{1}{\alpha_n + (1-\alpha_n)\lambda} \Big| \cdot \big\| (1-\alpha_n)\lambda T u_n + \alpha_n \omega \big\| \\
&\quad \le \frac{1}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} \big[\big\| (\alpha_{n+1} - \alpha_n)\omega \big\| + (1-\alpha_{n+1})\lambda \big\| u_{n+1} - u_n \big\| \\
&\qquad + \big| \alpha_{n+1} - \alpha_n \big| \lambda \big\| T u_n \big\| \big] \\
&\quad + \Big| \frac{1}{\alpha_{n+1} + (1-\alpha_{n+1})\lambda} - \frac{1}{\alpha_n + (1-\alpha_n)\lambda} \Big| \cdot \big\| (1-\alpha_n)\lambda T u_n + \alpha_n \omega \big\|.
\end{aligned}$$

Since $\liminf_{n\to\infty} r_n > 0$, without loss of generality, we may assume that there exists a real number c such that $r_n > c > 0$ for all $n \ge 1$. According to Lemma 2.3, it follows that

$$
\begin{aligned}
&\|y_{n+1}-y_n\| \\
&\quad\le \frac{1}{\alpha_{n+1}+(1-\alpha_{n+1})\lambda}\big[\big\|(\alpha_{n+1}-\alpha_n)\omega\big\| + [(1-\alpha_{n+1})\lambda+\alpha_{n+1}]\big\|u_{n+1}-u_n\big\| \\
&\quad+\big|\alpha_{n+1}-\alpha_n\big|\lambda\big\|Tu_n\big\|\big] \\
&\quad+\Big|\frac{1}{\alpha_{n+1}+(1-\alpha_{n+1})\lambda}-\frac{1}{\alpha_n+(1-\alpha_n)\lambda}\Big|\cdot\big\|(1-\alpha_n)\lambda Tu_n+\alpha_n\omega\big\| \\
&\quad\le \frac{1}{\alpha_{n+1}+(1-\alpha_{n+1})\lambda}\big[\big\|(\alpha_{n+1}-\alpha_n)\omega\big\| + [(1-\alpha_{n+1})\lambda+\alpha_{n+1}]\big(\big\|x_{n+1}-x_n\big\| \\
&\qquad+\frac{|r_{n+1}-r_n|}{c}\big\|u_{n+1}-x_{n+1}\big\|\big) + |\alpha_{n+1}-\alpha_n|\lambda\big\|Tu_n\big\|\big] \\
&\qquad+\Big|\frac{1}{\alpha_{n+1}+(1-\alpha_{n+1})\lambda}-\frac{1}{\alpha_n+(1-\alpha_n)\lambda}\Big|\cdot\big\|(1-\alpha_n)\lambda Tu_n+\alpha_n\omega\big\| \\
&\quad\le \big\|x_{n+1}-x_n\big\| + \frac{1}{\alpha_{n+1}+(1-\alpha_{n+1})\lambda}\big[\big\|(\alpha_{n+1}-\alpha_n)\omega\big\| \\
&\qquad+[(1-\alpha_{n+1})\lambda+\alpha_{n+1}]\frac{|r_{n+1}-r_n|}{c}\big\|u_{n+1}-x_{n+1}\big\| + |\alpha_{n+1}-\alpha_n|\lambda\big\|Tu_n\big\|\big] \\
&\qquad+\Big|\frac{1}{\alpha_{n+1}+(1-\alpha_{n+1})\lambda}-\frac{1}{\alpha_n+(1-\alpha_n)\lambda}\Big|\cdot\big\|(1-\alpha_n)\lambda Tu_n+\alpha_n\omega\big\|.
\end{aligned}
$$

Therefore, we get

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0.$$

It follows from Lemma 2.4 that $\lim_{n\to\infty}\|y_n-x_n\|=0$. Thus,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}[\alpha_n+(1-\alpha_n)\lambda]\|y_n-x_n\|=0. \tag{3.4}$$

For any $z\in\Omega$, we have

$$
\begin{aligned}
\|u_n-z\|^2 &= \|T_{r_n}x_n-T_{r_n}z\|^2 \\
&\le \langle x_n-z, u_n-z\rangle \\
&= \frac{1}{2}\left[\|x_n-z\|^2+\|u_n-z\|^2-\|x_n-u_n\|^2\right],
\end{aligned}
$$

which yields

$$\|u_n-z\|^2\le\|x_n-z\|^2-\|x_n-u_n\|^2. \tag{3.5}$$

By (3.1), one has

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|(1-\alpha_n)[\lambda Tu_n+(1-\lambda)x_n]+\alpha_n\omega-z\|^2\\
&= \|(1-\alpha_n)[\lambda Tu_n+(1-\lambda)x_n-z]+\alpha_n(\omega-z)\|^2\\
&\le (1-\alpha_n)\|\lambda Tu_n+(1-\lambda)x_n-z\|^2+\alpha_n\|\omega-z\|^2\\
&\le (1-\alpha_n)\|\lambda(Tu_n-z)+(1-\lambda)(x_n-z)\|^2+\alpha_n\|\omega-z\|^2\\
&\le (1-\alpha_n)[\lambda\|u_n-z\|^2+(1-\lambda)\|x_n-z\|^2]+\alpha_n\|\omega-z\|^2\\
&\le (1-\alpha_n)[\lambda\|x_n-z\|^2-\lambda\|x_n-u_n\|^2+(1-\lambda)\|x_n-z\|^2]\\
&\quad +\alpha_n\|\omega-z\|^2\\
&\le \|x_n-z\|^2-(1-\alpha_n)\lambda\|x_n-u_n\|^2+\alpha_n\|\omega-z\|^2,
\end{aligned}$$

which implies

$$\begin{aligned}
(1-\alpha_n)\lambda\|x_n-u_n\|^2 &\le \|x_n-z\|^2-\|x_{n+1}-z\|^2+\alpha_n\|\omega-z\|^2\\
&\le \|x_n-x_{n+1}\|(\|x_n-z\|+\|x_{n+1}-z\|)+\alpha_n\|\omega-z\|^2.
\end{aligned}$$

Consequently, equality (3.4) and $\lim_{n\to\infty}\alpha_n=0$ imply

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.6}$$

Observe that

$$\begin{aligned}
&\|x_n-Tx_n\|\\
&\le \|x_n-x_{n+1}\|+\|x_{n+1}-Tx_n\|\\
&\le \|x_n-x_{n+1}\|+\|(1-\alpha_n)[\lambda Tu_n+(1-\lambda)x_n]+\alpha_n\omega-Tx_n\|\\
&\le \|x_n-x_{n+1}\|+\|(1-\alpha_n)[\lambda Tu_n+(1-\lambda)x_n-Tx_n]+\alpha_n(\omega-Tx_n)\|\\
&\le \|x_n-x_{n+1}\|+\|(1-\alpha_n)[\lambda(Tu_n-Tx_n)+(1-\lambda)(x_n-Tx_n)]\\
&\quad +\alpha_n(\omega-Tx_n)\|\\
&\le \|x_n-x_{n+1}\|+(1-\alpha_n)\lambda\|u_n-x_n\|+(1-\alpha_n)(1-\lambda)\|x_n-Tx_n\|\\
&\quad +\alpha_n\|\omega-Tx_n\|,
\end{aligned}$$

which deduces

$$\begin{aligned}
&[1-(1-\alpha_n)(1-\lambda)]\|x_n-Tx_n\|\\
&\qquad \le \|x_n-x_{n+1}\|+(1-\alpha_n)\lambda\|u_n-x_n\|+\alpha_n\|\omega-Tx_n\|.
\end{aligned}$$

It follows from (3.4), (3.6) and $\lim_{n\to\infty}\alpha_n=0$ that

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{3.7}$$

We shall prove that

$$\limsup_{n\to\infty}\langle \omega - z_0, x_n - z_0\rangle \leq 0,$$

where $z_0 = P_\Omega \omega$. In order to show this inequality, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \omega - z_0, x_n - z_0\rangle = \lim_{i\to\infty}\langle \omega - z_0, x_{n_i} - z_0\rangle. \tag{3.8}$$

Based on the boundedness of $\{x_{n_i}\}$, without loss of generality, we assume that $x_{n_i} \rightharpoonup p$. Now we show that $p \in \Omega$. As $\{x_n\} \subset C$ and C is a closed convex set, one has $p \in C$. We firstly prove that $p \in EP(f)$. According to (3.1), we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \text{ for all } y \in C.$$

The monotonicity of f yields

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq f(y, u_n) \text{ for all } y \in C.$$

Replacing n by n_i, we obtain

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq f(y, u_{n_i}) \text{ for all } y \in C.$$

It follows from (3.6) and (A4) that

$$f(y, p) \leq 0 \text{ for all } y \in C.$$

For $0 < t \leq 1$, $y \in C$, set $y_t = ty + (1-t)p$. Then $y_t \in C$ and $f(y_t, p) \leq 0$. Hence

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, p) \leq tf(y_t, y).$$

Dividing by t, we see that

$$f(y_t, y) \geq 0.$$

Letting $t \downarrow 0$, we get

$$f(p, y) \geq 0 \text{ for all } y \in C.$$

That is, $p \in EP(f)$.

Now we prove that $p \in \mathrm{Fix}(T)$. Otherwise, assume that $p \notin \mathrm{Fix}(T)$, that is, $p \neq Tp$. The Opial's condition and (3.7) imply

$$\begin{aligned}
\liminf_{i\to\infty}\|x_{n_i} - p\| &< \liminf_{i\to\infty}\|x_{n_i} - Tp\| \\
&= \liminf_{i\to\infty}\|x_{n_i} - Tx_{n_i} + Tx_{n_i} - Tp\| \\
&= \liminf_{i\to\infty}\|Tx_{n_i} - Tp\| \\
&\leq \liminf_{i\to\infty}\|x_{n_i} - p\|.
\end{aligned}$$

This is a contradiction. Thus $p \in \mathrm{Fix}(T)$. By (3.8) and the property of metric projection, we have

$$\begin{aligned} \limsup_{n\to\infty}\langle \omega - z_0, x_n - z_0\rangle &= \lim_{i\to\infty}\langle \omega - z_0, x_{n_i} - z_0\rangle \\ &= \langle \omega - z_0, p - z_0\rangle \le 0. \end{aligned} \tag{3.9}$$

Using (3.1) again, we obtain

$$\begin{aligned} \|x_{n+1} - z_0\|^2 &= \|(1-\alpha_n)[\lambda T u_n + (1-\lambda)x_n - z_0] + \alpha_n(\omega - z_0)\|^2 \\ &\le (1-\alpha_n)\|\lambda(Tu_n - z_0) + (1-\lambda)(x_n - z_0)\|^2 \\ &\quad + 2\langle \alpha_n(\omega - z_0), x_{n+1} - z_0\rangle \\ &\le (1-\alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle \omega - z_0, x_{n+1} - z_0\rangle. \end{aligned}$$

It follows from (3.9) and Lemma 2.6 that $\{x_n\}$ converges strongly to z_0. ∎

Remark 1. The iterative algorithm in Theorem 3.1 weakens the conditions in Theorem 3.1 of Wang et al. [15] and Theorem 3.2 of Takahashi and Takahashi [6] in the following aspects:

(i) the condition $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ is removed;

(ii) the condition $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ is weakened by the condition

$$\lim_{n\to\infty} |r_{n+1} - r_n| = 0.$$

We immediately obtain the following corollaries by Theorem 3.1.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Suppose that $T : C \to C$ is a nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$. *For $\lambda \in (0,1)$ and $\omega \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ x_{n+1} = (1-\alpha_n)[\lambda T x_n + (1-\lambda)x_n] + \alpha_n\omega, \end{cases} \tag{3.10}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:

(1) $\lim_{n\to\infty} \alpha_n = 0$;

(2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(T)}\omega$.

Proof. Letting $f(x, y) \equiv 0$ for all $x, y \in C$ and $r_n = 1$ in Theorem 3.1, we obtain the desired result. ∎

Remark 2. Corollary 3.2 includes, as a special case, Theorem 3.2 of Cai, Tang and Liu [3].

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1) − (A4) *such that $EP(f) \neq \emptyset$. For $\lambda \in (0, 1)$ and $\omega \in C$, let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ u_n \in C, \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \text{ for all } y \in C, \\ x_{n+1} = (1 - \alpha_n)[\lambda u_n + (1 - \lambda)x_n] + \alpha_n \omega, \end{cases} \tag{3.11}$$

where the sequences $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(2) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $P_{EP(f)}\omega$.

Proof. Putting $T = I$ in Theorem 3.1, we get the result. ∎

4. Applications

In this section, we apply the results in the preceding section to convex minimization problem and variational inequality problem.

Now, we consider the convex minimization problem

$$\begin{cases} \min \phi(x) \\ x \in C, \end{cases} \tag{4.1}$$

where $\phi(x)$ is a proper lower semicontinuous convex function of H into $(-\infty, +\infty]$ such that C is included in $\text{dom}\phi = \{x \in H : \phi(x) < +\infty\}$. We denote by $\text{Sol}(\phi, C)$ the set of solutions of (4.1). Let $f : C \times C \to \mathbb{R}$ be a bifunction defined by

$$f(x, y) = \phi(y) - \phi(x).$$

It is clear that $f(x, y)$ satisfies $(A1) - (A4)$ and $EP(f) = \text{Sol}(\phi, C)$. Therefore the following result is obtained.

Theorem 4.1. *Let $\phi(x)$ be a proper lower semicontinuous convex function of H into $(-\infty, +\infty]$ and C a nonempty closed convex subset of H such that C is included in* $\mathrm{dom}\phi$. *Suppose that* $\mathrm{Sol}(\phi, C) \neq \emptyset$. *For $\lambda \in (0,1)$ and $\omega \in C$, let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\tag{4.2} \begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ u_n = \arg\min_{y \in C}\{\phi(y) + \frac{1}{2r_n}\|y - x_n\|^2\}, \\ x_{n+1} = (1-\alpha_n)[\lambda u_n + (1-\lambda)x_n] + \alpha_n \omega, \end{cases}$$

where the sequences $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

(1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then the sequence $\{x_n\}$ converges strongly to an element of $\mathrm{Sol}(\phi, C)$.

Proof. Letting $f(x, y) = \phi(y) - \phi(x)$ in Corollary 3.3, we get the conclusion. ∎

Next, we study the variational inequality problem. Let $A : C \to C$ be a mapping. The variational inequality problem for A is to find $z \in C$ such that

$$\tag{4.3} \langle Az, y - z\rangle \geq 0, \text{ for all } y \in C.$$

The set of its solutions is denoted by $VI(C, A)$. For $\lambda > 0$, it is easy to see that a point z is a solution of variational inequality (4.3) if and only if $z \in \mathrm{Fix}(P_C(I - \lambda A))$. Given a positive constant α, a mapping $A : C \to C$ is said to be $\alpha-$inverse strongly monotone if

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2 \text{ for all } x, y \in C.$$

Let $A : C \to C$ be $\alpha-$inverse strongly monotone and $0 < \lambda \leq 2\alpha$. Then mapping $I - \lambda A$ is nonexpansive (see [16]). Using Corollary 3.2, we obtain the strong convergence theorem for variational inequality problem.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $A : C \to C$ be an $\alpha-$inverse strongly monotone mapping with $\alpha > 0$. Suppose that $VI(C, A) \neq \emptyset$. For $\lambda \in (0,1)$ with $\lambda \leq 2\alpha$ and $\omega \in C$, let $\{x_n\}$ be a sequence generated by*

$$\tag{4.4} \begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ x_{n+1} = (1-\alpha_n)[\lambda P_C(I - \lambda A)x_n + (1-\lambda)x_n] + \alpha_n \omega, \end{cases}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:

(1) $\lim_{n\to\infty} \alpha_n = 0$;

(2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ *converges strongly to an element of* $VI(C, A)$.

Proof. Since $0 < \lambda \leq 2\alpha$, the mapping $I - \lambda A$ is nonexpansive. So is $P_C(I - \lambda A)$. Corollary 3.2 yields the result. ∎

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