ON NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR GENERALIZED CONVEX FUNCTIONS

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Abstract. In this article, we obtain some inequalities of Hermite-Hadamard type for functions whose third derivatives absolute values are $\phi$-convex, log $\phi$-convex and quasi-$\phi$-convex.

Keywords: Hermite-Hadamard inequality, $\phi$-convex functions, log-$\phi$-convex, quasi-$\phi$-convex function, Holder’s integral inequality.

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1. Introduction

One of the cornerstones of analysis is the Hadamard inequality, if $[a, b]$ ($a < b$) is a real interval and $f : [a, b] \rightarrow R$ a convex function, then

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2} \tag{1.1}$$

Over the last decade this has been extended in a number of ways. An important question is the estimating the difference between the middle and rightmost term in the (1.1). The following identity is a useful building block.

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For several results which generalize, improve and extend the inequalities (1.1), we refer the interested reader to [1,2], [10-16].

We recall that the notion of quasi-convex functions generalized the notion of convex functions. More precisely, a function \( f : [a, b] \to \mathbb{R} \) is said to be quasi-convex on \([a, b] \) if

\[
f (\lambda x + (1 - \lambda) y) \leq \max \{ f (x) , f (y) \}, \quad \forall x, y \in [a, b].
\]

Any convex function is a quasi-convex function but the reverse are not true, because there exist quasi-convex functions which are not convex, (see, e.g., [2]).

Recently, D.A. Ion [3] obtained two inequalities of the right hand side of Hermite-Hadamard’s type functions whose derivatives in absolute values are quasi-convex functions, as follows:

**Theorem 1.** Let \( f : I^0 \subseteq R \to R \) be a differentiable function on \( I^0 \), \( a, b \in I^0 \), with \( a < b \), and, if \( |f'| \) is quasi-convex on \([a, b] \), then the following inequality holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)}{4} \max \{|f'(a)|, |f'(b)|\}.
\]

**Theorem 2.** Let \( f : I^0 \subseteq R \to R \) be a differentiable function on \( I^0 \), \( a, b \in I^0 \), with \( a < b \), and, if \( |f'|^{p/(p-1)} \) is quasi-convex on \([a, b] \), then the following inequality holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{2} \left( \max \{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}\} \right)^{(p-1)/p}.
\]

Alomari, Darus and Dragomir in [4] introduced the following theorems for twice differentiable quasi-convex functions:

**Theorem 3.** Let \( f : I^0 \subseteq R \to R \) be a twice differentiable function on \( I^0 \), \( a, b \in I^0 \), with \( a < b \), and, if \( |f''| \) is quasi-convex on \([a, b] \), then the following inequality holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{12} \max \{|f''(a)|, |f''(b)|\}.
\]

**Theorem 4.** Let \( f : I^0 \subseteq R \to R \) be a twice differentiable function on \( I^0 \), \( a, b \in I^0 \), with \( a < b \), and, if \( |f''|^{p/(p-1)} \) is quasi-convex on \([a, b] \), then the following inequality holds:
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \\
\leq \frac{(b-a)^2}{8} \left( \frac{\sqrt{\pi}}{2} \right)^{1/p} \left( \frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \right)^{1/p} \left( \max \{|f''(a)|^q + |f''(b)|^q\} \right)^{1/q}.
\]

**Theorem 5.** Let \( f : I^0 \subseteq R \to R \) be a twice differentiable function on \( I^0 \) \( a, b \in I^0 \), with \( a < b \), and, if \( |f''|^q \) is quasi-convex on \( [a, b] \), \( q \geq 1 \), then the following inequality holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{12} \left( \max \{|f''(a)|^q, |f''(b)|^q\} \right)^{1/q}.
\]

This paper is in the direction of the results discussed in [5] but here we use \( \phi \)-convex, log \( \phi \)-convex and quasi-\( \phi \)-convex functions instead of s-convex function. After this introduction, in section 2 we found some new integral inequalities of the type of Hermite Hadamard’s for generalized convex functions.

### 2. Main results

To establish our principal results, we first obtain the following definitions.

Let \( K \) be a closed set \( R^n \) and let \( f, \phi : K \to R \) and \( \phi : K \times K \to R \) be continuous functions. We recall the following results, which are due to Noor [6], [7], Noor [8], [9] as follows:

**Definition 2.1.** Let \( x \in K \). Then the set \( K \) is said to be \( \phi \)-convex at \( x \) with respect to \( \phi \), if

\[
x + \lambda e^{i\phi} (y - x) \in K, \quad \forall x, y \in K, \quad \lambda \in [0, 1].
\]

**Observation 2.2.** We would like to mention that the Definition 2.1 of a \( \phi \)-convex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point \( x \) which is contained in \( K \). We don’t require that the point \( y \) should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that \( y \) should be an end point of the path for every pair of points, \( x, y \in K \), then \( e^{i\phi} (y - x) = y - x \) if and only if \( \phi = 0 \), and consequently \( \phi \)-convexity reduces to convexity. Thus, it is true that every convex set is also an \( \phi \)-convex set, but the converse is not necessarily true.

**Definition 2.3.** The function \( f \) on the \( \phi \)-convex set \( K \) is said to be \( \phi \)-convex with respect to \( \phi \), if

\[
f \left( x + \lambda e^{i\phi} (y - x) \right) \leq \left( 1 - \lambda \right) f(x) + \lambda f(y), \quad \forall x, y \in K, \quad \lambda \in [0, 1].
\]
The function \( f \) is said to be \( \phi \)-concave if and only if \(-f\) is \( \phi \)-convex.

It is to be noted that every convex function is \( \phi \)-convex function, but the converse is not true.

**Definition 2.4.** The function \( f \) on the quasi \( \phi \)--convex set \( K \) is said to be quasi \( \phi \)-convex with respect to \( \phi \), if

\[
f \left( x + \lambda \phi_i (y - x) \right) \leq \max \{ f(x), f(y) \}.
\]

**Definition 2.5.** The function \( f \) on the quasi \( \phi \)--convex set \( K \) is said to be logarithmic \( \phi \)-convex with respect to \( \phi \), if

\[
f \left( x + \lambda \phi_i (y - x) \right) \leq (f(x))^{1-\lambda} (f(y))^{\lambda}, \quad x, y \in K, \ \lambda \in [0, 1],
\]

where \( f(\cdot) > 0 \).

From the above definitions, we have

\[
f \left( x + \lambda \phi_i (y - x) \right) \leq (f(x))^{1-\lambda} (f(y))^{\lambda} \\
\leq (1 - \lambda) f(x) + \lambda f(y) \\
\leq \max \{ f(x), f(y) \}.
\]

**Lemma 2.6.** Suppose \( f : K = [a, a + e^{i\phi} (b-a)] \to (0, \infty) \) be a \( \phi \)-convex function on the interval of real numbers \( K^0 \) (the interior of \( K \)) and \( a, b \in K^0 \) with \( a < a + e^{i\phi} (b-a) \) and \( 0 \leq \phi \leq \frac{\pi}{2} \). Then the following inequality holds:

\[
\frac{f(a) + f(a + e^{i\phi} (b-a))}{2} - \frac{1}{e^{i\phi} (b-a)} \int_a^{a+e^{i\phi} (b-a)} f(x)dx - \frac{e^{i\phi} (b-a)}{12} \left[ f'(a + e^{i\phi} (b-a)) - f'(a) \right] \\
= \frac{e^{i\phi} (b-a)^3}{12} \left( \psi (1 - \psi) (2\psi - 1) f'''(a + \psi e^{i\phi} (b-a)) d\psi \right)
\]

A simple proof of this inequality can be done by integrating by parts on the right hand side. The details are left to the interested reader. The next theorem gives a new result of the Hermite-Hadamard inequality for \( \phi \)--convex function.

**Theorem 2.7.** Let \( K \subset R \) be an open interval, \( a, a + e^{i\phi} (b-a) \in K \) with \( a < a + e^{i\phi} (b-a) \). Suppose \( f : K = [a, a + e^{i\phi} (b-a)] \to (0, \infty) \) be a three times differentiable mapping such that \( f''' \) is integrable and \( 0 \leq \phi \leq \frac{\pi}{2} \). If \( |f'''| \) is \( \phi \)-convex function on \( [a, a + e^{i\phi} (b-a)] \), then following inequality holds:

\[
\left| \frac{f(a) + f(a + e^{i\phi} (b-a))}{2} - \frac{1}{e^{i\phi} (b-a)} \int_a^{a+e^{i\phi} (b-a)} f(x)dx - \frac{e^{i\phi} (b-a)}{12} \left[ f'(a + e^{i\phi} (b-a)) - f'(a) \right] \right| \\
\leq \frac{(e^{i\phi} (b-a))^3}{384} \max \{ |f'''(a)|, |f'''(b)| \}.
\]
Proof. From Lemma 2.6 and using the $\phi$-convexity of $|f'''|$, we get
\[
\left| \frac{f(a)+f(a+e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_a^{a+e^{i\phi}(b-a)} f(x)dx - \frac{e^{i\phi}(b-a)}{12} [f'(a+e^{i\phi}(b-a)) - f'(a)] \right|
\leq \frac{(e^{i\phi}(b-a))^3}{12} \left| \int_0^1 (1-\psi) |(2\psi - 1)| f'''(a+\psi e^{i\phi}(b-a)) d\psi \right|
\leq \frac{(e^{i\phi}(b-a))^3}{12} \left| \int_0^1 (1-\psi) |(2\psi - 1)| [f'''(a)] + \psi |f'''(b)| d\psi \right|
\leq \frac{(e^{i\phi}(b-a))^3}{96} \left[ |f'''(a)| + |f'''(b)| \right],
\]
which completes the proof.

Observation 2.8. If we take $e^{i\phi}(b-a) = b - a$ in Theorem 2.7, then inequality reduces to the [Corollary 3.1.1(2), 5].

Theorem 2.9. Suppose $f : K = [a, a + e^{i\phi}(b-a)] \to (0, \infty)$ be a three times differentiable mapping on $K$ and $f'''$ is integrable on $[a, a + e^{i\phi}(b-a)]$. Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'''|^p \phi$ is $\phi$-convex function on the interval of real numbers $K$ (the interior of $K$) and $a, b \in K$ with $a < a + e^{i\phi}(b-a)$ and $0 \leq \phi \leq \frac{\pi}{2}$. Then following inequality holds:
\[
\left| \frac{f(a)+f(a+e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_a^{a+e^{i\phi}(b-a)} f(x)dx - \frac{e^{i\phi}(b-a)}{12} [f'(a+e^{i\phi}(b-a)) - f'(a)] \right|
\leq \frac{(e^{i\phi}(b-a))^3}{96} \left( \frac{1}{p+1} \right)^{1/p} \left( |f'''(a)|^{1/p} + |f'''(b)|^{1/p} \right)^{p-1}. 
\]

Proof. Suppose that $a, a + e^{i\phi}(b-a) \in K$. By assumption, Holder’s inequality, then we have
\[
\left| \frac{f(a)+f(a+e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_a^{a+e^{i\phi}(b-a)} f(x)dx - \frac{e^{i\phi}(b-a)}{12} [f'(a+e^{i\phi}(b-a)) - f'(a)] \right|
\leq \frac{(e^{i\phi}(b-a))^3}{12} \left| \int_0^1 (1-\psi) |(2\psi - 1)| f'''(a+\psi e^{i\phi}(b-a)) d\psi \right|
\leq \frac{(e^{i\phi}(b-a))^3}{12} \left( \int_0^1 \psi^p (1-\psi)^p |(2\psi - 1)| d\psi \right)^{1/p} \left( \int_0^1 |f'''(a+\psi e^{i\phi}(b-a))|^{p-1} d\psi \right)^{1/p}
\leq \frac{(e^{i\phi}(b-a))^3}{12} \left( \frac{1}{p+1} \right)^{1/p} \left( \int_0^1 \psi^p (1-\psi)^p |(2\psi - 1)| d\psi \right)^{p-1}
\cdot \left( \int_0^1 \psi^p |f'''(a)|^{p-1} + \psi |f'''(b)|^{p-1} d\psi \right)^{p-1}
\leq \frac{(e^{i\phi}(b-a))^3}{96} \left( \frac{1}{p+1} \right)^{1/p} \left( |f'''(a)|^{p-1} + |f'''(b)|^{p-1} \right)^{p-1}
\leq \frac{(e^{i\phi}(b-a))^3}{96} \left( \frac{1}{p+1} \right)^{1/p} \left( |f'''(a)|^{p-1} + |f'''(b)|^{p-1} \right)^{p-1},
\]
where we use the fact that $\int_0^1 \psi^p (1-\psi)^p |2\psi - 1| d\psi = \frac{1}{2(p+1)^2}$.
Observation 2.10. If we take $e^{i\phi}(b-a) = b-a$ in Theorem 2.9, then inequality reduces to the [Corollary 3.2.1, 5].

**Theorem 2.11.** Let $K \subset R$ be an open interval, $a, a + e^{i\phi}(b-a) \in K$ with $a < a + e^{i\phi}(b-a)$. Suppose $f : K = [a, a + e^{i\phi}(b-a)] \rightarrow (0, \infty)$ be a three times differentiable mapping such that $f''$ is integrable and $0 \leq \phi \leq \frac{\pi}{2}$. If $|f'''|$ is log $\phi$-convex function on $[a, a + e^{i\phi}(b-a)]$. Then, the following inequality holds:

$$\left| \frac{f(a)+f(a+e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_a^{a+e^{i\phi}(b-a)} f(x)dx - \frac{e^{i\phi}(b-a)}{12} [f'(a + e^{i\phi}(b-a)) - f'(a)] \right|$$

$$\leq \frac{(e^{i\phi}(b-a))^3}{12} \left| \int_0^{1/2} \psi(1-\psi) \left| (2\psi - 1) \right| |f'''(a + \psi e^{i\phi}(b-a))| d\psi \right|$$

$$\leq \frac{(e^{i\phi}(b-a))^3}{12} \left| \int_0^{1/2} \psi(1-\psi) \left| (2\psi - 1) \right| \left[ |f''(a)|^{1/\psi} \cdot |f'''(b)|^{1-\psi} \right] \right|$$

$$= \frac{(e^{i\phi}(b-a))^3}{12} \left[ \frac{2(\frac{6}{3} |f'''(b)| + |f'''(a)|) - 12 |f'''(b)| - |f'''(a)|}{(\log |f'''(b)| - \log |f'''(a)|)^2} \right]$$

$$= \frac{(e^{i\phi}(b-a))^3}{(\log |f'''(b)| - \log |f'''(a)|)^2} \left[ A \left( |f'''(b)|, |f'''(a)| \right) - L \left( |f'''(b)|, |f'''(a)| \right) \right],$$

which completes the proof.

**Theorem 2.12.** Suppose $f : K = [a, a + e^{i\phi}(b-a)] \rightarrow (0, \infty)$ be a three times differentiable mapping on $K'$ and $f'''$ is integrable on $[a, a + e^{i\phi}(b-a)]$. Assume $p \in R$ with $p > 1$. If $|f'''|^{p/(p-1)}$ is log $\phi$-convex function on the interval of real numbers $K^0$ (the interior of $K$) and $a, b \in K^0$ with $a < a + e^{i\phi}(b-a)$ and $0 \leq \phi \leq \frac{\pi}{2}$. Then, the following inequality holds:

$$\left| \frac{f(a)+f(a+e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_a^{a+e^{i\phi}(b-a)} f(x)dx - \frac{e^{i\phi}(b-a)}{12} [f'(a + e^{i\phi}(b-a)) - f'(a)] \right|$$

$$\leq \frac{(e^{i\phi}(b-a))^3}{96} \left( \frac{1}{p+1} \right)^{1/p} \left( \frac{p-1}{p} \right)^{p-1} \left( \frac{|f'''(a)|^{p/(p-1)} + |f'''(b)|^{p/(p-1)}}{\log |f'''(b)| + \log |f'''(a)|} \right)^{p-1}.$$
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\[
\left| \frac{f(a) + f(a + e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_a f(x) \, dx - \frac{e^{i\phi}(b-a)}{12} \left[ f'(a + e^{i\phi}(b-a)) - f'(a) \right] \right| \\
\leq \frac{(e^{i\phi}(b-a))^3}{12^3} \int_0^1 \psi(1 - \psi) \left| \int f''(a + \psi e^{i\phi}(b-a)) \, d\psi \right| d\psi \\
\leq \frac{(e^{i\phi}(b-a))^3}{12^3} \left( \frac{1}{\psi^p} \int_0^1 \psi^p (1 - \psi)^p \, d\psi \right) \frac{1}{f''(a) + f''(b)} \cdot \frac{1}{p-1} \psi \cdot d\psi \\
= \frac{(e^{i\phi}(b-a))^3}{12^3} \left( \frac{1}{p+1} \right)^{1/p} \left( \frac{1}{p-1} \right)^{1/p} \left( \frac{1}{f''(a) + f''(b)} \right)^{p-1} \psi \cdot d\psi \\
\]

where we use the fact that
\[
\int_0^1 \psi^p (1 - \psi)^p \, d\psi = \frac{1}{2^{2+p+1}(p+1)},
\]
which completes the proof.

\[\square\]

**Theorem 2.13.** Suppose \( f : K = [a, a + e^{i\phi}(b-a)] \to (0, \infty) \) be a three times differentiable mapping on \( K^0 \) and \( f''' \) is integrable on \([a, a + e^{i\phi}(b-a)]\). If \( |f'''| \) is quasi \( \phi \)-convex function on the interval of real numbers \( K^0 \) (the interior of \( K \)) and \( a, b \in K^0 \) with \( a < a + e^{i\phi}(b-a) \) and \( 0 \leq \phi \leq \frac{\pi}{2} \). Then, the following inequality holds:

\[
\left| \frac{f(a) + f(a + e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_a f(x) \, dx - \frac{e^{i\phi}(b-a)}{12} \left[ f'(a + e^{i\phi}(b-a)) - f'(a) \right] \right| \\
\leq \frac{(e^{i\phi}(b-a))^3}{192} \max \{|f'''(a)|, |f'''(b)|\}.
\]

**Proof.** From Lemma 2.6 and using the quasi-\( \phi \)-convexity of \(|f'''|\), we get

\[
\left| \frac{f(a) + f(a + e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_a f(x) \, dx - \frac{e^{i\phi}(b-a)}{12} \left[ f'(a + e^{i\phi}(b-a)) - f'(a) \right] \right| \\
\leq \frac{(e^{i\phi}(b-a))^3}{12} \int_0^1 \psi(1 - \psi) \left| \int f''(a + \psi e^{i\phi}(b-a)) \, d\psi \right| d\psi \\
\leq \frac{(e^{i\phi}(b-a))^3}{12} \max \{|f'''(a)|, |f'''(b)|\} \cdot \int_0^1 \psi(1 - \psi) \left| \int f''(a + \psi e^{i\phi}(b-a)) \, d\psi \right| d\psi \\
\leq \frac{(e^{i\phi}(b-a))^3}{192} \max \{|f'''(a)|, |f'''(b)|\},
\]
which completes the proof.

\[\square\]
Theorem 2.14. Suppose $f : K = [a, a + e^{i\phi}(b - a)] \to (0, \infty)$ be a three times differentiable mapping on $K^0$ and $f''$ is integrable on $[a, a + e^{i\phi}(b - a)]$. Assume $p \in R$ with $p > 1$. If $|f'''|^q/(p-1)$ is quasi-\phi-convex function on the interval of real numbers $K^0$ (the interior of $K$) and $a, b \in K^0$ with $a < a + e^{i\phi}(b - a)$ and $0 \leq \phi \leq \frac{\pi}{2}$. Then, the following inequality holds:

$$
\left| \frac{f(a) + f(a + e^{i\phi}(b - a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_a^{a + e^{i\phi}(b-a)} f(x)dx - e^{i\phi}(b-a) \right| \leq \left( \frac{e^{i\phi}(b-a)}{96} \right)^3 \left( \frac{1}{p+1} \right) \frac{1}{p} \left[ \max \left\{ \int_0^1 |f'''(a)|^{\frac{p}{p-1}}, |f'''(b)|^{\frac{p}{p-1}} \right\} \right] \frac{1}{p}.
$$

Proof. From Lemma 2.6, and using the well known Holder integral inequality, we have

$$
\left| \frac{f(a) + f(a + e^{i\phi}(b - a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_a^{a + e^{i\phi}(b-a)} f(x)dx - e^{i\phi}(b-a) \right| \leq \left( \frac{e^{i\phi}(b-a)}{12} \right)^3 \int_0^1 \psi (1 - \psi) \left| f'''(a + \psi e^{i\phi}(b - a)) \right| d\psi
$$

$$
\leq \left( \frac{e^{i\phi}(b-a)}{12} \right)^3 \left( \int_0^1 \psi^p (1 - \psi)^p (2\psi - 1) d\psi \right)^{1/p} \left( \int_0^1 \left| f'''(a + \psi e^{i\phi}(b - a)) \right| \frac{p}{p-1} d\psi \right)^{1/p}.
$$

Since $|f'''|^q$ is quasi-\phi-convex, we have

$$
\int_0^1 \left| f'''(a + \psi e^{i\phi}(b - a)) \right| \frac{p}{p-1} d\psi \leq \left\{ \max \left\{ |f'''(a)|^{\frac{p}{p-1}}, |f'''(b)|^{\frac{p}{p-1}} \right\} \right\}.
$$

Hence

$$
\leq \left( \frac{e^{i\phi}(b-a)}{96} \right)^3 \left( \frac{1}{p+1} \right) \frac{1}{p} \left[ \max \left\{ |f'''(a)|^{\frac{p}{p-1}}, |f'''(b)|^{\frac{p}{p-1}} \right\} \right] \frac{1}{p},
$$

where we use the fact that

$$
\int_0^1 \psi^p (1 - \psi)^p (2\psi - 1) d\psi = \frac{1}{2^{2p+1} (p+1)},
$$

which completes the proof.

Theorem 2.15. Suppose $f : K = [a, a + e^{i\phi}(b - a)] \to (0, \infty)$ be a three times differentiable mapping on $K^0$ and $f'''$ is integrable on $[a, a + e^{i\phi}(b - a)]$. Assume $q \in R$ with $q \geq 1$. If $|f'''|^q$ is quasi-\phi-convex function on the interval of real numbers $K^0$ (the interior of $K$) and $a, b \in K^0$ with $a < a + e^{i\phi}(b - a)$ and $0 \leq \phi \leq \frac{\pi}{2}$. Then, the following inequality holds

$$
\left| \frac{f(a) + f(a + e^{i\phi}(b - a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_a^{a + e^{i\phi}(b-a)} f(x)dx - e^{i\phi}(b-a) \right| \leq \left( \frac{e^{i\phi}(b-a)}{192} \right)^3 \left( \max \left\{ |f'''(a)|^q, |f'''(b)|^q \right\} \right)^{1/q}.
$$
Proof. Suppose that $q \geq 1$. From Lemma 2.6 and using the well known power mean inequality, we have

$$
\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)^{a+e^{i\phi}(b-a)}} \int_a^{a+e^{i\phi}(b-a)} f(x)dx - \frac{e^{i\phi}(b-a)}{12} \left[f'(a+e^{i\phi}(b-a)) - f'(a)\right]\right|
$$

$$\leq \frac{(e^{i\phi}(b-a))^{3}}{12} \int_0^1 \psi(1-\psi) |(2\psi-1)| f'''(a+\psi e^{i\phi}(b-a))d\psi
$$

$$\leq \frac{(e^{i\phi}(b-a))^{3}}{12} \left(\int_0^1 \psi(1-\psi) |(2\psi-1)|d\psi\right)^{1-1/q}
$$

$$\cdot \left(\int_0^1 \psi(1-\psi) |(2\psi-1)| f'''(a+\psi e^{i\phi}(b-a))|\frac{q}{q}d\psi\right)^{1/q}
$$

$$\leq \frac{(e^{i\phi}(b-a))^{3}}{12} \left(\frac{1}{16}\right)^{1-1/q} \cdot \left(\frac{1}{16} \max \{|f'''(a)|^q, |f'''(b)|^q\}\right)^{1/q}
$$

$$= \frac{(e^{i\phi}(b-a))^{3}}{192} \left(\max \{|f'''(a)|^q, |f'''(b)|^q\}\right)^{1/q},
$$

where we use the fact

$$\int_0^1 \psi(1-\psi) |(2\psi-1)|d\psi = \frac{1}{16}.
$$

References


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