ANALYSIS OF BLOOD FLOW THROUGH AN ARTERY WITH MILD STENOSIS: A TWO-LAYERED MODEL

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Abstract. In this paper, a model of blood flow through a constricted arterial segment has been considered. We have proposed a trapezium shaped geometry of mild axisymmetric stenosis. The flow of blood with artery has been represented by a two-layered model consisting of a core layer and a peripheral layer. It has been observed that the resistance to flow and wall shear stress increase as the peripheral layer viscosity increases. The results are compared graphically with those of previous investigators.

Keywords. arterial wall, blood flow, peripheral layer viscosity, stenosis.

AMS Classification: 92C35.

1. Introduction

Stenosis is an arterial disease in which the internal lumen of blood vessel is affected by some abnormal growth. This leads to blockage of the artery and then myocardial infarction. Keeping in mind the consequences of this type of blockages various researchers analyzed the problem experimentally and theoretically. The study of blood flow through stenotic arteries plays an important role in the diagnostic and fundamental understanding of cardiovascular diseases. The fluid mechanical behaviour of an artery having stenosis in its lumen had been given considerable attention by Lee and Fung [5], Rodbard [11], Young and Tsai [18]. The fluid behaviour of biological systems was thoroughly discussed by Leyton [6] and McDonald [7]. To understand the effects of mild stenosis several researchers Haldar [2], Misra and Chakravarty [8], Young [17] investigated the flow of blood through constricted tube treating blood as a Newtonian fluid. Nakamura and Sawada [9], Shukla et al. [12] extended the model by assuming that blood behaves like a non-Newtonian fluid. In all above models, the blood flow was represented by a single-layered model. Bugliarello and Sevilla [1] have shown experimentally that the blood flowing through narrow tubes can be well represented by a two-layered model instead of one. In this type of models there is a peripheral layer of plasma and a core region of suspension of red blood cells. Shukla et al. [13] have taken two-layered model to analyze the peripheral layer viscosity. Singh et al. [15] discussed a model of blood flow through an artery formulated for generalized geometry of multiple, mild and radially non-symmetric stenosis. Ponalagusamy [10] focused on slip velocity, thickness of peripheral layer and core layer viscosity at the vessel wall. Srivastava [16] studied analytically and numerically the effects of mild stenosis on blood flow characteristics in a two-fluid model. Recently, Joshi et al. [3] analyzed the flow of blood by taking a two-layered model with composite shaped geometry of constriction. In this paper the flow of blood has been analyzed with an artery having trapezium shaped mild stenosis. The results obtained are compared with previous investigators.

2. Formulation of the Model

In this paper, the flow of blood in a cylindrical tube having axisymmetric mild stenosis have been presented by a two-layered model. The external layer shows peripheral layer of plasma and the internal core layer describes the suspension of red blood cells. The schematic diagram is as follows:

The geometry of the stenotic tube without peripheral layer is described as follows,

$$(1)\ R(z)=\begin{cases} R_0-\dfrac{2\delta_s}{(L_0-\alpha}(z-d), & d\le z\le d+\dfrac{L_0-\alpha}{2}\\ R_0-\delta_s, & d+\dfrac{L_0-\alpha}{2}\le z\le d+\dfrac{L_0+\alpha}{2}\\ R_0-\delta_s+\frac{2\delta_s}{(L_0-\alpha)}\left(z-d-\dfrac{L_0+\alpha}{2}\right), & d+\dfrac{L_0+\alpha}{2}\le z\le d+L_0\\ R_0, & \text{elsewhere.}\end{cases}$$

where the symbols stand for

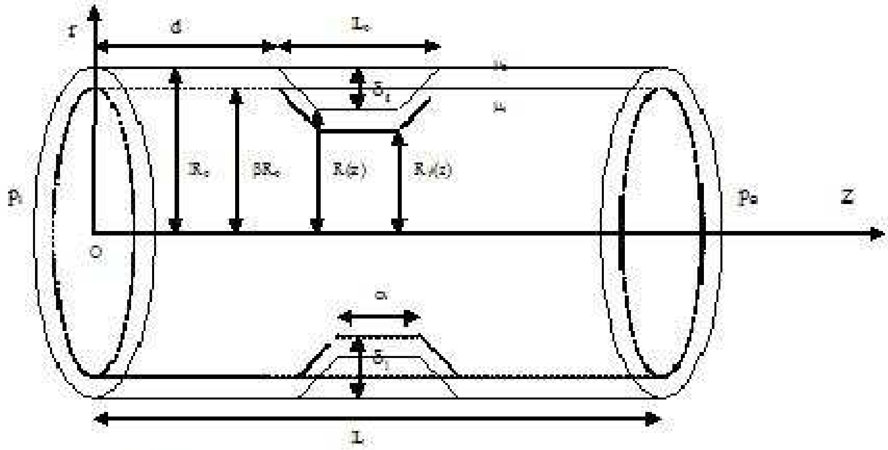


Figure 1: Geometry of Stenosed Artery

R_0 : Radius of the non-stenotic region
$R(z)$: Radius of the stenotic region
$R_1(z)$: Radius of the central core layer in stenotic region
L : The length of the artery
L_0 : The length of the stenosis
d : Location of stenosis
p_i : Inlet fluid pressure
p_0 : Instantaneous outlet fluid pressure
δ_s : Instantaneous maximum height of the stenosis
δ_i : Maximum bulging of interface
μ_1 : Viscosity of fluid in central core layer
μ_2 : Viscosity of fluid in peripheral layer
α : A parameter ($\alpha \geq 0$)
β : Ratio of central core radius to the tube radius

The governing equation of blood flow is given by Kapur [4],

$$0 = -\frac{dp}{dz} + \frac{1}{r}\frac{\partial}{\partial r}\left\{\mu\left(r\right) r \frac{\partial w}{\partial r}\right\}, \tag{2}$$

where w is axial velocity, p is fluid pressure and $\mu\left(r\right)$ is viscosity of fluid.

The boundary conditions are,

$$w = 0 \quad at \quad r = R(z) \tag{3}$$

and

$$\frac{\partial w}{\partial r} = 0 \quad at \quad r = 0. \tag{4}$$

Solving equation (2) under boundary conditions (3) and (4), we get

$$w = \left(-\frac{1}{2}\frac{dp}{dz}\right)\int_r^R \frac{r}{\mu\left(r\right)}\,dr. \tag{5}$$

The volumetric flow rate is given by

$$Q = \int_0^R 2\,\pi\, r\, w\, dr, \tag{6}$$

which on using equation (5) gives,

$$Q = \left(-\frac{\pi}{2}\frac{dp}{dz}\right)\int_0^R \frac{r^3 dr}{\mu\,(r)}. \tag{7}$$

Thus, the pressure gradient can be obtained as,

$$\frac{dp}{dz} = -\frac{2Q}{\pi I(z)}, \tag{8}$$

where

$$I(z) = \int_0^R \frac{r^3 dr}{\mu\,(r)}. \tag{9}$$

Integrating equation (8) using conditions $p = p_i$ at $z = 0$ and $p = p_0$ at $z = L$, we have

$$p_i - p_0 = \frac{2Q}{L}\int_0^L \frac{dz}{I(z)}. \tag{10}$$

The resistance to flow is defined by,

$$\lambda = \frac{p_i - p_0}{Q} \tag{11}$$

From equations (1), (10) and (11), we can find

$$\lambda = \frac{2}{\pi}\left[\frac{L - L_0}{I_0} + \int_d^{d+L_0} \frac{dz}{I(z)}\right], \tag{12}$$

where

$$I_0 = \int_0^{R_0} \frac{r^3}{\mu\,(r)}\, dr. \tag{13}$$

Now, the shear stress at wall is given by,

$$\tau_R = \left[-\mu\,(r)\,\frac{\partial\, w}{\partial\, r}\right]_{r=R(z)}. \tag{14}$$

By using equations (5) and (8) in (14), we can find shear stress at maximum height of stenosis as follows,

$$\tau_s = \left[\frac{R(z)Q}{\pi I(z)}\right]. \tag{15}$$

For finding the effects of peripheral layer viscosity, the viscosity function $\mu(r)$ can be defined as,

$$\mu(r) = \begin{cases} \mu_1; & 0 \le r \le R_1(z), \\ \mu_2; & R_1(z) \le r \le R(z), \end{cases} \tag{16}$$

where μ_1 and μ_2 are the viscosities of the central and the peripheral layers respectively. The function $R_1(z)$ represents the shape of the central layer with stenosis. The mathematical representation of this model can be described as,

$$R_1(z)=\begin{cases}\beta R_0-\dfrac{2\delta_i}{(L_0-\alpha)}(z-d), & d\leq z\leq d+\dfrac{L_0-\alpha}{2}\\ \beta R_0-\delta_i, & d+\dfrac{L_0-\alpha}{2}\leq z\leq d+\dfrac{L_0+\alpha}{2}\\ \beta R_0-\delta_i+\dfrac{2\delta_i}{(L_0-\alpha)}\left(z-d-\dfrac{L_0+\alpha}{2}\right), & d+\dfrac{L_0+\alpha}{2}\leq z\leq d+L_0\\ \beta R_0, & \text{elsewhere.}\end{cases} \tag{17}$$

By using equation (16) in equation (5), velocities w_c,w_p and then the corresponding volumetric flow rates Q_c, Q_p can be obtained as follows,

$$Q_c=\int_0^{R_1}2\pi r w_c\,dr=\left(-\frac{\pi}{8\mu_2}\frac{dp}{dz}\right)2{R_1}^2\left[R^2-\left(1-\frac{\bar{\mu}_2}{2}\right)R_1^2\right] \tag{18}$$

$$Q_p=\int_{R_1}^{R}2\pi r w_p\,dr=\left(-\frac{\pi}{8\mu_2}\frac{dp}{dz}\right)\left(R^2-R_1^2\right)^2 \tag{19}$$

where $\bar{\mu}_2=\mu_2/\mu_1$.

Thus, the total volumetric flow rate Q is defined as,

$$Q=Q_c+Q_p=\left(-\frac{\pi}{8\mu_2}\frac{dp}{dz}\right)\left(R^4-(1-\overline{\mu_2})R_1^4\right). \tag{20}$$

Equation (20) can also be obtained by using equation (16) in equation (7) which shows that Q is a constant.

Integrating equation (18), (19) and (20) by assuming that the pressure drop is same in each case across the length of artery. We obtain,

$$Q_c=\frac{(p_i-p_0)\,\pi R_0^4M_1}{4\mu_2L\left(1-\frac{L_0}{L}+M_1G_1\right)}, \tag{21}$$

where

$$M_1=\beta^2\left[1-\left(1-\frac{\overline{\mu_2}}{2}\right)\beta^2\right. \tag{22}$$

and

$$G_1=g_1+g_2+g_3, \tag{23}$$

where

$$g_1=\frac{1}{L}\int_d^{d+\frac{L_0-\alpha}{2}}\frac{dz}{\left(\frac{R_1}{R_0}\right)^2\left[\left(\frac{R}{R_0}\right)^2-\left(1-\frac{\overline{\mu_2}}{2}\right)\left(\frac{R_1}{R_0}\right)^2\right]} \tag{24}$$

$$g_2=\frac{1}{L}\int_{d+\frac{L_0-\alpha}{2}}^{d+\frac{L_0+\alpha}{2}}\frac{dz}{\left(\frac{R_1}{R_0}\right)^2\left[\left(\frac{R}{R_0}\right)^2-\left(1-\frac{\overline{\mu_2}}{2}\right)\left(\frac{R_1}{R_0}\right)^2\right]} \tag{25}$$

$$g_3=\frac{1}{L}\int_{d+\frac{L_0+\alpha}{2}}^{d+L_0}\frac{dz}{\left(\frac{R_1}{R_0}\right)^2\left[\left(\frac{R}{R_0}\right)^2-\left(1-\frac{\overline{\mu_2}}{2}\right)\left(\frac{R_1}{R_0}\right)^2\right]} \tag{26}$$

and

$$Q_p=\frac{(p_i-p_0)\ \pi R_0^4 M_2}{8\mu_2 L\left(1-\frac{L_0}{L}+M_2G_2\right)}, \tag{27}$$

where

$$M_2=\left(1-\beta^2\right)^2 \tag{28}$$

$$G_2=g_4+g_5+g_6, \tag{29}$$

where

$$g_4=\frac{1}{L}\int_{d}^{d+\frac{L_0-\alpha}{2}}\frac{dz}{\left[\left(\frac{R}{R_0}\right)^2-\left(\frac{R_1}{R_0}\right)^2\right]^2} \tag{30}$$

$$g_5=\frac{1}{L}\int_{d+\frac{L_0-\alpha}{2}}^{d+\frac{L_0+\alpha}{2}}\frac{dz}{\left[\left(\frac{R}{R_0}\right)^2-\left(\frac{R_1}{R_0}\right)^2\right]^2} \tag{31}$$

$$g_6=\frac{1}{L}\int_{d+\frac{L_0+\alpha}{2}}^{d+L_0}\frac{dz}{\left[\left(\frac{R}{R_0}\right)^2-\left(\frac{R_1}{R_0}\right)^2\right]^2}. \tag{32}$$

Now,

$$Q=\frac{(p_i-p_0)\ \pi R_0^4 M}{8\mu_2 L\left(1-\frac{L_0}{L}+MG\right)}, \tag{33}$$

where

$$M=1-(1-\overline{\mu_2})\beta^4 \tag{34}$$

$$G=g_7+g_8+g_9, \tag{35}$$

where

$$g_7=\frac{1}{L}\int_{d}^{d+\frac{L_0-\alpha}{2}}\frac{dz}{\left[\left(\frac{R}{R_0}\right)^4-(1-\overline{\mu_2})\left(\frac{R_1}{R_0}\right)^4\right]} \tag{36}$$

$$g_8=\frac{1}{L}\int_{d+\frac{L_0-\alpha}{2}}^{d+\frac{L_0+\alpha}{2}}\frac{dz}{\left[\left(\frac{R}{R_0}\right)^4-(1-\overline{\mu_2})\left(\frac{R_1}{R_0}\right)^4\right]} \tag{37}$$

$$g_9=\frac{1}{L}\int_{d+\frac{L_0+\alpha}{2}}^{d+L_0}\frac{dz}{\left[\left(\frac{R}{R_0}\right)^4-(1-\overline{\mu_2})\left(\frac{R_1}{R_0}\right)^4\right]}. \tag{38}$$

From equations (21), (27) and (33) and using $Q=Q_c+Q_p$, we can find

$$\frac{M}{\left(1-\frac{L_0}{L}+MG\right)}=\frac{2M_1}{\left(1-\frac{L_0}{L}+M_1G_1\right)}+\frac{M_2}{\left(1-\frac{L_0}{L}+M_2G_2\right)}. \tag{39}$$

Now, using $R_1 = \beta R$ in equation (17), we get

$$(40)\quad R(z) = \begin{cases} R_0 - \dfrac{2\delta_i}{\beta(L_0-\alpha)}(z-d), & d \le z \le d + \dfrac{L_0-\alpha}{2} \\ R_0 - \dfrac{\delta_i}{\beta}, & d + \dfrac{L_0-\alpha}{2} \le z \le d + \dfrac{L_0+\alpha}{2} \\ R_0 - \dfrac{\delta_i}{\beta} + \dfrac{2\delta_i}{\beta(L_0-\alpha)}\left(z-d-\dfrac{L_0+\alpha}{2}\right), & d + \dfrac{L_0+\alpha}{2} \le z \le d+L_0 \\ R_0, & \text{elsewhere.} \end{cases}$$

On comparing equation (1) and (40), we can observe that

$$(41)\qquad \delta_i = \beta\delta_s.$$

Now by keeping in mind equation (16), the dimensionless resistance to flow $\overline{\lambda}$ and the dimensionless shear stress $\overline{\tau}_s$ can be obtained by using equation (33) in equations (11) and (15) respectively,

$$(42)\qquad \overline{\lambda} = \frac{\lambda}{\lambda_0} = \frac{\overline{\mu_2}}{M}\left(1 - \frac{L_0}{L} + MG\right),$$

where $\overline{\mu_2} = \frac{\mu_2}{\mu_1}$ and $\lambda_0 = \frac{8\mu_1 L}{\pi R_0^4}$, and

$$(43)\qquad \overline{\tau_s} = \frac{\tau_s}{\tau_0} = \frac{\overline{\mu_2}\left(1 - \frac{\delta_s}{R_0}\right)}{\left[\left(1 - \frac{\delta_s}{R_0}\right)^4 - (1-\overline{\mu_2})\left(\beta - \frac{\delta_i}{R_0}\right)^4\right]},$$

where $\tau_0 = \frac{4\mu_1 Q}{\pi R_0^3}$, and λ_0, τ_0 are the resistance to flow and wall shear stress for the case of no stenosis respectively, with $\overline{\mu_2} = 1$.

Evaluating the integrals (36), (37) and (38) after using equation (41) and rewriting the expressions for $\overline{\lambda}$ and $\overline{\tau_s}$ which are as follows,

$$(44)\qquad \overline{\lambda} = \tfrac{\overline{\mu_2}}{M}\left[1 - \tfrac{L_0}{L} + \tfrac{1}{L}\left\{(L_0-\alpha)\left(1 + 2\left(\tfrac{\delta_s}{R_0}\right) + \tfrac{10}{3}\left(\tfrac{\delta_s}{R_0}\right)^2 + \cdots\right) \right.\right. \\ \left.\left. +\alpha\left(1 + 4\left(\tfrac{\delta_s}{R_0}\right) + 10\left(\tfrac{\delta_s}{R_0}\right)^2 + \cdots\right)\right\}\right]$$

and

$$(45)\qquad \overline{\tau_s} = \frac{\overline{\mu_2}}{\left[\left(1 - \frac{\delta_s}{R_0}\right)^3 M\right]},$$

here $\overline{\tau_s}$ obtained is same as in Shukla et al. [13]. If $\overline{\mu_2} = 1$ in equations (44) and (45), we get

$$(46)\qquad \overline{\lambda} = 1 - \tfrac{L_0}{L} + \tfrac{1}{L}\left\{(L_0-\alpha)\left(1 + 2\left(\tfrac{\delta_s}{R_0}\right) + \tfrac{10}{3}\left(\tfrac{\delta_s}{R_0}\right)^2 + \cdots\right) \right. \\ \left. +\alpha\left(1 + 4\left(\tfrac{\delta_s}{R_0}\right) + 10\left(\tfrac{\delta_s}{R_0}\right)^2 + \cdots\right)\right\},$$

which is same as the ratio obtained by Singh et al. [14], and

$$\overline{\tau_s} = \left(1 - \frac{\delta_s}{R_0}\right)^{-3}, \tag{47}$$

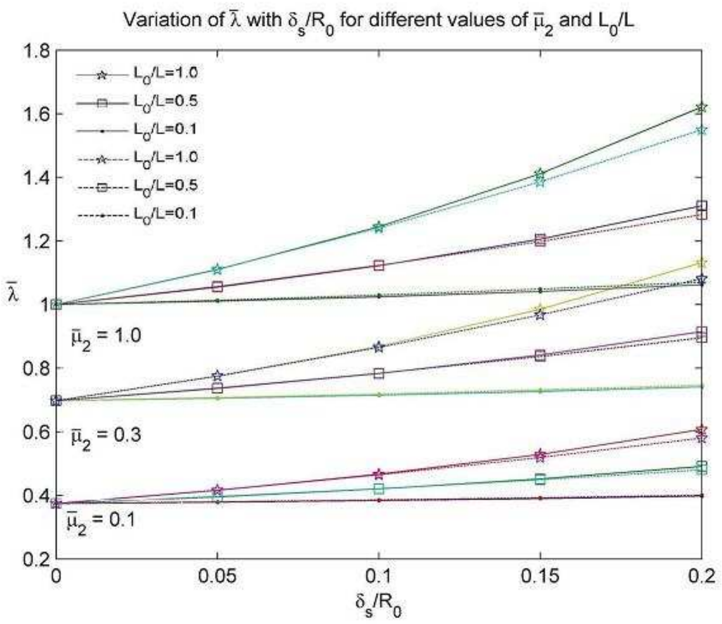
which is same as obtained by Young [17].

3. Conclusion

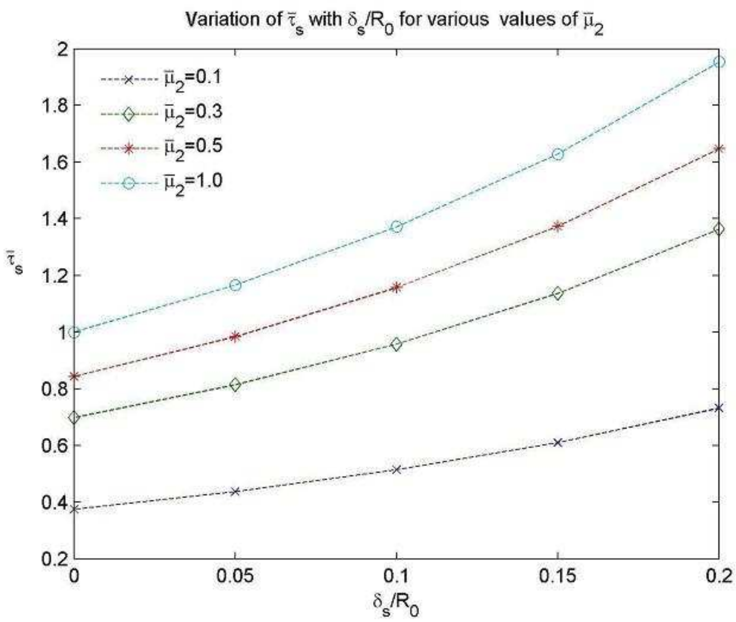
In this paper, we consider a two-layered model of blood flow through a stenosed artery. It is assumed that when blood flows in a cylindrical tube there exists two layers. The central core layer consists of erythrocytes surrounded by a peripheral plasma layer. Both fluids have different viscosities. The expressions for dimensionless resistance to flow $\overline{\lambda}$ and dimensionless shear stress $\overline{\tau_s}$ have been plotted by using MATLAB software for different values of parameters. Graphs in appendix represent the variations of $\overline{\lambda}$and $\overline{\tau_s}$ with $\frac{\delta_s}{R_0}$ for different values of $\overline{\mu_2}$ and $\frac{L_0}{L}$. It has been observed that $\overline{\lambda}$ and $\overline{\tau_s}$ increase with the increase in the height of stenosis, this increase have also been noted when $\overline{\mu_2}$ is increased. It is further noted that resistance to flow $\overline{\lambda}$(denoted by broken lines) is lower in the present model as compared to previous investigation of Shukla et al. [13]. Also, using the data $\overline{\mu_2} = 0.3$, $\frac{L_0}{L} = 1.0$, $\alpha = 0.25$, $\beta = 0.95$ and $\frac{\delta_s}{R_0} = 0.1$ in equations (44) and (45), it can be noted that $\overline{\lambda}$ and $\overline{\tau_s}$ are decreased by 13% and 4% respectively when compared with the case of no stenosis with $\overline{\mu_2} = 1$. Again, in the absence of peripheral layer these characteristics are increased by 24% and 37% respectively for the same stenosis size and $\overline{\mu_2} = 1$. These values are almost same to previous ones thus it seems that the results of present analysis of two-layered model can be useful to explain the flow behaviour of stenotic arteries.

Acknowledgment. We are thankful to the anonymous reviewer for valuable suggestions.

Appendix



Variation of $\bar{\lambda}$ with δ_s/R_0 for different values of $\bar{\mu}_2$ and L_0/L
$L_0/L=1.0$
$L_0/L=0.5$
$L_0/L=0.1$
$L_0/L=1.0$
$L_0/L=0.5$
$L_0/L=0.1$
$\bar{\mu}_2 = 1.0$
$\bar{\mu}_2 = 0.3$
$\bar{\mu}_2 = 0.1$
$\bar{\lambda}$
δ_s/R_0



Variation of $\bar{\tau}_s$ with δ_s/R_0 for various values of $\bar{\mu}_2$
$\bar{\mu}_2=0.1$
$\bar{\mu}_2=0.3$
$\bar{\mu}_2=0.5$
$\bar{\mu}_2=1.0$
$\bar{\tau}_s$
δ_s/R_0

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Accepted: 25.05.2010