INVERTIBLE ELEMENTS IN BCK-ALGEBRAS

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Abstract. In this article, we introduce the notion of cyclic BCK-algebra and study some of its main properties. In addition, we obtain a structure theorem for bounded commutative of finite order and use it to prove a Lagrange-like theorem for the above class of algebras. Finally, we use the notion of invertible elements to obtain a new characterization of implicative BCK-algebras and study the intersection of all maximal ideals of bounded BCK-algebras.

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1. Introduction

Introduced in the mid 60's by Iseki and Imai, the theory of BCK-algebras has been the object of intense development for the past four decades. BCK-algebras are in many respect similar to rings, and most of the theory has been developed in perfect agreement with the classical ring theory. For instance, there is a notion of ideal in BCK-algebras, and they have been studied extensively ([1], [5], [7]). The notion of zero-divisor was introduced in [1] and used to characterize prime ideals. In the same line of ideas, we introduced the notion of invertible elements a BCK-algebra and used it to characterize maximal ideals [2].

In the present work, we introduced Cyclic BCK-algebras, which seems to be structurally the simplest, as they are representable using only one element. We explore some of the applications of invertible elements. We prove that for every order, there exists a unique (up to isomorphism) cyclic BCK-algebra of that order. This approach gives a new description and characterization of bounded commutative BCK-chains of finite order as treated in [4]. We also prove that sub-algebras of a cyclic BCK-algebra are also cyclic. Using the newly introduced notion and some results from [4], we obtain a structure theorem for bounded commutative BCK-algebras. As a consequence of this theorem, we obtain a Lagrange-like theorem for finite bounded commutative BCK-algebras. A new characterization of implicative algebras among bounded commutative algebras in terms of invertible elements is also established. In the final section, we revisit the notion of J-semisimple for bounded BCK-algebras as introduced in 1978 by J. Ahsan and E. Deeba (for details on J-semisimple see [3, §5.3]). The notion J-semisimplicity for BCK-algebras corresponds to the well known semisimplicity in classical ring theory. Having introduced the notion of invertibility, we are able to use this to offer a fresh treatment of J-simisimplicity in BCK-algebras. Among other things, we prove that every finite bounded commutative BCK-algebra is J-semisimple.

2. Preliminaries and notations

A *BCK-algebra* is a set X with a binary operation $\star$ and a constant 0 satisfying the following axioms:

(i) $(x \star y) \star (x \star z) \leq z \star y$

(ii) $x \star (x \star y) \leq y$

(iii) $x \leq x$

(iv) $0 \leq x$

(v) $x \leq y$ and $y \leq x$ implies that $x = y$

where $x \leq y$ if and only if $x \star y = 0$. A key property of BCK-algebras as proved in [[6], Eq.3] is the following:

$$(x \star y) \star z = (x \star z) \star y \tag{3}$$

In addition, if there is an element $1 \in X$ such that $x \leq 1$ for all $x \in X$, then X is said to be *bounded* and we set $Nx = 1 \star x$. If we denote $x \wedge y$ by $y \star (y \star x)$, then X is said to be *commutative* if $x \wedge y = y \wedge x$ for all $x, y \in X$. A BCK-algebra is *implicative* if $x \star (y \star x) = x$ for all $x, y \in X$.

A subset I of a BCK-algebra X is an *ideal*, if $0 \in I$ and if $x, y \star x \in I$, then $y \in I$. Let A be a subset of a BCK-algebra X, the ideal generated by A, denoted by $< A >$ is the set of all $x \in X$ such that $(\dots(x \star a_1) \star a_2) \star \dots a_n) = 0$ for some $a_1, a_2, \dots a_n \in A$ [7, Theorem 3]. If $A = \{a\}$, we denote $< A >$ by $< a >$ the ideal generated by a. We denote the expression $(\ldots(x \star a) \star a) \star \ldots a)$ by $x \star a^n$, where n is the number of times a appears in the expression, in particular $x \star a^0 = x$. In which case $< a >= \{x \in X : \exists n > 0, x \star a^n = 0\}$. Let I be an ideal of X. The relation $\sim$ defined on X by $x \sim y$ if and only if $x \star y, y \star x \in I$ is an equivalence relation. Let C_x denote the class of $x \in X$ and X/I the set of equivalence classes C_x, $x \in X$. It is clear that $C_0 = I$. Define on X/I a binary operation $\star$ given by $C_x \star C_y = C_{x \star y}$ and $C_x \leq C_y$ if and only if $x \star y \in I$. Then X/I together with $\star$ and its constant I is a BCK-algebra called the quotient BCK-algebra of X determined by I. The maximal ideals of BCK-algebras have the usual meaning. A proper ideal P of a commutative BCK-algebra X is called *prime* if $a \wedge b \in P$ implies $a \in P$ or $b \in P$.

If X and Y are BCK-algebras, a homomorphism from X to Y is a map $f : X \to Y$ such that $f(x \star y) = f(x) \star f(y)$.

If X and Y are bounded, we require that homomorphisms between X and Y map 1 to 1. By isomorphism between two BCK-algebras, we mean a bijective homomorphism. We recall the following definitions and results which are mainly from [2].

Definition 2.1 Let X be a BCK-algebra. An $u \in X$ is *invertible* if $< u >= X$.

Example 2.2

1. It is clear that 1 is always an invertible element of any bounded BCK-algebra as $\langle 1 \rangle = X$.

2. Examples of nontrivial invertible elements of a BCK-algebra can be found under this section (see Example 2.5).

We denote by $\mathcal{U}_X$ the set of units of a bounded BCK-algebra X or simply $\mathcal{U}$ when there is no risk of confusion.

Proposition 2.3 *If X is a bounded BCK-algebra, then u is invertible in X if and only if there exists $n > 0$ such that $1 \star u^n = 0$.*

Proof. If u is an invertible element of X, then $1 \in< u >= X$. So there exists $n > 0$ such that $1 \star u^n = 0$. Conversely, if there exists $n > 0$ such that $1 \star u^n = 0$, then $1 \in< u >$ and hence $< u >= X$. ∎

In [7], Iseki and Tanaka introduced *simple* BCK-algebras as BCK-algebras having only two ideals, $\{0\}$ and X. A trivial example of simple bounded BCK-algebra is $X = \{0, 1\}$. We will provide more examples later. We have the following easy characterization of simple algebras using invertible elements.

Lemma 2.4 *A BCK-algebra X is simple if and only if all non-zero elements of X are invertible.*

Proof. Suppose X is simple. Then $< x >= X$ for all $x \in X \setminus \{0\}$. So x is invertible. Conversely, if all $x \in X \setminus \{0\}$ are invertible, it is clear that X is simple. ∎

Example 2.5

1. **Example of a simple non-commutative bounded BCK-algebra**
   For $n > 1$, consider $X = \{a_1, a_2 \ldots, a_n\}$ and order X by $a_i \leq a_{i+1}$ for all $i \geq 1$. Define $\star$ on X by:

$$a_i \star a_j = \begin{cases} 0 & : \ i \leq j \\ a_2 & : \ 1 < j < i \\ a_i & : \ j = 1 \end{cases}$$

   Then X is a bounded non-commutative simple BCK-algebra. For the simplicity, note that for every $i > 1$, $a_n \star a_i^2 = 0$, hence a_i is invertible. Therefore, every non-zero element of X is invertible, and by Lemma 2.4, X is simple.

2. **Example of a simple commutative bounded BCK-algebra**
   For $n > 1$, consider $\mathbb{X}_n = \{a_0, a_1, a_2 \ldots, a_{n-1}\}$ and order X by $a_i \leq a_{i+1}$ for all $i \geq 0$. Define $\star$ on X by: $a_i \star a_j = 0$ if $i \leq j$ and $a_i \star a_j = a_k$ if $i - j = k > 0$. Then $\mathbb{X}_n$ is a bounded commutative simple BCK-algebra.

3. Cyclic BCK-algebras

Recall that the order of a BCK-algebra X is the cardinality of X and is often denoted by $o(X)$ or $|X|$. We start by introducing the order of elements in bounded BCK-algebras.

Definition 3.1 Let X be a bounded BCK-algebra with unit 1. The *order* of $x \in X$, denoted by $o(x)$, is the smallest positive integer n such that $1 \star x^n = 0$. If such n does not exist, we say that x has infinite order.

It is obvious that 1 is the only invertible element of order 1. It is also clear from the definition that an $x \in X$ has finite order if and only if x is invertible. If X is a bounded implicative different than $\{0, 1\}$, then each $x \in X \setminus \{1\}$ has infinite order. This is because, in a bounded implicative BCK-algebra, we have $1 \star x^n = 1 \star x$ for all $n \geq 1$.

Lemma 3.2 *Let u be an invertible element of X of order n. Then the following elements $0, 1, 1 \star u^i$, $i = 1, \ldots, n-1$, of X are pairwise distinct.*

Proof. By contradiction, suppose that $1 \star u^i = 1 \star u^j$ for some $0 \leq i < j \leq n$. Then

$$\begin{aligned} 0 &= 1 \star u^n \\ &= (1 \star u^j) \star u^{n-j} \\ &= (1 \star u^i) \star u^{n-j} && \text{since } 1 \star u^i = 1 \star u^j \\ &= 1 \star u^{n-j+i} \end{aligned}$$

But $n - j + i < n$ which is a contradiction with the order of u being n. So, $1 \star u^i \neq 1 \star u^j$ for $1 \leq i < j \leq n$, as required. ■

Next, we give a result that provides an upper bound for the orders in finite BCK-algebras.

Proposition 3.3 *Let X be a finite bounded BCK-algebra. Then for every invertible element u of X, $o(u) < o(X)$.*

Proof. This is immediate from Lemma 3.2. ■

Remark 3.4 The upper bound given in Proposition 3.3 is the sharpest possible. In fact consider the BCK-algebra of Example 2.5 2, then it is easy to see that a_1 has order n.

We introduce the following definition.

Definition 3.5 A *cyclic* BCK-algebra X is a finite bounded BCK-algebra with an invertible element $u \in X$ such that $o(u) = o(X) - 1$.

The above definition is motivated by the following result.

Proposition 3.6 *Let X be cyclic BCK-algebra of order n. Then, there exists $u \in X$ invertible such that*

$$X = \{1 \star u^i : i = 0, 1, 2, \ldots, n-1\}$$

where $1 \star u^0 = 1$.

Proof. Since X is cyclic of order n, there exists an invertible element $u \in X$ of order $n - 1$. It follows from Lemma 3.2 that the elements $1 \star u^i$'s are pairwise distinct. Therefore, $|\{1 \star u^i : i = 0, 1, 2, \ldots, n-1\}| = n$. Thus as $|X| = n$, we obtain that $X = \{1 \star u^i : i = 0, 1, 2, \ldots, n-1\}$ as needed. ■

Recall that if X is a BCK-algebra and $a \in X \setminus \{0\}$, then a is called an *atom* of X if for every $x \in X$, if $x \leq a$, then $x = 0$ or $x = a$. We start with the following easy lemma.

Lemma 3.7 *Every Cyclic BCK-algebra X is a BCK-chain. Furthermore, the atom of X is the generator for X, in particular the generator is unique.*

Proof. Let X be a cyclic algebra of order n, by Proposition 3.6, there exists $u \in X$ such that $X = \{1 \star u^i : i = 0, 1, 2, \ldots, n-1\}$. Note that $i \geq j$ implies $1 \star u^i \leq 1 \star u^j$, therefore,

$$0 = 1 \star u^{n-1} \leq 1 \star u^{n-2} \leq \cdots \leq 1 \star u \leq 1 \star u^0 = 1$$

Hence X is a chain.

It remains to show that $1 \star u^{n-2} = u$. Note that, since $u \neq 0$, and $X = \{1 \star u^i : i = 0, 1, 2, \ldots, n-1\}$, then $u = 1 \star u^i$ for some $i = 0, 1, \ldots, n-2$. But since $u \star u = 0$, then $1 \star u^{i+1} = 0$, thus by the definition of order, we get $n - 1 \leq i + 1$. Hence $i \geq n - 2$, thus $i = n - 2$ and $u = 1 \star u^{n-2}$ as needed. Note that $1 \star u^{n-2}$ is the atom of X. ■

Remark 3.8 It is straightforward to see that properties such as cyclicity, commutativity, and simplicity are preserved by BCK-isomorphisms. It is also clear that orders of elements are preserved by BCK-isomorphisms as well.

Theorem 3.9 *Every Cyclic BCK-algebra of order n is isomorphic to the BCK-algebra $\mathbb{X}_n$ of Example 2.5 2. In particular, every cyclic BCK-algebra is simple and commutative.*

Proof. From Lemma 3.7, we have $X = \{0 \leq u \leq 1 \star u^{n-3} \cdots \leq 1 \star u \leq 1\}$. For simplicity, we denote $1 \star u^{n-i-1}$ by x_i, so $x_0 = 0$, $x_1 = u$, and so on.

We need to show that $x_i \star x_j = 0$ if $i \leq j$ and $x_i \star x_j = x_k$ if $i - j = k > 0$.

It is clear that $x_i \star x_j = 0$ if $i \leq j$.

It remains to show that $x_i \star x_j = x_k$ if $i - j = k > 0$.

For this, we first prove the case $k = 1$, that is $i = j + 1$.

We have

$$\begin{aligned}(x_i \star x_j) \star x_1 &= (x_i \star x_j) \star u \\ &= (x_i \star u) \star x_j \\ &= x_j \star x_j \qquad \text{since } x_j = x_i \star u \\ &= 0\end{aligned}$$

Hence $x_i \star x_j \leq u$, so $x_i \star x_j = 0$ or $x_i \star x_j = u$. But $x_i \star x_j = 0$ implies $x_i = x_j$ which is impossible. Therefore, $x_i \star x_j = u = x_1$ as needed.

For the general step, assume $k > 1$ and $i = j + k$. For simplicity, name $x_i \star x_j$ by a. Since $X = \{1 \star u^i : i = 0, 1, 2, \ldots, n-1\}$, we know that $a = 1 \star u^r$ for some $r = 0, 1, \ldots, n-1$.

On the other hand, we have

$$\begin{aligned}a \star u^{k-1} &= (x_i \star x_j) \star u^{k-1} \\ &= (x_i \star u^{k-1}) \star x_j \\ &= x_{j+1} \star x_j \qquad \text{since } x_{j+1} = x_i \star u^{k-1} \\ &= u \qquad \text{by the step above}\end{aligned}$$

Thus $u = a \star u^{k-1} = 1 \star u^{r+k-1}$, therefore $1 \star u^{r+k} = u \star u = 0$, thus $n - 1 \leq r + k$. Hence $r \geq n - k - 1(1)$. On the other hand, since $u \neq 0$ and $u = 1 \star u^{r+k-1}$, then $r + k - 1 \leq n - 2$ as $1 \star u^s = 0$ for all $s \geq n - 1$. Therefore, $r \leq n - k - 1$, which coupled with (1) imply that $r = n - k - 1$. Thus, $x_i \star x_j = a = 1 \star u^{n-k-1} = x_k$ as required.

It is now clear that $x_i \mapsto a_i$ is an isomorphism between X and $\mathbb{X}_n$, so $X \cong \mathbb{X}_n$ as needed.

Now, since $\mathbb{X}_n$ is commutative and simple, it follows from Remark 3.8 that X is commutative and simple. ∎

Remark 3.10 As stated in Theorem 3.9, cyclic BCK-algebras are commutative and simple. Note however that a finite bounded simple BCK-algebra needs not be cyclic [e.g., Example 2.5(1)]. Neither is every finite bounded commutative BCK-algebra cyclic [e.g., B_{4-2-3} from [3]], but every finite bounded commutative and simple BCK-algebra is cyclic as we now prove.

Proposition 3.11 *A finite bounded BCK-algebra is cyclic if and only if it is commutative and simple.*

Proof. Note that the necessity is from Theorem 3.9 and the sufficiency is an immediate consequence of [Cor. 2.3.2, Thm. 2.3.3] from [3]. ∎

A combination of [4, Theorem 7], [4, Theorem 8] and Proposition 3.11 yields the following structure result for bounded commutative BCK-algebras.

Theorem 3.12 *Every finite non-zero bounded commutative BCK-algebra is a product of cyclic BCK-algebras.*

It follows from Theorem 3.12 that every bounded commutative BCK-algebra of prime order is cyclic. This result is for bounded commutative BCK-algebras what the fundamental theorem of finite Abelian groups is for Abelian groups.

Proposition 3.13 *Every non-zero sub-algebra of a cyclic BCK-algebra is cyclic.*

Proof. Let X be a cyclic BCK-algebra algebra and S a non-zero sub-algebra of X. Then, by Theorem 3.9, X is commutative and simple. Therefore, by Theorem 2.3.8 of [3], S is simple. But being a sub-algebra of a commutative algebra, S is also commutative. On the other hand by Lemma 3.7, X is a finite BCK-chain, thus S is bounded. Therefore, S is bounded, commutative and simple, hence cyclic by Proposition 3.11. ∎

We have the following result that can rightfully be called the Lagrange's theorem for bounded commutative BCK-algebras.

Proposition 3.14 *Let X be a bounded commutative BCK-algebra of finite order. Then, for every ideal I of X,*

$$|X| = |I||X/I|$$

In particular, the order $|I|$ of I divides $|X|$.

Proof. First, note that from Theorem 3.12, there exist simple algebras X_i, i=1, ..., n, such that $X=\prod_{i=1}^{n} X_i$. For each k=1, ..., n, let $I_k:=\{a \in X_k : \iota_k(a) \in I\}$, where $\iota_k : X_k \to X$ is the natural inclusion.

Claim. $I = I_1 \times \cdots \times I_n$.

Let $x = (x_1, \ldots, x_n) \in I$. Then for each $k = 1, \ldots, n$, $x \star \widehat{x} = \iota_k(x_k)$, where $\widehat{x}$ is obtained from x by keeping every coordinate, but replacing the k^{th}-coordinate by 0. Since $x \in I$, then $x \star \widehat{x} \in I$ and therefore $\iota_k(x_k) \in I$ for all $k = 1, \ldots, n$. Thus, $x_k \in I_k$ for all $k = 1, \ldots, n$ and hence $x \in I_1 \times \cdots \times I_n$.

Conversely, if $x = (x_1, \ldots, x_n) \in I_1 \times \cdots \times I_n$, then

$$(\ldots((x \star \iota_n(x)) \star \iota_{n-1}(x)) \ldots \star \iota_1(x)) = 0 \in I.$$

Since $\iota_k(x) \in I$ for all $k = 1, \ldots, n$ and I is an ideal of X, one obtains that $x \in I$. In addition, since each X_k is simple, then $I = I_1 \times \cdots \times I_n$ with $I_k = \{0\}$ or X_k.

Now, let $A = \{k \in \{1, \ldots, n\} | I_k = \{0\}\}$, so $I_k = X_k$ for all $k \notin A$. For every $x = (x_k) \in X$, it is easy to see that $C_x = \{(a_k) \in X | a_k = x_k \text{ for all } k \in A\}$. It follows that the map $(a_k) \mapsto (b_k)$ (where $b_k = 0$ for $k \in A$ and $b_k = a_k$ otherwise) is a bijection between C_x and I. Therefore, for every $x \in X$, $|C_x| = |I|$. Since $X = \biguplus_{X/I} C_x$, then

$$|X| = \sum_{X/I} |C_x| = \sum_{X/I} |I| = |I||X/I|.$$

Hence $|X| = |I||X/I|$, in particular $|I|$ divides $|X|$ as required. ∎

Remark 3.15 It is important to point out that to prove Proposition 3.14, we have just proved that any ideal I of X is the product of ideals coming from each component X_i, but this is not true in general for infinite product. In fact if we consider the direct sum of algebras, it is an ideal of the direct product that is not the direct product of ideals.

Remark 3.16 Note that Proposition 3.14 holds for any BCK-algebra that is a product of finite simple algebras as this was the only aspect of finite bounded commutative BCK-algebras used in the proof.

Proposition 3.17 *Let $X_1, X_2, \ldots, X_n$ be bounded BCK-algebras and $X = \prod_{i=1}^{n} X_i$. Then*

$$\mathcal{U}_X = \times_{i=1}^{n} \mathcal{U}_{X_i}$$

Proof. Let $x = (x_1, \ldots, x_n) \in X$ such that each x_i is invertible in X_i. Then, for each i, there exists n_i such that $1 \star x_i^{n_i} = 0$. If $m = \max\{n_i\}$, then $1 \star x^m = 0$, so x is invertible. The reverse inclusion is even simpler. ∎

Remark 3.18 Proposition 3.17 is not true in general for infinite product. For instance, consider the real interval $X_i = [0, 1]$ with $\star$ defined as $x \star y = \max\{0, x - y\}$ for each $i \in \mathbb{N}$. Now, let $X = \prod_{i\in\mathbb{N}} X_i$ and $x = (1/i)_{i\in\mathbb{N}} \in X$. Now, for each i, $1/i$ is invertible in X_i as X_i is simple, but we claim that x is not invertible. Suppose that x is invertible. Then, there is a positive integer n such that $1 \star x^n = 0$, i.e., for all $i \in \mathbb{N}$, $1 \star (1/i)^n = 0$ $(*)$. But there exists an $i_0 > n$ such that $(k + 1)/i_0 < 1$ for each $k = 1, ..., n$. Thus, $1 - k/i_0 > 1/i_0$ for each $k = 1, \ldots, n$. So then $1 \star (1/i_0)^n = 1 - n/i_0 \neq 0$, which is a contradiction with $(*)$.

Most of the theory of BCK-algebras studied so far in this article is in perfect agreement with the classical ring theory. But this is a major difference between the two theories because the units of direct product of rings is the product of units.

We obtain the following characterization of implicative BCK-algebras using invertible elements.

Proposition 3.19 *Let X be a non-zero bounded commutative BCK-algebra. Then X is implicative if and only if $\mathcal{U}_X = \{1\}$.*

Proof. Note that if X is a non-zero bounded implicative, and $x \in \mathcal{U}_X$, then there exists $n \geq 1$ such that $1 \star x^n = 0$. But since $1 \star x^n = 1 \star x$ [6, Prop. 6], hence $x = 1$ and $\mathcal{U}_X = \{1\}$.

Conversely, suppose that X is a non-zero bounded commutative BCK-algebra such that $\mathcal{U}_X = \{1\}$. Since X is commutative and bounded, in order to show that X is implicative, it is enough by [6, Prop. 6] to show that $x \vee Nx = 1$ for all $x \in X$. Let $x \in X$, then since $Nx \leq x \vee Nx$, we have $1 \star (x \vee Nx) \leq 1 \star Nx = 1 \wedge x = x$. Hence, $1 \star (x \vee Nx) \leq x$, which implies $1 \star (x \vee Nx)^2 \leq x \star (x \vee Nx) = 0$ as $x \leq x \vee Nx$. Therefore, $1 \star (x \vee Nx)^2 = 0$, hence $x \vee Nx \in \mathcal{U}_X = \{1\}$ and $x \vee Nx = 1$ as needed. ∎

Remark 3.20 Proposition 3.19 is not true in general for bounded non-commutative algebras. In fact there are bounded positive implicative algebras X with $\mathcal{U}_X = \{1\}$ that are not implicative, see for example the BCK-algebra B_{5-2-12} from [3].

4. J-Semisimple BCK-algebras

Given a bounded BCK-algebra X, we denote by $J(X)$ the intersection of all maximal ideals of X.

Definition 3.1 A bounded BCK-algebra X is *J-semisimple* if $J(X) = 0$.

Example 3.2

1. For every $n \geq 1$, $J(\mathbb{X}_n) = 0$, because $\mathbb{X}_n$ is simple. So, for all n, $\mathbb{X}_n$ is J-semisimple.

2. Let X is an unbounded BCK-algebra, and $\widehat{X}$ be the Iséki's extension i.e., the BCK-algebra obtained by adjunction of a unit [see [5], Section II]. Then, since X is the unique maximal ideal of $\widehat{X}$, then $J(\widehat{X}) = X$. Thus, $\widehat{X}$ is not J-semisimple.

3. [3, Ex.1.2.2] In this example, $\mathbb{Z}^+$ denotes the set of all non-negative integers. Let $A := \{a_n : n \in \mathbb{Z}^+\}$ such that $A \cap \mathbb{Z}^+ = \emptyset$ and let $X = A \cup \mathbb{Z}^+$. Define a binary operation $\star$ on X as follows: For every $m, n \in \mathbb{Z}^+$,

$$\begin{aligned} m \star n &= \max\{0, m-n\} \\ m \star a_n &= 0 \\ a_m \star n &= a_{m+n} \\ a_m \star a_n &= n \star m \end{aligned}$$

   Then, it is easy to see that X is a bounded commutative BCK-algebra with unit a_0. In fact, the BCK-order on X is given by the chain

$$0 \leq 1 \leq \cdots a_2 \leq a_1 \leq a_0.$$

One can verify that $\mathcal{U}_X = A$ and that $\mathbb{Z}^+$ is the ideal of X generated by 1. Therefore, by [2, Thm. 4.3], X is a local BCK-algebra with unique maximal ideal $\mathbb{Z}^+$. In fact, it is quite straightforward to see that X has only three ideals: $0, \mathbb{Z}^+$ and X. In either case, $J(X) = \mathbb{Z}^+$ and X is not J-semisimple.

4. Let $X = \prod_{n\in\mathbb{N}} \mathbb{X}_n$. Since each $\mathbb{X}_n$ is simple, then by Proposition 3.7 below, X is an infinite J-semisimple BCK-algebra.

Recall that in ring theory the Jacobson radical of a ring is the intersection of all maximal ideals of the ring. So the notion of Jacobson radical coincides with $J(X)$ in the context of bounded BCK-algebras. Motivated by the fact that the Jacobson radical of a ring is characterized in terms of units of the ring, we give a characterization of $J(X)$ in terms of invertible elements of any bounded BCK-algebra X.

Proposition 3.3 *Let X be a bounded BCK algebra with unit 1. Then*

$$J(X) = \{x \in X : 1 \star x^n \in \mathcal{U}_X \text{ for all } n \in \mathbb{N}\}$$

Proof. First note that for any $x \in X$, if there exists $n \geq 1$ so that $1 \star x^n \notin \mathcal{U}_X$, then there exists a maximal ideal M of X such that $1 \star x^n \in M$. Therefore, since $1 \notin M$, it follows that $x \notin M$. Hence $x \notin J(X)$.

Conversely, suppose that $1 \star x^n$ is invertible for all n. Suppose that there is a maximal ideal M such that $x \notin M$. Then since X/M is simple [2, Prop. 3.6], it follows by [2, Lemma 3.5] that $C_x \in X/M$ is invertible. So there is an m such that $C_{1\star x^m} = C_0$. Thus $1 \star x^m \in M$ a contradiction. ∎

Corollary 3.4 *Let X be a bounded BCK-algebra that is J-semisimple and A be a subalgebra of X. If $1 \in A$, then A is J-semisimple.*

Proof. Since $1 \in A$, it follows from Proposition 3.3 that $J(A) \subseteq J(X) = 0$, hence $J(A) = 0$. ∎

Recall that a bounded BCK-algebra X is *multiply implicative* if, for all $x, y \in X$, there is a positive integer $n = n(x, y)$ such that $x \star (y \star x^n) = x$. In particular, any bounded implicative BCK-algebra is multiply implicative.

Corollary 3.5 *Every bounded multiply implicative BCK-algebra is J-semisimple.*

Proof. Let X be a bounded multiply implicative BCK-algebra and let $x \in J(X)$. Then there is an $n = n(x, 1)$ such that $x \star (1 \star x^n) = x$. Letting $u = 1 \star x^n$, we have $x \star u = x$ and so $u \in I := \{y \in X : x \star y = x\}$. It is not hard to see that I is an ideal of X. In fact, $x \star 0 = x$ so $0 \in I$. Now let $y \star z, z \in I$. On one hand, we know that $x \star y \leq x$, on the other hand, $x = x \star (y \star z) = (x \star z) \star (y \star z) \leq x \star y$ [3, Cor.1.1.6]. Thus $x = x \star y$ and $y \in I$ as needed.

But then, by Proposition 3.3, $u = 1 \star x^n$ is invertible and it then follows that $I = X$. Thus $1 \in I$ and $x = x \star 1 = 0$. Hence $J(X) = 0$ and X is J-semisimple. ∎

Remark 3.6 It is well known that the converse of Corollary 3.5 is not true (see Example 5.3.1 [3]).

Proposition 3.7 *Let $(X_i)_{i\in I}$ be a family of bounded simple BCK-algebras and $X = \prod_{i\in I} X_i$ the direct product of the family $(X_i)_{i\in I}$. Then X is J-semisimple.*

Proof. We claim that $J(X) \subseteq \prod_{i\in I} J(X_i)$ in which case $J(X) = 0$ as each $J(X_i) = 0$ since, for each $i \in I$, X_i is a bounded simple BCK-algebra. To prove the claim, let $x = (x_i)_{i\in I} \in J(X)$. Then, by Proposition 3.3, $1 \star x^n = (1 \star x_i^n)_{i\in I}$ is invertible for all positive integer n. By definition of invertible, it is clear that if $y = (y_i)_{i\in I} \in X$ is invertible, then for each $i \in I$ y_i is invertible in X_i. So then, for each $i \in I$, $1 \star x_i^n$ is invertible for all positive integer n in X_i. So $x_i \in J(X_i)$. ■

Remark 3.8

1. The same argument in the proof of proposition 3.7 will show that a direct product of J-semisimple BCK-algebras is again J-semisimple.
2. To prove Proposition 3.7, we used the fact that $J(\prod_{i\in I} X_i) \subseteq \prod_{i\in I} J(X_i)$. In fact, for finite product, one can easily, combining Proposition 3.17 and Proposition 3.3 , see that

$$J\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} J(X_i).$$

An open question is whether equality still holds for infinite product.

Since every bounded commutative BCK-algebra of finite order is a product of simple BCK-algebras [Theorem 3.12] we obtain as a consequence of Proposition 3.7 the following:

Corollary 3.9 *Every bounded commutative BCK-algebra of finite order is J-semisimple.*

Remark 3.10

1. The Iséki's extension $\widehat{X}$ in Example 3.2.2. is bounded non commutative and may be chosen to be finite, but $J(\widehat{X}) = X \neq 0$.
2. If a finite bounded local BCK-algebra is not simple, then it is non-commutative.
3. It is worth pointing out that Corollary 3.9 is not true in general for infinite algebras. In fact, there exists infinite bounded commutative BCK-algebras that are not J-semisimple. For instance, the BCK-algebra X of Example 3.2.3. is bounded and commutative, but with $J(X) \neq 0$.

4. Also there are bounded BCK-algebras that are J-semisimple but not commutative nor simple. For instance picking $X = B_{5-4-1} \times B_{5-4-2}$ it is clear that $J(X) = 0$ and that X is not commutative and not simple. In fact, taking any product of non commutative simple BCK-algebras would provide such examples.

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