SUPER PRINCIPAL FIBER BUNDLE WITH SUPER ACTION

M.R. Farhangdoost
Department of Mathematics
College of Sciences
Shiraz University
Shiraz, 71457-44776
Iran
e-mail: farhang@shirazu.ac.ir

Abstract. We introduce super action for supermanifolds to devoted to principal fiber bundle with structural generalized Lie groups. We present a product super fiber bundle, also we extend the coordinate bundle in the sense of Steenrod to show that super coordinate bundles are equivalent if and only if their super actions agree.

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1. Introduction

In physics and mathematics, supermanifolds are generalizations of the manifold concept based on ideas coming from supersymmetry, also a super Lie group is a group object in the category of supermanifolds. An affine super algebraic group is a group object in the category of affine supervarieties.

In this paper we introduce super action for manifolds and supermanifolds to devoted to principal fiber bundle with structural generalized Lie groups, also we extend coordinate bundle in the sense of **Steenrod** ([7]), for the generalized Lie group T. We show that our definition of super fiber bundle is an equivalence class of super coordinate bundles, which is another way of saying that super coordinate bundles are equivalent if and only if their super actions agree. Therefore, we have a set of super transition functions defined for an open covering of a manifold M, which determine a super principal bundle whose transition maps relative to the covering are old super transitions. Note that this generalization is a different structure from fiber bundle of top spaces introduced in ([6,2]). The concept of generalized Lie groups is a different structure from super Lie group, but results may be useful for researchers who working in $Superspace\ time$ and $Quantum\ field\ theory$. Moreover, we present a useful theorem of super product fiber bundle.

Definition 1.1. Let M be a real manifold and * be an involution over the fiber turning it into a * algebra. Then the resulting algebra is called a real supermanifold.

Note that, in physics the points of this algebra isn't a point set space and so doesn't "really" exist. A supermanifold is a concept in noncommutative geometry.

Example 1.2. Each supermanifold is a real C^∞ manifold.

Now, we present the concept of generalized Lie group(top space), which is both a generalized group and a manifold, such that the generalized group operations are C^∞. The several authors studied various aspects and concepts of generalized group, well known from standard groups, [4,5].

Definition 1.3. ([6],[1]) A non-empty Hausdorff smooth d-dimensional manifold T is called a top space if there is an associative action "$\cdot$" on T such that $t \cdot s \in T$, for every $t, s \in T$, and satisfies in the following conditions (Note that $t \cdot s$ showed that by ts.):

(i) For each $t \in T$, there is a unique $e(t) \in T$ such that $te(t) = e(t)t = t$;
(ii) For each $t \in T$, there is $s \in T$ such that $ts = st = e(t)$;
(iii) For all $t, s \in T$, $e(ts) = e(t)e(s)$;
(iv) The mappings

$$\begin{array}{rcl} m_1 : T \times T & \longrightarrow & T \\ (t,s) & \longmapsto & ts \end{array} \quad \text{and} \quad \begin{array}{rcl} m_2 : T & \longrightarrow & T \\ t & \longmapsto & t^{-1} \end{array}$$

are smooth mappings.

Note that, $e(t)$ is called the identity of t. Moreover, s is called the inverse of t, and denoted by t^{-1}.

Example 1.4. ([6]) Each Lie group is top space.

There is a Lie groups which is not top space, for example:

Example 1.5. ([6]) The $n-$torus $\mathbb{T}^n = \frac{\mathbb{R}}{\mathbb{Z}}$ with the product:

$$((r^1, ..., r^n) + \mathbb{Z}, ((s^1, ..., s^n) + \mathbb{Z}) \mapsto ((r^1 + s^1, ..., r^{n-1} + s^{n-1}, r^n) + \mathbb{Z})$$

is a top space, which is not a Lie group.

2. Super action of top spaces

All of us knew how important actions of Lie groups on manifolds are, in this section at first we introduce a super action for supermanifolds and generalized Lie groups as a generalization of usual action for Lie groups, and then we shall introduce the notion of a super principal fiber bundle, also we shall present on the most important class of super fiber bundle, moreover we shall make a super coordinate bundle, in the sense of Steenrod, to find a balance with respect to the super actions.

Definition 2.1. ([2]) A super action(or generalized action) of top space T on supermanifold (or manifold) M is differentiable map $\lambda : M \times T \longrightarrow M$ which satisfies the conditions:

(i) For any $m \in M$, there is $e(t)$ in T such that $\lambda(m, e(t)) = m$;

(ii) $\lambda(\lambda(m, t_1), t_2) = \lambda(m, t_1 t_2)$ for all $t_1, t_2 \in T$ and $m \in M$.

In this case, we called T *superacts* on M.

Remark. We often use the notion tm instead of $\lambda(m, t)$, so the second condition in this notation is $(mt_1)t_2 = m(t_1 t_2)$.

Note that, if T is a Lie group, then a super action is an action (C^∞ action).

Example 2.2. $T = R \times R - \{0\}$ with product $(a, b).(e, f) = (be, bf)$ is top space [Remark: $e((a, b)) = (a/b, 1)$ and $(a, b)^{-1} = (a/b^2, 1/b)$]. The map $\lambda : R \times T \longrightarrow R$ defined by $\lambda(c, (a, b)) = ac/b$ is a super action.

T is said to super acts *effectively*, if $m \cdot t = m$, for all $t \in T$, implies that $t \in e(T)$. Also, T supper acts *transitivity* to the right if for every $m, n \in M$, there is $t \in T$ such that $mt = n$.

T super acts *freely* if the only elements of T having fixed point on M are belong to the identity set $e(T)$, where $e(T)$ the set of all identity elements in T, also the set $O(m) = \{mt | t \in T\}$ is called the *orbit* of m.

Now, we introduce a product super fiber bundle:

Definition 2.3. A super principal fiber bundle is a set (P, T, M), where P, M are differentiable manifolds, T is a top space such that:

(i) T super acts freely to the right on P;

(ii) M is the quotient space of P, by equivalence under T, and projection $\pi : P \to M$ is differentiable;

(iii) For every $m \in M$, there is an open set U of m and differentiable map $F_U : \pi^{-1}(U) \to T$ such that $F_U(pt) = e(t)F_U(p)t$, for all $t \in T$ and $p \in U$.

Also, the map of $\pi^{-1}(U) \to U \times T$ given by $p \mapsto (\pi(p), F_U(p))$ is a diffeomorphism. In this case, P is called the super bundle space, M the base space, and T the structural top space.

Note that, we can easily extend this definition for spurmanifolds P and M, moreover, if T is a Lie group, then the concept of super principal fiber bundle is the concept of principal fiber bundle, it clear that, there is a super principal fiber bundle which is not a principal fiber bundle (see Example 1.5). Now, by Definition 2.3 we make a product super bundle, which is very important case in the sense of Steenrod ([6]):

Theorem 2.4. *Let T be a top space, M a manifold (or supermanifold), and $P = T \times M$, also P provided with the right super action of T on itself in the second factor, then (P, T, M) is a super principal fiber bundle.*

Proof. At first, we define a suitable super action:

Define $P \times T \to P$ by $((m,t),t') \mapsto (m, e(t')tt')$.

We show that this C^∞-map satisfies in the conditions of Definition 2.3.

Let $(m,t) \in P$, since $t \in T$, then $e(t) \in T$.

Moreover, $(m,t)e(t) = (m, e(e(t))te(t))$.

By uniqueness of identity in Definition 1.3, we have $e(e(t)) = e(t)$.

Then $(m, e(e(t))te(t)) = (m,t)$.

Moreover, $(((m,t),t_1),t_2) = ((m, e(t_1)tt_1), t_2) = (m, e(t_2)e(t_1)tt_1t_2)$.

By definition of top spaces, we knew that $e(t_1)e(t_2) = e(t_1t_2)$, so

$$(m, e(t_2)e(t_1)tt_1t_2) = (m, e(t_1t_2)tt_1t_2) = ((m,t), t_1t_2).$$

Therefore, T super acts on P.

Step I. Let $(m,t) \in M \times T$, and there exist $t' \in T$ such that

$$(m,t)t' = (m, e(t')tt') = (m,t).$$

Then $e(t')tt' = t$, then $e(e(t')tt') = e(t)$.

By uniqueness of identity, we have $e(e(t')tt') = e(t)$.

Then $e(t') = e(t))$, and so $e^{-1}(e(t')) = e^{-1}(e(t))$.

Therefore, $e(t')tt' = e(t)tt' = tt'$.

Since $e(t')tt' = t$, then $tt' = t$, and then, by multiplying t' from the left side, we have $t^{-1}tt' = t^{-1}t$. Then $e(t)t' = e(t)$.

Since $e(t) = e(t')$, then $e(t)t' = e(t')t' = t'$. Hence $t' = e(t)$.

Therefore, $t' \in e(T)$, thus T super acts freely on P.

Step II. Let $\sim$ be a relation on P, defined by:

$$(m_1,t_1) \sim (m_2,t_2) \text{ iff there is } t \in T \text{ such that } (m_1, e(t)t_1t) = (m_2,t_2),$$

this relation is an equivalence relation, because:

Let (m,t) belong to P. Since $e(t) \in T$ and $(m,t)e(t) = (m, e(t)te(t)) = (m,t)$, then $\sim$ is a reflexive relation.

Let (m_1,t_1), (m_2,t_2) belong to P, and $(m_1,t_1) \sim (m_2,t_2)$. Then there is $t \in T$ such that $(m_1, e(t)t_1t) = (m_2,t_2)$. Therefore,

$$(*) \qquad m_1 = m_2 \text{ and } e(t)t_1t = t_2.$$

By multiplying $e(t_1)$ from the left side, we have $e(t_1)e(t)t_1e(t) = e(t_1)t_2$. Since

$$\begin{aligned} e(t_1)e(t)t_1t &= e(t_1)e(t)e(t_1)t_1t; && \text{(because: } e(t_1)t_1 = t_1) \\ &= e(t_1)t_1t && \text{(because: } e(t_1)e(t)e(t_1) = e(t_1)) \\ &= t_1t \end{aligned}$$

and

$$t_1t = e(t_1)t_2 \qquad \text{(because: } e(t_1)e(t)t_1t = e(t_1)t_2,$$

then $t_1 t = e(t_1)t_2$).

By multiplying t^{-1} from the right side, we have: $t_1 e(t) = e(t_1)t_2 t^{-1}$.

By multiplying $e(t_1)$ from the right side, we have:

$$(**) \qquad t_1 e(t)e(t_1) = e(t_1)t_2 t^{-1} e(t_1).$$

We knew that

$$t_1 e(t)e(t_1) = (t_1 e(t_1))e(t)e(t_1) = t_1(e(t_1)e(t)e(t_1)) = t_1 e(t_1) = t_1.$$

Then, by $(**)$ we have $t_1 = e(t_1)t_2 t^{-1} e(t_1)$.
By $(*)$, we knew that $e(t) = e(t_2)$(because: $e(e(t)t_1 t) = e(t)$) and then

$$\begin{aligned} e(t_1)t_2 t^{-1} e(t_1) &= e(t_1)e(t_2)t_2 t^{-1} e(t_1) \\ &= e(t_1)e(t)t_2 t^{-1} e(t_1) && \text{(because: } e(t_2) = e(t)) \\ &= e(t_1)e(t^{-1})t_2 t^{-1} e(t_1) && \text{(because: } e(t^{-1}) = e(t)) \\ &= e((t^{-1}e(t_1))^{-1})t_2 t^{-1} e(t_1) && \text{(because: } (t^{-1}e(t_1))^{-1} = e(t_1)t) \\ &= e((t^{-1}e(t_1))t_2 t^{-1} e(t_1)). \end{aligned}$$

Then, there is $t^{-1}e(t_1) \in T$ such that:

$$t_1 = e((t^{-1}e(t_1))^{-1})t_2 t^{-1} e(t_1),$$

and so $(m_2, t_2) \sim (m_1, t_1)$, therefore $\sim$ is a symmetric relation.

Let (m_1, t_1), (m_2, t_2) and (m_3, t_3) belong to P, and $(m_1, t_1) \sim (m_2, t_2)$, $(m_2, t_2) \sim (m_3, t_3)$, then $m_1 = m_2$ and $m_2 = m_3$, respectively.

Also, there are $t', t'' \in T$ such that: $e(t')t_1 t' = t_2$ and $e(t'')t_2 t'' = t_3$.

Therefore, $e(t'')e(t')t_1 t' t'' = t_3$. Then $e(t''t')t_1 t' t'' = t_3$, and so

$$e((t''t')^{-1})t_1 t' t'' = t_3 \quad \text{(because: } e(s) = e(s^{-1}), \text{ for all } s \in T.)$$

Then $e((t')^{-1}(t'')^{-1})t_1 t' t'' = t_3$. Hence $e(t')e(t'')t_1 t' t'' = e((t')^{-1}(t'')^{-1})t_1 t' t'' = t_3$. Therefore, $e(t' t'')t_1 t' t'' = e(t')e(t'')t_1 t' t'' = t_3$. Then $(m_1, t_1) \sim (m_3, t_3)$, therefore $\sim$ is a transitive relation.

Step III. Now, we show that M is the quotient space of P by this equivalence relation under T.

Let $(m, t_1), (m, t_2)$ belong to P, since $e(t_2), (t_1)^{-1}t_2$ belong to T, and

$$\begin{aligned} e((t_1)^{-1}t_2)t_1(t_1)^{-1}t_2 &= = e(((t_1)^{-1}t_2)^{-1})e(t_1)t_2 \\ &= e((t_2)^{-1})e((t_1)^{-1})^{-1}e(t_1)t_2 \\ &= e(t_2)e(t_1)e(t_2)t_2 \end{aligned}$$

and, since $e(t_2)t_2 = t_2$ and $e(t^{-1})e(t_1) = e(t_1)$), we have

$$e(t_2)e(t_1)e(t_2)t_2 = e(t_2)t_2 = t_2.$$

Then

$$(m, t_1) \sim (m, t_2).$$

Also, by Definition 1.3, it is clear that the projection $\pi : P \to M$ is a C^∞ map.

Now, let $m \in M$ be given, it is clear that for each open set U of m in M, we have F_U commutes with right super action, i.e., $F_U(pt^{'}) = F_U(p)t^{'}$ for every $t^{'} \in T$ and $p \in P$, because:

$$\begin{aligned} F_U((m,t),t^{'}) &= F_U(m, e(t^{'})tt^{'}) \\ &= e(t^{'})tt^{'} \\ &= e(t^{'})F_U((m,t))t^{'}. \end{aligned}$$

Also, the map $p \mapsto (\pi(p), F_U(p))$ from $\pi^{-1}(U)$ into $U \times T$, which is the identity map, is a diffeomorphism.

Then (P, T, M) is a super fiber bundle, which is called product super fiber bundle. ∎

Now, we present the structure of super coordinate bundle with structural top space T, where T is a top space with the finite identity elements, i.e., the cardinality of $e(T)$ is finite, note that we knew that $e^{-1}(e(t))$ is Lie group, for all $t \in T$, and also $e^{-1}(e(t))$ is diffeomorphism to $e^{-1}(e(t^{'}))$, for all $t, t^{'} \in T$ [2, Corollaries 3.2 and 3.3].

Let (P, T, M) be a super principal fiber bundle, and let $\{U_i\}$ be an open covering of M such that $\pi^{-1}(U_i)$ can be represented as a product space via the function $F_i : \pi^{-1}(U_i) \to T$. We define a map $G_{ij}^{e(t)} : U_i \cap U_j \to T$ as follows:

If $m \in U_i \cap U_j$, let $p \in \pi^{-1}(m) \cap \{p_0t_0 | t_0 \in e^{-1}(e(t)), p_0 \in \pi^{-1}(m)\}$, and put

$$(G_{ij})^{e(t)}(m) = e(t)F_j(p)e(t)(F_i(p))^{-1}.$$

Now, we want to show that the definition of $(G_{ij})^{e(t)}$ is independent of the choice of p.

If $p^{'} \in \pi^{-1}(m) \cap \{p_0t_0 | t_0 \in e^{-1}(e(t)), p_0 \in \pi^{-1}(m)\}$, then there is $t_1 \in e^{-1}(e(t))$ such that $p^{'} = pt_1$.

Now, we have:

$$\begin{aligned} F_j(p^{'})(F_i(p^{'}))^{-1} &= F_j(pt_1)(F_i(pt_1))^{-1} \\ &= e(t_1)F_j(p)t_1e(t_1)(F_i(p))^{-1}(t_1)^{-1} \\ &= e(t_1)F_j(p)t_1(F_i(p))^{-1}(t_1)^{-1} \\ &= e(t_1)F_j(p)t_1(t_1)^{-1}(F_i(p))^{-1} \\ &= e(t_1)F_j(p)e(t_1)(F_i(p))^{-1}. \end{aligned}$$

Since $e^{-1}(e(t_1)) = e^{-1}(e(t))$ and $e(t_1) = e(t)$, then

$$e(t_1)F_j(p)e(t_1)(F_i(p))^{-1} = e(t_1)F_j(p)e(t)(F_i(p))^{-1}.$$

Moreover, it is easy to show that:

$$(G_{ik})^{e(t)}(m)(G_{kj})^{e(t)}(m) = (G_{ij})^{e(t)}(m),$$

for all $m \in U_i \cap U_j \cap U_k \cap e^{-1}(e(t))$ and $t \in T$.

The functions $(G_{ij})^{e(t)}$ are called the *super transition functions* corresponding to the covering $\{U_i\}$.

By this covering we extend a coordinate principal bundle in the sense of Steenrod ([6]), for all top spaces T.

Since T is a top space with the finite identity element $e(T)$, then

$$T = \overset{\circ}{\bigcup_{e(t)\in e(T)}} (e^{-1}(e(t))), \qquad [2, \text{Corollary } 3.3].$$

Now, suppose M is covered by domains of coordinate systems(i.e., $\{U_i\}$), let

$$P = \overset{\circ}{\bigcup_{e(t)\in e(T)}} \bigcup_i U_i \times e^{-1}(e(t))$$

and

$$\bigcup_i (U_i \times e^{-1}(e(t))) \times e^{-1}(e(t)) \to P$$

defined by $((u,t),t') \mapsto (u,tt')$.

For all $t \in T$, by the usual manner, we can induce a C^∞ structure on $\bigcup_i (U_i \times e^{-1}(e(t))$.

Since $e^{-1}(e(t))$ is diffeomorphism to $e^{-1}(e(t'))$ [2, Corollary 3.2], for all $t, t' \in T$, then all of the C^∞ structures are diffeomorphic.

Then, by the projection map and the usual manner, we have a differentiable structure on

$$\overset{\circ}{\bigcup_{e(t)\in e(T)}} \bigcup_i \bigcup (U_i \times e^{-1}(e(t))),$$

where $\pi_{e(t)} : \bigcup_i (U_i \times e^{-1}(e(t))) \to M$ is a projection map, for all $t \in T$.

Hence, in the sense of Steenrod, our definition of super fiber bundle is an equivalence class of super coordinate bundle.

Example 2.5. The Euclidean subspace $R^* = R - \{0\}$, with the product $(a,b) \mapsto a|b|$, is a top space with the identity element $e(T) = \{+1, -1\}$, then $\lambda : R^* \times R \longrightarrow R$ defined by $\lambda(a,m) = am$ is a super action of top space R^* on Euclidean manifold R.

Let

$$U_1 = (-2, \infty) \quad \text{and} \quad U_2 = (-\infty, 2).$$

Now, let

$$P = \overset{\circ}{\bigcup_{e(t)\in e(T)}} \bigcup_i (U_i \times e^{-1}(e(t)))$$

and

$$\overset{\circ}{\bigcup_{e(t)\in e(T)}} \bigcup_i ((U_i \times e^{-1}(e(t))) \times e^{-1}(e(t))) \to R$$

defined by

$$((u,t),t^{'}) \mapsto u.$$

It is clear that, by the usual manner, we can induce a differentiable structure by the projection maps:

$$\pi_{e(1)} : ((-2,\infty)\times R^{+})\bigcup((-\infty,2)\times R^{+}) \to R$$

and

$$\pi_{e(-1)} : ((-2,\infty)\times R^{-})\bigcup((-\infty,2)\times R^{-}) \to R.$$

Since $((-2,\infty)\times R^{+})\bigcup((-\infty,2)\times R^{+})$ and $((-2,\infty)\times R^{-})\bigcup((-\infty,2)\times R^{-})$ are disjoint diffeomorphic manifolds, then we have a C^{∞} structure on P such that the projection maps be C^{∞} maps, it is clear that (P, R^{*}, R) is super fiber bundle.

Conclusion

In the extension mode of coordinate bundle, we found the important following condition:

- All of the super coordinate bundles, in the sense of Steenrod, are equivalent if and only if their super actions agree.

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