A NOTE ON NON-FRAGMENTABLE SUBSPACE OF $\ell^c_\infty(\Gamma)$

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Abstract. In this paper we consider $\ell^c_\infty(\Gamma)$ where $\Gamma$ is uncountable and introduce subspaces $\{A_P\}_{P \in \Sigma}$ of $\ell^c_\infty(\Gamma)$ which are fragmented by a metric that generates the discrete topology but $A = \bigcup_{P \in \Sigma} A_P$ is not countable unions of fragmentable subspaces.

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1. Introduction

A topological space $X$ is fragmentable if there exists a metric $d(.,.)$ on $X$ such that for every $\varepsilon > 0$ and every nonempty set $A \subseteq X$ there exists a nonempty subset $B \subseteq A$ which is relatively open in $A$ and $d-diam(B) = \sup\{d(x, y) : x, y \in B\} < \varepsilon$. In such a case we say that the metric $d$ fragments $X$. Obviously subspaces of fragmentable space are fragmentable, metric spaces are fragmentable and if $\tau_1$ and $\tau_2$ are two topology on set $X$ such that $\tau_1$ is stronger than $\tau_2$ and $(X, \tau_2)$ is fragmentable then $(X, \tau_1)$ is fragmentable.

If $X$ is countable union of fragmentable closed subspaces then $X$ is fragmentable [1, Theorem 5.1.10]. This is not true when we replace countable by uncountable. Let $\Gamma$ be an uncountable set and $Y = \ell^c_\infty(\Gamma)$ be the space of all bounded real-valued functions with countable support defined on $\Gamma$. This space by supremum norm is closed subspace of $\ell^c_\infty(\Gamma)$. In next section we introduce a subspace of $Y$ which is uncountable unions of subspaces which (by weak topology)
are fragmentable by discrete metric but the space is not even countable unions of fragmentable spaces.

In [6] the following topological game was used to characterize the fragmentability of the space $X$. Two player $\mathcal{A}$ and $\mathcal{B}$ alternatively select subset of $X$. The player $\mathcal{A}$ starts the game by choosing some nonempty subset $A_1$ of $X$, then the player $\mathcal{B}$ chooses some nonempty relatively open subset $B_1$ of $A_1$. Then again $\mathcal{A}$ selects an arbitrary nonempty subset $A_2 \subseteq B_1$ and $\mathcal{B}$ responds by choosing some nonempty relatively open subset $B_2$ of $A_2$. Continuing this alternative selection of sets the two players generate a sequence of sets

$$A_1 \supseteq B_1 \supseteq A_2 \supseteq B_2 \supseteq \cdots$$

which we call a play and denote by $p = (A_i, B_i)_{i \geq 1}$. We say that the player $\mathcal{B}$ is winner whenever the set $\bigcap_{i \geq 1} A_i = \bigcap_{i \geq 1} B_i$ contains at most one point, otherwise the player $\mathcal{A}$ is winner. A strategy $w$ for the player $\mathcal{B}$ is a mapping which assigns to each partial play, $A_1 \supseteq B_1 \supseteq A_2 \supseteq B_2 \supseteq \cdots \supseteq A_k$, some nonempty set $B_k = w(A_1, B_1, \ldots, A_k)$ which is relatively open subset of $A_k$.

We call the play $p = (A_i, B_i)_{i \geq 1}$, a $w$-play if, $B_i = w(A_1, B_1, \ldots, A_i)$ for every $i \geq 1$. The strategy $w$ is a winning strategy for $\mathcal{B}$ if, the player $\mathcal{B}$ wins every $w$-play. We denote such a game by $G_f$.

The following theorem determines the relation between fragmentability and topological game:

**Theorem 1.1** [6, Theorem 1.1] The topological space $X$ is fragmentable if and only if the player $\mathcal{B}$ has a winning strategy for the game $G$.

The following theorem is proved in [7, Lemma 3] about spaces which are countable unions of fragmentable spaces:

**Theorem 1.2** If $X$ is countable unions of fragmentable subspaces then there exists a strategy $w$ for the player $\mathcal{B}$ in the game $G$ such that for every $w$-play $p = (A_i, B_i)_{i \geq 1}$ the set $\bigcap_{i \geq 1} B_i$ contains at most countable point.

Let $\tau_1, \tau_2$ be two (not necessarily distinct) topologies on the set $X$. We say that $(X, \tau_1)$ is fragmented by a metric $d$ which majorizes the topology $\tau_2$ if the topology generated by $d$ is stronger than or equal to the topology $\tau_2$.

**Theorem 1.3** [5, Theorem 1.2] Let $\tau_1, \tau_2$ be two (not necessarily distinct) topologies on the set $X$. The space $(X, \tau_1)$ is fragmented by a metric $d$ which majorizes $\tau_2$ if and only if there exists a strategy $w$ for the player $\mathcal{B}$ in the game $G$ in $(X, \tau_1)$ such that, for every $w$-play $p = (A_i, B_i)_{i \geq 1}$ either $\bigcap_{i \geq 1} A_i = \bigcap_{i \geq 1} B_i = \emptyset$ or $\bigcap_{i \geq 1} B_i = \{x\}$ for some $x \in X$, and for every $\tau_2$-open set $U$ that contains $x$, there exists some integer $k > 0$ with $B_k \subseteq U$. 
Let $(X, \tau)$ be a topological space which fragmented by metric $d$. By use of recent Theorem we can determine that $d$ generates the topology $\tau$ or not.

In general, $d$ does not generate the topology $\tau$. For example $(X = \ell_\infty, \text{weak})$ is fragmentable since $(B_X, \text{weak}^*)$ is metrizable and $X = \bigcup_{n \in \mathbb{N}} nB_X$ but it is proved in [4, Example 3.2] that this space is not fragmented by a metric which majorizes the weak topology.

If the topology on $X$ is discrete then obviously $X$ is fragmented by each metric on it and then $X$ is fragmented by a metric which generates the discrete topology.

2. Results

Let $x \in Y = \ell_\infty^c(\Gamma)$, $\text{supp}(x) = \{\alpha \in \Gamma : x(\alpha) \neq 0\}$, $A = \{x \in \ell_\infty^c(\Gamma) : x(\alpha) = 1$, $\alpha \in \text{supp}(x)\}$.

Let $\Sigma$ be the collection of all partitions of $\Gamma$ such that, for each partition $\mathcal{P} \in \Sigma$, if $I \in \mathcal{P}$, then $I$ is countable.

Let $\mathcal{P} \in \Sigma$, define $A_\mathcal{P} = \{x \in A : \text{supp}(x) = I$, for some $I \in \mathcal{P}\}$, obviously $A = \bigcup_{\mathcal{P} \in \Sigma} A_\mathcal{P}$.

**Theorem 2.1** If $\mathcal{P} \in \Sigma$ then $(A_\mathcal{P}, \text{weak})$ is discrete.

**Proof.** Let $x_0 \in A_\mathcal{P}$. We show that $(\{x_0\}, \text{weak})$ is open in $A_\mathcal{P}$.

If $\alpha \in \text{supp}(x_0)$, then $x_0(\alpha) = 1$ and $x(\alpha) = 0$ for other $x \in A_\mathcal{P}$, that implies $x_0 \notin A_\mathcal{P} \setminus \{x_0\}$. Therefore, there exists $f \in Y^*$ such that $f(x_0) = 1$ and $f(x) = 0$ for other $x \in A_\mathcal{P}$. Put $B = \{x \in A_\mathcal{P} : |f(x - x_0) < \frac{1}{2}\}$. $B$ is open in $A_\mathcal{P}$ by weak topology and contains just $x_0$.

For every $\mathcal{P} \in \Sigma$, the set $A_\mathcal{P}$ by weak topology is closed in $A$. Theorem 2.1 implies the following theorem:

**Theorem 2.2** If $\mathcal{P} \in \Sigma$ then $(A_\mathcal{P}, \text{weak})$ is fragmented by a metric which generates the discrete topology.

It is proved in [2, Theorem 3.1] that $(A, \text{weak})$ is not fragmentable. By use of property of $Y^*$, we show that $(A, \text{weak})$ is not countable unions of fragmentable subspaces.

**Lemma 2.3** Let $\Gamma_1$ be an uncountable subset of $\Gamma$ and $y \in Y^*$, then there exists an uncountable subset $J_y(\Gamma_1)$ of $\Gamma_1$ such that $y(x) = 0$ for each $x \in A$ where $\text{supp}(x) \subseteq J_y(\Gamma_1)$.

**Proof.** It is proved in [3] that for $y \in Y^*$, there exists a countable subset $I_y$ of $\Gamma$ such that $y(x) = 0$ for $x \in A$ where $\text{supp}(x) \subseteq I_y^\circ$. If $J_y(\Gamma_1) = \Gamma_1 \cap I_y^\circ$, then $y(x) = 0$, for $x \in A$ where $\text{supp}(x) \subseteq J_y(\Gamma_1)$.
Corollary 2.4 Let $\Gamma_1 \subseteq \Gamma$ be uncountable and $y_1, y_2, \ldots, y_n \in Y^*$, then there exists an uncountable subset $J_{y_1, y_2, \ldots, y_n}(\Gamma_1)$ of $\Gamma_1$ such that

$$y_1(x) = y_2(x) = \cdots = y_n(x) = 0 \text{ for each } x \in A,$$

where

$$\text{supp}(x) \subseteq J_{y_1, y_2, \ldots, y_n}(\Gamma_1).$$

Proof. We get $J_1 = J_{y_1}(\Gamma_1)$ and $J_2 = J_{y_2}(J_1)$ and continue this process to get

$$J_n = J_{y_n}(J_{n-1}).$$

Put $J_{y_1, y_2, \ldots, y_n}(\Gamma_1) = J_n$, then

$$y_1(x) = y_2(x) = \cdots = y_n(x), \text{ for } x \in A,$$

where

$$\text{supp}(x) \subseteq J_{y_1, y_2, \ldots, y_n}(\Gamma_1).$$

\[\blacksquare\]

Theorem 2.5 $(A, \text{weak})$ is not countable unions of fragmentable subspaces.

Proof. By Theorem 1.2 it is enough to show that there exists a play $p = (A_i, B_i)_{i \geq 1}$ in the game $G$ such that $\bigcap_{i \geq 1} B_i$ has uncountable point. Let player $A$ select $A_1 = A$ and player $B$ select non-empty and relatively open subset $B_1 \subseteq A_1$. Let $x_1 \in B_1$, then there are $y_{11}, y_{12}, \ldots, y_{1m_1} \in Y^*$ and $\varepsilon_1 > 0$ such that $B_1' \subseteq B_1$, where $B_1' = \{x \in A_1 : |y_{11}(x-x_1)| < \varepsilon_1, \ldots, |y_{1m_1}(x-x_1)| < \varepsilon_1\}$. Put $I_1 = \text{supp}(x_1)$ and $J_1 = J_{y_1, y_2, \ldots, y_{1m_1}}(I_1^c)$. Let

$$A_2 = \{x \in B_1' : I_1 \subseteq \text{supp}(x) \subseteq I_1 \cup J_1\}.$$

Let $B_2 \subseteq A_2$ (non-empty and relatively open) be selected. Let $x_2 \in B_2$, then there are $y_{21}, y_{22}, \ldots, y_{2m_2} \in Y^*$ and $\varepsilon_2 > 0$ such that $B_2' \subseteq B_2$ where

$$B_2' = \{x \in A_2 : |y_{21}(x-x_2)| < \varepsilon_2, \ldots, |y_{2m_2}(x-x_2)| < \varepsilon_2\}.$$

Put $I_2 = \text{supp}(x_2)$ and $J_2 = J_{y_{21}, y_{22}, \ldots, y_{2m_2}}(I_2^c \cap J_1)$. Let

$$A_3 = \{x \in B_2' : I_2 \subseteq \text{supp}(x) \subseteq I_2 \cup J_2\}.$$

We get $B_3'$ similarly. Following this process, in $m$th stage we have $I_m = \text{supp}(x_m)$ and $J_m = J_{y_{m1}, y_{m2}, \ldots, y_{mm}}(I_m^c \cap J_{m-1})$ and

$$A_m = \{x \in B_{m-1}' : I_m \subseteq \text{supp}(x) \subseteq I_m \cup J_m\}.$$

We have

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m \subseteq \cdots, J_1 \supseteq J_2 \supseteq \cdots \supseteq J_m \supseteq \cdots.$$
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Since $I_n$ and $J_n \setminus J_{n+1}$ are countable for each $n \in \mathbb{N}$, $\left( \bigcup_{n \in \mathbb{N}} I_n \right)$ is countable and $J$ is uncountable where $J = \bigcap_{n \in \mathbb{N}} J_n$. For each $n \in \mathbb{N}$, we have $I_n \cap J_n = \emptyset$, then

$$\left( \bigcup_{n \in \mathbb{N}} I_n \right) \cap \left( \bigcap_{n \in \mathbb{N}} J_n \right) = \emptyset.$$

Let $x \in A$ such that $x(\alpha) = 1$, for every $\alpha \in \bigcup_{n \in \mathbb{N}} I_n$ and for one $\alpha \in J$ and $x(\alpha) = 0$, for other $\alpha$. We have $x = x_1 + x'_1$ where $x'_1 \in A$ and $\text{supp}(x'_1) \subseteq J_1$. Then $y_{1i}(x - x_1) = y_{1i}(x'_1) = 0$, for each $1 \leq i \leq n_1$, that follows $x \in B'_1$ and $x \in A_2$. Also we have $x = x_2 + x'_2$, where $x'_2 \in A$ and $\text{supp}(x'_2) \subseteq J_2$, then $y_{2i}(x - x_2) = y_{2i}(x'_2) = 0$, for each $1 \leq i \leq n_2$, that follows $x \in B'_2$ and $x \in A_3$. By continuing this process we have $x \in A_n$ for each $n \in \mathbb{N}$, then

$$x \in \bigcap_{n \in \mathbb{N}} A_n.$$

Since $J$ is uncountable, there are uncountable choices for $x$, then

$$\bigcap_{n \in \mathbb{N}} A_n = \bigcap_{n \in \mathbb{N}} B_n$$

contains uncountable point.

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