ON NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR GENERALIZED CONVEX FUNCTIONS

Shahid Qaisar[1]
Chuanjiang He

College of Mathematics and Statistics
Chongqing University
Chongqing, 401331
P.R. China

Sabir Hussain

Department of Mathematics
College of Science
Qassim University
P.O. Box 6644, Buraydah 51482
Saudi Arabia

Abstract. In this article, we obtain some inequalities of Hermite-Hadamard type for functions whose third derivatives absolute values are ϕ-convex, log ϕ-convex and quasi-ϕ-convex.

Keywords: Hermite-Hadamard inequality, ϕ-convex functions, log-ϕ-convex, quasi-ϕ-convex function, Holder's integral inequality.

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1. Introduction

One of the cornerstones of analysis is the Hadamard inequality, if $[a, b]$ $\quad (a < b)$ is a real interval and $f : [a, b] \to R$ a convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

Over the last decade this has been extended in a number of ways. An important question is the estimating the difference between the middle and rightmost term in the (1.1). The following identity is a useful building block.

[1a] Corresponding author. E-mail address: shahidqaisar90@yahoo.com

For several results which generalize, improve and extend the inequalities (1.1), we refer the interested reader to [1,2], [10-16].

We recall that the notion of quasi-convex functions generalized the notion of convex functions. More precisely, a function $f : [a, b] \to R$ is said to be quasi-convex on $[a, b]$ if

$$f(\lambda x + (1-\lambda) y) \leq \max\{f(x), f(y)\}, \quad \forall x, y \in [a, b].$$

Any convex function is a quasi-convex function but the reverse are not true, because there exist quasi-convex functions which are not convex, (see, e.g., [2]).

Recently, D.A. Ion [3] obtained two inequalities of the right hand side of Hermite-Hadamard's type functions whose derivatives in absolute values are quasi-convex functions, as follows:

Theorem 1. *Let $f : I^0 \subseteq R \to R$ be a differentiable function on I^0, $a, b \in I^0$, with $a < b$, and, if $|f'|$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{4} \max\{|f'(a)|, |f'(b)|\}.$$

Theorem 2. *Let $f : I^0 \subseteq R \to R$ be a differentiable function on I^0, $a, b \in I^0$, with $a < b$, and, if $|f'|^{p/(p-1)}$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2(p+1)^{1/p}}\left(\max\left\{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}\right\}\right)^{(p-1)/p}.$$

Alomari, Darus and Dragomir in [4] introduced the following theorems for twice differentiable quasi-convex functions:

Theorem 3. *Let $f : I^0 \subseteq R \to R$ be a twice differentiable function on I^0, $a, b \in I^0$, with $a < b$, and, if $|f''|$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12} \max\{|f''(a)|, |f''(b)|\}.$$

Theorem 4. *Let $f : I^0 \subseteq R \to R$ be a twice differentiable function on I^0, $a, b \in I^0$, with $a < b$, and, if $|f''|^{p/(p-1)}$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{1/p}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{1/p}\left(\max\left\{|f''(a)|^q+|f''(b)|^q\right\}\right)^{1/q}.$$

Theorem 5. *Let $f: I^0 \subseteq R \to R$ be a twice differentiable function on I^0 $a,b \in I^0$, with $a<b$, and, if $|f''|^q$ is quasi-convex on $[a,b]$, $q \geq 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12}\left(\max\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{1/q}.$$

This paper is in the direction of the results discussed in [5] but here we use ϕ-convex, log ϕ-convex and quasi-ϕ-convex functions instead of s-convex function. After this introduction, in section 2 we found some new integral inequalities of the type of Hermite Hadamard's for generalized convex functions.

2. Main results

To establish our principal results, we first obtain the following definitions.

Let K be a closed set R^n and let $f,\phi : K \to R$ and $\phi : K\times K \to R$ be continuous functions . we recall the following results, which are due to Noor [6], [7], Noor [8], [9] as follows:

Definition 2.1. Let $x \in K$. Then the set K is said to be ϕ-convex at x with respect to ϕ, if

$$x+\lambda e^{i\phi}(y-x) \in K,\ \forall x,y \in K,\ \lambda \in [0,1].$$

Observation 2.2. We would like to mention that the Definition 2.1 of a ϕ-convex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point x which is contained in K. We don't require that the point y should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that y should be an end point of the path for every pair of points, $x,y \in K$, then $e^{i\phi}(y-x)=y-x$ if and only if $\phi=0$, and consequently $\phi-$convexity reduces to convexity. Thus, it is true that every convex set is also an $\phi-$convex set, but the converse is not necessarily true.

Definition 2.3. The function f on the $\phi-$convex set $K$is said to be ϕ-convex with respect to ϕ, if

$$f\left(x+\lambda e^{i\phi}(y-x)\right) \leq +(1-\lambda)f(x)+\lambda f(y),\ \forall x,y \in K,\ \lambda \in [0,1].$$

The function f is said to be ϕ-concave if and only if $-f$ is ϕ-convex.

It is to be noted that every convex function is ϕ-convex function, but the converse is not true.

Definition 2.4. The function f on the quasi $\phi-$convex set $K$is said to be quasi ϕ-convex with respect to ϕ, if

$$f\left(x+\lambda e^{i\phi}(y-x)\right)\leq\max\left\{f(x),f(y)\right\}.$$

Definition 2.5. The function f on the quasi $\phi-$convex set $K$is said to be logarithmic ϕ-convex with respect to ϕ, if

$$f\left(x+\lambda e^{i\phi}(y-x)\right)\leq(f(x))^{1-\lambda}(f(y))^{\lambda},\ x,y\in K,\ \lambda\in[0,1],$$

where $f(\cdot)>0$.

From the above definitions, we have

$$\begin{aligned} f\left(x+\lambda e^{i\phi}(y-x)\right) &\leq (f(x))^{1-\lambda}(f(y))^{\lambda}\\ &\leq (1-\lambda)f(x)+\lambda f(y)\\ &\leq \max\left\{f(x),f(y)\right\}. \end{aligned}$$

Lemma 2.6. *Suppose* $f:K=\left[a,a+e^{i\phi}(b-a)\right]\to(0,\infty)$ *be a* ϕ*-convex function on the interval of real numbers* K^0 *(the interior of* K*) and* $a,b\in K^0$ *with* $a<a+e^{i\phi}(b-a)$ *and* $0\leq\phi\leq\frac{\pi}{2}$*. Then the following inequality holds:*

$$\begin{aligned} &\tfrac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\tfrac{1}{e^{i\phi}(b-a)}\int\limits_a^{a+e^{i\phi}(b-a)}f(x)dx-\tfrac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right)-f'(a)\right]\\ &=\tfrac{\left(e^{i\phi}(b-a)\right)^3}{12}\int\limits_0^1\psi(1-\psi)(2\psi-1)f'''\left(a+\psi e^{i\phi}(b-a)\,d\psi\right) \end{aligned}$$

A simple proof of this inequality can be done by integrating by parts on the right hand side. The details are left to the interested reader. The next theorem gives a new result of the Hermite-Hadamard inequality for$\phi-$convex function.

Theorem 2.7. *Let* $K\subset R$ *be an open interval,* $a,a+e^{i\phi}(b-a)\in K$ *with* $a<a+e^{i\phi}(b-a)$*. Suppose* $f:K=\left[a,a+e^{i\phi}(b-a)\right]\to(0,\infty)$ *be a three times differentiable mapping such that* f''' *is integrable and* $0\leq\phi\leq\frac{\pi}{2}$*. If* $|f'''|$ *is* ϕ*-convex function on* $\left[a,a+e^{i\phi}(b-a)\right]$*, then following inequality holds:*

$$\begin{aligned} &\left|\tfrac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\tfrac{1}{e^{i\phi}(b-a)}\int\limits_a^{a+e^{i\phi}(b-a)}f(x)dx-\tfrac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right)-f'(a)\right]\right|\\ &\leq\tfrac{\left(e^{i\phi}(b-a)\right)^3}{384}\max\left\{|f'''(a)|,|f'''(b)|\right\}. \end{aligned}$$

Proof. From Lemma 2.6 and using the ϕ-convexity of $|f'''|$, we get

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\frac{1}{e^{i\phi}(b-a)}\int\limits_a^{a+e^{i\phi}(b-a)} f(x)dx-\frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right)-f'(a)\right]\right|$$
$$\leq \frac{\left(e^{i\phi}(b-a)\right)^3}{12}\left|\int\limits_0^1 \psi(1-\psi)\,|(2\psi-1)|f'''\left(a+\psi e^{i\phi}(b-a)\,d\psi\right)\right|$$
$$\leq \frac{\left(e^{i\phi}(b-a)\right)^3}{12}\int\limits_0^1 \psi(1-\psi)\,|(2\psi-1)|\left[(1-\psi)\,|f'''(a)|+\psi\,|f'''(b)|\right]d\psi$$
$$\leq \frac{\left(e^{i\phi}(b-a)\right)^3}{384}\left[|f'''(a)|+|f'''(b)|\right],$$

which completes the proof. ■

Observation 2.8. If we take $e^{i\phi}(b-a)=b-a$ in Theorem 2.7, then inequality reduces to the [Corollary 3.1.1(2), 5].

Theorem 2.9. *Suppose $f:K=\left[a,a+e^{i\phi}(b-a)\right]\to(0,\infty)$ be a three times differentiable mapping on K^0 and f''' is integrable on $\left[a,a+e^{i\phi}(b-a)\right]$. Assume $p\in R$ with $p>1$. If $|f'''|^{p/(p-1)}$ is ϕ-convex function on the interval of real numbers K^0 (the interior of K) and $a,b\in K^0$ with $a<a+e^{i\phi}(b-a)$ and $0\leq\phi\leq\frac{\pi}{2}$. Then following inequality holds:*

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\frac{1}{e^{i\phi}(b-a)}\int\limits_a^{a+e^{i\phi}(b-a)} f(x)dx-\frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right)-f'(a)\right]\right|$$
$$\leq \frac{\left(e^{i\phi}(b-a)\right)^3}{96}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{|f'''(a)|^{\frac{p}{p-1}}+|f'''(b)|^{\frac{p}{p-1}}}{2}\right)^{\frac{p-1}{p}}.$$

Proof. Suppose that $a,a+e^{i\phi}(b-a)\in K$. By assumption, Holder's inequality, then we have

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\frac{1}{e^{i\phi}(b-a)}\int\limits_a^{a+e^{i\phi}(b-a)} f(x)dx-\frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right)-f'(a)\right]\right|$$
$$\leq \frac{\left(e^{i\phi}(b-a)\right)^3}{12}\int\limits_0^1 \psi(1-\psi)\,|(2\psi-1)|\left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right|d\psi$$
$$\leq \frac{\left(e^{i\phi}(b-a)\right)^3}{12}\left(\int\limits_0^1 \psi^p(1-\psi)^p\,|(2\psi-1)|d\psi\right)^{1/p}\left(\int\limits_0^1 \left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right|^{\frac{p-1}{p}}d\psi\right)^{\frac{p-1}{p}}$$
$$\leq \frac{\left(e^{i\phi}(b-a)\right)^3}{12}\left(\int\limits_0^1 \psi^p(1-\psi)^p\,|(2\psi-1)|d\psi\right)^{1/p}$$
$$\cdot\left(\int_0^1\left[(1-\psi)\,|f'''(a)|^{\frac{p}{p-1}}+\psi\,|f'''(b)|^{\frac{p}{p-1}}\right]d\psi\right)^{\frac{p-1}{p}}$$
$$=\frac{\left(e^{i\phi}(b-a)\right)^3}{96}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{|f'''(a)|^{\frac{p}{p-1}}+|f'''(b)|^{\frac{p}{p-1}}}{2}\right)^{\frac{p-1}{p}},$$

where we use the fact that $\int\limits_0^1 \psi^p(1-\psi)^p\,|2\psi-1|\,d\psi=\frac{1}{2^{2p+1}(p+1)}$,, which completes the proof. ■

Observation 2.10. If we take $e^{i\phi}(b-a)=b-a$ in Theorem 2.9, then inequality reduces to the [Corollary 3.2.1, 5].

Theorem 2.11. *Let $K\subset R$ be an open interval, $a, a+e^{i\phi}(b-a)\in K$ with $a<a+e^{i\phi}(b-a)$. Suppose $f:K=\left[a,a+e^{i\phi}(b-a)\right]\to(0,\infty)$ be a three times differentiable mapping such that f''' is integrable and $0\le\phi\le\frac{\pi}{2}$. If $|f'''|$ is log ϕ-convex function on $\left[a,a+e^{i\phi}(b-a)\right]$. Then, the following inequality holds:*

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\frac{1}{e^{i\phi}(b-a)}\int_a^{a+e^{i\phi}(b-a)}f(x)dx-\frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right)-f'(a)\right]\right|$$
$$\le\frac{\left(e^{i\phi}(b-a)\right)^3}{(\log|f'''(a)|-\log|f'''(b)|)^3}\left[A\left(|f'''(b)|,|f'''(a)|\right)-L\left(|f'''(b)|,|f'''(a)|\right)\right].$$

Proof. From Lemma 2.6 and using the log-ϕ-convexity of $|f'''|$, we get

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\frac{1}{e^{i\phi}(b-a)}\int_a^{a+e^{i\phi}(b-a)}f(x)dx-\frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right)-f'(a)\right]\right|$$
$$\le\frac{\left(e^{i\phi}(b-a)\right)^3}{12}\int_0^1\psi(1-\psi)\left|(2\psi-1)\right|\left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right|d\psi$$
$$\le\frac{\left(e^{i\phi}(b-a)\right)^3}{12}\int_0^1\psi(1-\psi)\left|(2\psi-1)\right|\left(|f'''(a)|^{1-\psi}.|f'''(b)|^{\psi}\right)$$
$$=\frac{\left(e^{i\phi}(b-a)\right)^3}{12}.\left[\frac{6(|f'''(b)|+|f'''(a)|)}{(\log|f'''(b)|-\log|f'''(a)|)^3}-\frac{12(|f'''(b)|-|f'''(a)|)}{(\log|f'''(b)|-\log|f'''(a)|)^4}\right]$$
$$=\frac{\left(e^{i\phi}(b-a)\right)^3}{(\log|f'''(b)|-\log|f'''(a)|)^3}\left[A\left(|f'''(b)|,|f'''(a)|\right)-L\left(|f'''(b)|,|f'''(a)|\right)\right],$$

which completes the proof. ∎

Theorem 2.12. *Suppose $f:K=\left[a,a+e^{i\phi}(b-a)\right]\to(0,\infty)$ be a three times differentiable mapping on K^0 and f''' is integrable on $\left[a,a+e^{i\phi}(b-a)\right]$. Assume $p\in R$ with $p>1$. If $|f'''|^{p/(p-1)}$ is log ϕ-convex function on the interval of real numbers K^0 (the interior of K) and $a,b\in K^0$ with $a<a+e^{i\phi}(b-a)$ and $0\le\phi\le\frac{\pi}{2}$. Then, the following inequality holds:*

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\frac{1}{e^{i\phi}(b-a)}\int_a^{a+e^{i\phi}(b-a)}f(x)dx-\frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right)-f'(a)\right]\right|$$
$$\le\frac{\left(e^{i\phi}(b-a)\right)^3}{96}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{p-1}{p}\right)^{\frac{p-1}{p}}\left(\frac{|f'''(a)|^{\frac{p}{p-1}}+|f'''(b)|^{\frac{p}{p-1}}}{\log|f'''(b)|+\log|f'''(a)|}\right)^{\frac{p-1}{p}}.$$

Proof. From Lemma 2.6, and using the well known Holder integral inequality, we have

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\frac{1}{e^{i\phi}(b-a)}\int_a^{a+e^{i\phi}(b-a)}f(x)dx-\frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right)-f'(a)\right]\right|$$
$$\leq\frac{\left(e^{i\phi}(b-a)\right)^3}{12}\int_0^1\psi(1-\psi)\left|(2\psi-1)\right|\left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right|d\psi$$
$$\leq\frac{\left(e^{i\phi}(b-a)\right)^3}{12}\left(\int_0^1\psi^p(1-\psi)^p\left|(2\psi-1)\right|d\psi\right)^{1/p}\left(\int_0^1\left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right|^{\frac{p}{p-1}}d\psi\right)^{\frac{p-1}{p}}$$
$$\leq\frac{\left(e^{i\phi}(b-a)\right)^3}{12}\left(\int_0^1\psi^p(1-\psi)^p\left|(2\psi-1)\right|d\psi\right)^{1/p}$$
$$\cdot\left(\int_0^1\left|f'''(a)\right|^{\frac{p}{p-1}(1-\psi)}+\left|f'''(b)\right|^{\frac{p}{p-1}\psi}d\psi\right)^{\frac{p-1}{p}}$$
$$=\frac{\left(e^{i\phi}(b-a)\right)^3}{96}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{p-1}{p}\right)^{\frac{p-1}{p}}\left(\frac{\left|f'''(a)\right|^{\frac{p}{p-1}}+\left|f'''(b)\right|^{\frac{p}{p-1}}}{\log\left|f'''(b)\right|+\log\left|f'''(a)\right|}\right)^{\frac{p-1}{p}},$$

where we use the fact that

$$\int_0^1\psi^p(1-\psi)^p\left|2\psi-1\right|d\psi=\frac{1}{2^{2p+1}(p+1)},$$

which completes the proof. ∎

Theorem 2.13. *Suppose* $f:K=\left[a,a+e^{i\phi}(b-a)\right]\to(0,\infty)$ *be a three times differentiable mapping on* K^0 *and* f''' *is integrable on* $\left[a,a+e^{i\phi}(b-a)\right]$. *If* $|f'''|$ *is quasi* ϕ*-convex function on the interval of real numbers* K^0 *(the interior of* K*) and* $a,b\in K^0$ *with* $a<a+e^{i\phi}(b-a)$ *and* $0\leq\phi\leq\frac{\pi}{2}$. *Then, the following inequality holds:*

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\frac{1}{e^{i\phi}(b-a)}\int_a^{a+e^{i\phi}(b-a)}f(x)dx-\frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right)-f'(a)\right]\right|$$
$$\leq\frac{\left(e^{i\phi}(b-a)\right)^3}{192}\max\left\{\left|f'''(a)\right|,\left|f'''(b)\right|\right\}.$$

Proof. From Lemma 2.6 and using the quasi-ϕ-convexity of $|f'''|$, we get

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2}-\frac{1}{e^{i\phi}(b-a)}\int_a^{a+e^{i\phi}(b-a)}f(x)dx-\frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right)-f'(a)\right]\right|$$
$$\leq\frac{\left(e^{i\phi}(b-a)\right)^3}{12}\int_0^1\psi(1-\psi)\left|(2\psi-1)\right|\left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right|d\psi$$
$$\leq\frac{\left(e^{i\phi}(b-a)\right)^3}{12}\max\left\{\left|f'''(a)\right|,\left|f'''(b)\right|\right\}.\int_0^1\psi(1-\psi)\left|(2\psi-1)\right|d\psi$$
$$\leq\frac{\left(e^{i\phi}(b-a)\right)^3}{192}\max\left\{\left|f'''(a)\right|,\left|f'''(b)\right|\right\},$$

which completes the proof. ∎

Theorem 2.14. *Suppose $f : K = \left[a, a + e^{i\phi}(b-a)\right] \to (0,\infty)$ be a three times differentiable mapping on K^0 and f''' is integrable on $\left[a, a + e^{i\phi}(b-a)\right]$. Assume $p \in R$ with $p > 1$. If $|f'''|^{p/(p-1)}$ is quasi ϕ-convex function on the interval of real numbers K^0 (the interior of K) and $a, b \in K^0$ with $a < a + e^{i\phi}(b-a)$ and $0 \le \phi \le \frac{\pi}{2}$. Then, the following inequality holds:*

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_a^{a+e^{i\phi}(b-a)} f(x)dx - \frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right) - f'(a)\right]\right|$$
$$\le \frac{\left(e^{i\phi}(b-a)\right)^3}{96}\left(\frac{1}{p+1}\right)^{1/p}\left[\max\left\{|f'''(a)|^{\frac{p}{p-1}}, |f'''(b)|^{\frac{p}{p-1}}\right\}\right]^{\frac{p-1}{p}}.$$

Proof. From Lemma 2.6, and using the well known Holder integral inequality, we have

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_a^{a+e^{i\phi}(b-a)} f(x)dx - \frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right) - f'(a)\right]\right|$$
$$\le \frac{\left(e^{i\phi}(b-a)\right)^3}{12}\int_0^1 \psi(1-\psi)\left|(2\psi-1)\right|\left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right| d\psi$$
$$\le \frac{\left(e^{i\phi}(b-a)\right)^3}{12}\left(\int_0^1 \psi^p(1-\psi)^p\left|(2\psi-1)\right| d\psi\right)^{1/p}\left(\int_0^1 \left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right|^{\frac{p}{p-1}} d\psi\right)^{\frac{p}{p-1}}.$$

Since $|f'''|^q$ is quasi-ϕ-convex, we have

$$\int_0^1 \left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right|^{\frac{p}{p-1}} d\psi \le \left\{\max|f'''(a)|^{\frac{p}{p-1}}, |f'''(b)|^{\frac{p}{p-1}}\right\}.$$

Hence

$$\le \frac{\left(e^{i\phi}(b-a)\right)^3}{96}\left(\frac{1}{(p+1)}\right)^{\frac{1}{p}}\left\{\max|f'''(a)|^{\frac{p}{p-1}}, |f'''(b)|^{\frac{p}{p-1}}\right\}^{\frac{1}{q}},$$

where we use the fact that

$$\int_0^1 \psi^p(1-\psi)^p\left|2\psi-1\right| d\psi = \frac{1}{2^{2p+1}(p+1)},$$

which completes the proof. ∎

Theorem 2.15. *Suppose $f : K = \left[a, a + e^{i\phi}(b-a)\right] \to (0,\infty)$ be a three times differentiable mapping on K^0 and f''' is integrable on $\left[a, a + e^{i\phi}(b-a)\right]$. Assume $q \in R$ with $q \ge 1$.If $|f'''|^q$ is quasi-ϕ-convex function on the interval of real numbers K^0 (the interior of K) and $a, b \in K^0$ with $a < a + e^{i\phi}(b-a)$ and $0 \le \phi \le \frac{\pi}{2}$. Then, the following inequality holds*

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int_a^{a+e^{i\phi}(b-a)} f(x)dx - \frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right) - f'(a)\right]\right|$$
$$= \frac{\left(e^{i\phi}(b-a)\right)^3}{192}\left(\max\left\{|f'''(a)|^q, |f'''(b)|^q\right\}\right)^{\frac{1}{q}}.$$

Proof. Suppose that $q \geq 1$. From Lemma 2.6 and using the well known power mean inequality, we have

$$\left|\frac{f(a)+f(a+e^{i\phi}(b-a))}{2} - \frac{1}{e^{i\phi}(b-a)} \int\limits_a^{a+e^{i\phi}(b-a)} f(x)dx - \frac{e^{i\phi}(b-a)}{12}\left[f'\left(a+e^{i\phi}(b-a)\right) - f'(a)\right]\right|$$
$$\leq \frac{\left(e^{i\phi}(b-a)\right)^3}{12} \int\limits_0^1 \psi(1-\psi)\left|(2\psi-1)\right| \left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right| d\psi$$
$$\leq \frac{\left(e^{i\phi}(b-a)\right)^3}{12} \left(\int\limits_0^1 \psi(1-\psi)\left|(2\psi-1)\right| d\psi\right)^{1-1/q}$$
$$\cdot \left(\int\limits_0^1 \psi(1-\psi)\left|(2\psi-1)\right| \left|f'''\left(a+\psi e^{i\phi}(b-a)\right)\right|^q d\psi\right)^{1/q}$$
$$\leq \frac{\left(e^{i\phi}(b-a)\right)^3}{12} \left(\tfrac{1}{16}\right)^{1-1/q} \cdot \left(\tfrac{1}{16}\max\left\{\left|f'''(a)\right|^q, \left|f'''(b)\right|^q\right\}\right)^{1/q}$$
$$= \frac{\left(e^{i\phi}(b-a)\right)^3}{192} \left(\max\left\{\left|f'''(a)\right|^q, \left|f'''(b)\right|^q\right\}\right)^{1/q},$$

where we use the fact

$$\int\limits_0^1 \psi(1-\psi)\left|(2\psi-1)\right| d\psi = \frac{1}{16}.$$

References

[1] Pearce, C.E.M., Pecarić, J., *Inequalities for differentiable mappings with application to special means and quadrature formula*, Appl. Math. Lett., 13 (2000), 51–55.

[2] Alomari, M., Darus, M., *On the Hadamard's inequality for log-convex functions on the coordinates,* J. Ineq. Appl., vol. 2009, Article ID 283147, 13 pages. doi:10.1155/2009/283147.

[3] Ion, D.A., *Some estimates on the Hermite-Hadamard inequality through quasi-convex functions*, Annals of University of Craiova, Math. Comp. Sci. Ser., 34 (2007), 82-87.

[4] Alomari, M., Darus, M., Dragomir, S,S., *New inequalities of Hermite-Hadamard's type for functions whose second derivatives absolute values are quasiconvex*, Tamk. J. Math., 41 (2010), 353-359.

[5] Chun, L.,. Qi, F., *Integral inequalities for Hermite-Hadamard type for functions whose 3rd derivatives are s-convex*, Applied Mathematics, 3 (2012), 1680-1885.

[6] K. Inayat Noor, K., Aslam Noor, M., *Relaxed strongly nonconvex functions*, Appl. Math., E-Notes, 6 (2006), 259-267.

[7] Aslam Noor, M., *Some new classes of nonconvex functions*, Nonl. Funct. Anal. Appl., 11 (2006), 165-171.

[8] Noor, M.A., *Hermite-Hadamard integral inequalities for log-convex functions*, Nonl. Anal. Forum, (2009).

[9] Noor, M.A., *On Hadamard integral inequalities involving two log-preinvex functions*, J. Inequal. Pure Appl. Math., 8 (2007), No. 3, 1-6, Article 75.

[10] Dragomir, S.S., Agarwal, R.P., *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett., 11 (1998), 91–95.

[11] Pecaric, J., Proschan, F., Tong, Y.L., *Convex functions, partial ordering and statistical applications*, Academic Press, New York, 1991.

[12] Dragomir, S.S., Pearce, C.E.M., *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. Online: http://www.staff.vu.edu.au/RGMIA/monographs/hermite hadamard.html.

[13] Sarikaya, M.Z., Aktan, N., *On the generalization some integral inequalities and their applications Mathematical and Computer Modelling*, vol. 54, issues 9-10, November 2011, 2175-2182.

[14] Sarikaya, M.Z., Set, E., Ozdemir, M.E., *On some new inequalities of Hadamard type involving h-convex functions*, Acta Mathematica Universitatis Comenianae, vol. LXXIX, 2 (2010), 265-272.

[15] Sarikaya, M.Z., Bozkurt, H., *On Hadamard Type Integral Inequalities for nonconvex Functions*, (submitted), arXiv:1203.2282v1.

[16] Sarikaya, M.Z., Bozkurt, H., Alp, N., *On Hermite-Hadamard Type Integral Inequalities for preinvex and log-preinvex functions*, (submitted), arXiv:1203.4759v1.

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