GENERALIZED EXPONENTIAL OPERATORS AND DIFFERENCE EQUATIONS

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Abstract. The present paper deals with the generalization of exponential operators used by Dattoli and Levi for translation and diffusive operator which were utilized to establish analytical solutions of difference and integral equations. The generalization of their technique is expected to cover wide range of such utilization.

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1. Introduction

In 2000, Dattoli and Levi [1] discussed general methods for the solution of difference equations, arising in physical and biological problems.Their technique play crucial role in unifying the generalized families of the difference equations. The present paper deals with the generalization of exponential operators used in [1] to operators of the type $a^{\lambda q(x)\frac{d}{dx}}$, where base a $(a > 0,\ a \neq 1)$ is a real number. In particular when $a = e$, the operator reduces to the operators used by Dattoli et al. [1].

The action of the generalized exponential operator on a generic function $f(x)$ is defined as

$$a^{\lambda q(x)\dfrac{d}{dx}} f(x) = e^{(\lambda \ln(a))q(x)\frac{d}{dx}} f(x) = f(F^{-1}(\lambda \ln(a) + F(x))). \tag{1.1}$$

where $F(x)$ (called the Similarity Factor (SF)) denotes the function

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$$F(x) = \int^x \frac{d\xi}{q(\xi)},$$

and $F^{-1}(\sigma)$ is its inverse. For $q(x) = 1$, the SF is given by

$$F(x) = \int^x d\xi = x, \tag{1.2}$$

therefore $F^{-1}(x) = x$, then operator (1.1) reduces to the ordinary translation or shift operator as follows:

$$a^{\lambda \frac{d}{dx}} f(x) = f(F^{-1}(\lambda \ln(a) + x)) = f(\lambda \ln(a) + x). \tag{1.3}$$

Another example of application of operator (1.1), for $q(x) = x$, the SF is given by

$$F(x) = \int^x \frac{d\xi}{\xi} = \ln(x), \tag{1.4}$$

so that $F^{-1}(x) = e^x$, and hence operator (1.1) reduces to the dilatation operator

$$a^{\lambda x \frac{d}{dx}} f(x) = f(F^{-1}(\lambda \ln(a) + \ln(x))) = f(e^{\lambda \ln(a) + \ln(x)}) = f(a^\lambda x). \tag{1.5}$$

The ordinary shift operators and their properties play a central role within the context of the theory of difference equations [3]. One can, therefore, suspect that the above generalized exponential operators and the wealth of their properties can be exploited to develop tools which allow the solution of different forms of difference equations.

1(a). Particular case: The substitution of $a = e$, into equations (1.1), (1.3) and (1.5) reduce to equations (1), (2′) and (3) of Dattoli et al. [1].

A simple example of how the exponential operators can help us to solve difference equations may be illuminating. Let us consider the linear dilatation difference equation of the type

$$b_1 f(a^2 x) + b_2 f(ax) + b_3 f(x) = 0, \tag{1.6}$$

which, according to equation (1.5), equation (1.6) can be written in the following form

$$\left[b_1\ a^{2x \frac{d}{dx}} + b_2 a^{x \frac{d}{dx}} + b_3\right] f(x) = 0. \tag{1.7}$$

Suppose $f(x) = R^{\ln(x)}$, we have

$$a^{\lambda x \frac{d}{dx}} R^{\ln(x)} = e^{\lambda \ln(a) x \frac{d}{dx}} R^{\ln(x)},$$

where $q(x) = x$, so that $F(x) = \ln(x)$ and $F^{-1}(x) = e^x$ or $F^{-1}(\lambda \ln(a) + \ln(x)) = e^{\lambda \ln(a) + \ln(x)} = x a^\lambda$. Therefore,

$$a^{\lambda x \frac{d}{dx}} R^{\ln x} = R^{\ln(x a^\lambda)} = R^{\lambda \ln(a)} R^{\ln x}. \tag{1.8}$$

Hence we can associate with equation (1.7) the characteristic equation

$$(1.9)\qquad [b_1R^{2\ln(a)} + b_2R^{\ln(a)} + b_3]R^{\ln(x)} = 0, \quad \text{or} \quad b_1R^{2\ln(a)} + b_2R^{\ln(a)} + b_3 = 0,$$

whose roots $R_1^{\ln(a)}$ and $R_2^{\ln(a)}$ allow to write $f(x)$ in terms of the following linear combination of independent solutions:

$$(1.10)\qquad f(x) = c_1R_1^{\ln(x)} + c_2R_2^{\ln(x)} = \sum_{\alpha=1}^{2} c_\alpha R_\alpha^{\ln(x)}.$$

The above example indicates that we can extend well-established methods of solutions of difference equations to other types of equations reducible to ordinary difference equations, after a proper change of variable implicit in equations (1.1), (1.3).

1(b). Particular case: The replacement of a with e in equations (1.6), (1.7), (1.8) and (1.9) give raise to equations (5), (6), (7), and (8) of Dattoli et al. [1].

To give a further example of the flexibility of the formalism associated with exponential operators, let us consider the generalized Heat Equation of the following type

$$(1.11)\qquad \begin{cases} \dfrac{\partial}{\partial\lambda}Q(x,\lambda\ln(a)) = \ln(a)\left[q(x)\dfrac{\partial}{\partial x}\right]^2 Q(x,\lambda\ln(a)), \\ Q(x,0) = g(x), \end{cases}$$

which can formally be solved by rewriting equation (1.11) as

$$\frac{\partial}{\partial\lambda}Q(x,\lambda\ln(a)) - \ln(a)\left[q(x)\frac{\partial}{\partial x}\right]^2 Q(x,\lambda\ln(a)) = 0,$$

which can formally be solved by considering this as ordinary linear differential equation of order one, whose I.F. is determined as

$$e^{-\int \ln(a)\left[q(x)\frac{\partial}{\partial x}\right]^2 d\lambda} = e^{-\ln(a)\left[q(x)\frac{\partial}{\partial x}\right]^2\lambda} = a^{-\lambda\left[q(x)\frac{\partial}{\partial x}\right]^2},$$

we can, therefore, find its general solution as

$$Q(x,\lambda\ln(a))a^{-\lambda\left[q(x)\frac{\partial}{\partial x}\right]^2} = C,$$

where C in any constant and using the given initial condition, we get

$$Q(x,0) = g(x) = C,$$

and, finally, we obtain the solution of the Heat equation (1.11) as

$$(1.12)\qquad Q(x,\lambda\ln(a)) = a^{\lambda[q(x)\frac{\partial}{\partial x}]^2}g(x).$$

Using the identity

$$e^{b^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2+2b\xi} d\xi,$$

and replacing b^2 by $\lambda \ln(a)[q(x)\frac{\partial}{\partial x}]^2$, we have

$$e^{\lambda \ln(a)[q(x)\frac{\partial}{\partial x}]^2} = a^{\lambda [q(x)\frac{\partial}{\partial x}]^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2+2\sqrt{\lambda \ln(a)}\xi} d\xi. \tag{1.13}$$

Using equation (1.1), finally yields the solution of equation (1.11) in the form of an integral transform, which can be viewed as a generalized Gauss transform

$$a^{\lambda [q(x)\frac{\partial}{\partial x}]^2} g(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2} g(F^{-1}(2\xi\sqrt{\lambda \ln(a)} + F(x))) d\xi. \tag{1.14}$$

or, in other words, we have

$$Q(x, \lambda \ln(a)) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\xi^2} g(F^{-1}(2\xi\sqrt{\lambda \ln(a)} + F(x))) d\xi. \tag{1.14$'$}$$

It is evident that the formalism associated with generalized exponential operators can be exploited in many flexible ways in finding the general solution of a large number of problems. This paper is devoted to the discussion of methods which provide the solution of the classes of "difference" and generalized "Heat" equations and we shall see that the techniques we propose offer reliable analytical tools and efficient numerical algorithms.

1(c). Particular case: To put $a = e$, in the equations (1.11), (1.12), and (1.14) give raise the same forms of the equations (10), (11) and (13) respectively of Dattoli et al. [1].

2. Generalized difference equations

Before discussing the problem in its generality, let us consider the equation of the following type, as a further example, which reduces to ([1]; p. 655 (14)) when we consider $a = e$, we have

$$\sum_{\alpha=0}^{N} b_\alpha f(x \cos(\alpha \ln(a)) + \sqrt{1-x^2} \sin(\alpha \ln(a))) = 0, \tag{2.1}$$

which belongs to the families of generalized difference equation. This equation can be obtained by the action of the generalized exponential operator on the function $f(x)$.

$$\sum_{\alpha=0}^{N} b_\alpha f(\sin(\sin^{-1} x)\cos(\alpha\ln(a)) + \cos(\cos^{-1}\sqrt{1-x^2})\sin(\alpha\ln(a))) = 0,$$

$$\sum_{\alpha=0}^{N} b_\alpha f(\sin(\sin^{-1} x)\cos(\alpha\ln(a)) + \cos(\sin^{-1} x)\sin(\alpha\ln(a))) = 0,$$

$$\sum_{\alpha=0}^{N} b_\alpha f(\sin(\sin^{-1} x) + \alpha\ln(a)) = 0, \; \sum_{\alpha=0}^{N} b_\alpha a^{\alpha\sqrt{1-x^2}} f(x) = 0.$$

According to the discussion of the previous section, the use of the exponential operator

$$\sum_{\alpha=0}^{N} b_\alpha \widehat{A}^\alpha f(x) = 0,$$

where

$$\widehat{A} = a^{\sqrt{1-x^2}\frac{d}{dx}} \tag{2.2}$$

allows to cast (2.1) in the operator form

$$\Psi(\widehat{A}) f(x) = 0,$$

where

$$\Psi(\widehat{A}) = \sum_{\alpha=0}^{N} b_\alpha \widehat{A}^\alpha. \tag{2.3}$$

In this case, the SF associated with (2.2) is

$$F(x) = \sin^{-1}(x). \tag{2.4}$$

Independent solutions of (2.1) can, therefore, be constructed in terms of the function $R^{\sin^{-1}(x)}$, which satisfies the identity

$$\widehat{A}^\alpha R^{\sin^{-1}(x)} = R^{\alpha\ln(a)} R^{\sin^{-1}(x)}, \tag{2.5}$$

the general solution of (2.1) can finally be written as

$$f(x) = \sum_{\alpha=0}^{N} c_\alpha R_\alpha^{\sin^{-1}(x)}. \tag{2.6}$$

Similarly, if we consider the following example, we have

$$\sum_{\alpha=0}^{N} b_\alpha f(x\cos(\alpha\ln(a)) - \sqrt{1-x^2}\sin(\alpha\ln(a))) = 0, \tag{2.1$'$}$$

which belongs to the families of generalized difference equation. According to the discussion of the previous section, the use of the exponential operator

$$\widehat{A} = a^{-\sqrt{1-x^2}\frac{d}{dx}} \tag{2.2$'$}$$

allows to cast (2.1)$'$ in the operator form (2.3). In this case the SF associated with (2.2)$'$ is

$$F(x) = \cos^{-1}(x). \tag{2.4$'$}$$

Independent solutions of (2.1)$'$ can be therefore constructed in terms of the function $R^{\cos^{-1}(x)}$, which satisfies the identity

$$\widehat{A}^{\alpha} R^{\cos^{-1}(x)} = R^{\alpha \ln(a)} R^{\cos^{-1}(x)}. \tag{2.5$'$}$$

The general solution of (2.1)$'$ can finally be written as

$$f(x) = \sum_{\alpha=0}^{N} c_{\alpha} R_{\alpha}^{\cos^{-1}(x)}, \tag{2.6$'$}$$

where $R_{\alpha}^{\ln(a)}$ are the roots of the characteristic equation

$$\Psi(R^{\ln(a)}) = 0. \tag{2.7}$$

From the above discussion it is now clear that, whenever one deals with equations of the type

$$\sum_{\alpha=0}^{N} b_{\alpha} f(F^{-1}(\alpha \ln(a) + F(x))) = 0, \tag{2.8}$$

one can associate it with the generalized exponential operator

$$\widehat{A} = a^{q(x)\frac{d}{dx}}, \tag{2.9}$$

which allows to cast (2.8) in the operator form (2.3) and we get the relevant solution in the form

$$f(x) = \sum_{\alpha=0}^{N} c_{\alpha} R_{\alpha}^{\int^{x} \frac{d\xi}{q(\xi)}}. \tag{2.10}$$

2(a). Particular case: when we substitute $a = e$ in equations (2.1), (2.2), (2.3), (2.5), (2.7), (2.8) and (2.9) then these equation lead to equations (14), (15), (16), (18), (20), (21) and (22) respectively due to Dattoli et al. [1].

A useful example is given by the equation

$$\sum_{\alpha=0}^{N} b_\alpha f\left(\frac{x}{1-\alpha\ln(a)x}\right)=0, \tag{2.11}$$

by making use of the shift operator $a^{x^2\frac{d}{dx}}$, which allows to cast (2.11) in the operator form (2.3), i.e.,

$$\sum_{\alpha=0}^{N} b_\alpha f\left(-\frac{1}{\alpha\ln(a)-\frac{1}{x}}\right)=0,$$

$$\sum_{\alpha=0}^{N} b_\alpha a^{\alpha x^2\frac{d}{dx}} f(x)=0,$$

$$\sum_{\alpha=0}^{N} b_\alpha \widehat{A}^\alpha f(x)=0,$$

where

$$\widehat{A}=a^{x^2\frac{d}{dx}},\quad \Psi(\widehat{A})f(x)\ =\ 0 \text{ and } \quad \Psi(A)=\sum_{\alpha=0}^{N} b_\alpha\widehat{(A)^\alpha}.$$

In this case, the SF associated with (2.11) is $F(x)=-\dfrac{1}{x}$. Its solution can thus be written as

$$f(x)=\sum_{\alpha=1}^{N} c_\alpha R_\alpha^{-\frac{1}{x}}. \tag{2.12}$$

The validity of the above solutions is limited to the case in which $R_\alpha^{\ln(a)}$ is not a multiple root of the characteristic equation; this point will be discussed in the concluding section.

2(b). Particular case: Replacing a with e in equation (2.11) reduce to Dattoli et al. ([1]; p.656(24)).

In the tunes of Dattoli et al. ([1]; p. 656(26)), let us introduce the following operational identities:

$$\begin{cases} \widehat{A}^{\pm\alpha} b^{\int^x \frac{d\xi}{q(\xi)}} = b^{\pm\alpha\ln(a)} b^{\int^x \frac{d\xi}{q(\xi)}}, \\ \widehat{A}^{\pm\alpha}(b^{\int^x \frac{d\xi}{q(\xi)}}\phi(x)) = b^{\int^x \frac{d\xi}{q(\xi)}}(b^{\ln(a)}\widehat{A})^{\pm\alpha}\phi(x). \end{cases} \tag{2.13}$$

valid for exponential operators of the form (2.9).

We note that, according to the first of (2.13), the non-homogeneous equation

$$\Psi(\widehat{A})f(x)=Cb^{\int^x \frac{d\xi}{q(\xi)}}, \tag{2.14}$$

where C is a constant and $b^{\ln(a)}$ is not a root of the characteristic equation, admits the particular solution

$$f(x) = \frac{Cb^{\int^x \frac{d\xi}{q(\xi)}}}{\Psi(b^{\ln(a)})}. \tag{2.15}$$

In the slightly more complicated case

$$\Psi(\widehat{A})f(x) = Cb^{\int^x \frac{d\xi}{q(\xi)}}\phi(x), \tag{2.16}$$

the second of (2.13) yields

$$f(x) = Cb^{\int^x \frac{d\xi}{q(\xi)}} \frac{1}{\Psi(b^{\ln(a)}\widehat{A})}\phi(x). \tag{2.17}$$

Further comments shall be discussed in the concluding section.

2(c). Particular case: When $a = e$, equations (2.14), (2.15), (2.16) and (2.17) convert into equations (27), (28), (29) and (30) of Dattoli et al. [1].

3. Generalized shift operators and Jackson derivatives

In the previous section, we have considered linear equations involving discrete power of the generalized exponential operator. Here, we shall discuss examples in which the exponents are not necessarily integers. The introductory example is

$$\frac{f(a^\lambda x) - f(x)}{\lambda \ln(a)} = g(x), \tag{3.1}$$

where $f(x)$ is unknown, $\lambda \in C$, and $g(x)$ is an analytical function. The use of the dilatation operator allows to cast equation (2.17) in the form of a the Jackson derivative [4], namely

$$\frac{a^{\lambda \frac{d}{d\xi}} - 1}{\lambda \ln(a)} f(x) = g(x). \tag{3.2}$$

The operator on the left hand side can formally be inverted and by writing the differentiation variable in terms of the inverse of the SF we find

$$f(e^\xi) = \frac{\lambda \ln(a)}{a^{\lambda \frac{d}{d\xi}} - 1} g(e^\xi). \tag{3.3}$$

The operator on the r.h.s. of (3.3) can be expanded as

$$\frac{\lambda\ln(a)}{a^{\lambda\frac{d}{d\xi}}-1}=\frac{\lambda\ln(a)}{\lambda\ln(a)\frac{d}{d\xi}+\frac{1}{2!}(\lambda\ln(a)\frac{d}{d\xi})^2+\frac{1}{3!}(\lambda\ln(a)\frac{d}{d\xi})^3+\cdots}$$
$$=\frac{1}{\frac{d}{d\xi}\left[1+\frac{1}{2!}\left(\lambda\ln(a)\frac{d}{d\xi}\right)+\frac{1}{3!}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^2+\cdots\right]}$$
$$=D_\xi^{-1}\left[1+\left(\frac{1}{2!}\lambda\ln(a)\frac{d}{d\xi}+\frac{1}{3!}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^2+\cdots\right)\right]^{-1}$$
$$=D_\xi^{-1}\left[1-\left(\frac{1}{2!}\lambda\ln(a)\frac{d}{d\xi}+\frac{1}{3!}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^2+\cdots\right)\right.$$
$$\left.+\left(\frac{1}{2!}\lambda\ln(a)\frac{d}{d\xi}+\frac{1}{3!}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^2+\cdots\right)^2+\cdots\right]$$
$$=D_\xi^{-1}\left[1-\frac{1}{2}\lambda\ln(a)\frac{d}{d\xi}+\left(\frac{1}{4}-\frac{1}{6}\right)\left(\lambda\ln(a)\frac{d}{d\xi}\right)^2-\left(\frac{1}{24}+\frac{1}{8}\right)\left(\lambda\ln(a)\frac{d}{d\xi}\right)^3+\cdots\right]$$
$$=D_\xi^{-1}\left[1-\frac{1}{2}\lambda\ln(a)\frac{d}{d\xi}+\frac{1}{12}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^2-\frac{1}{6}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^3+\cdots\right]$$
$$=D_\xi^{-1}\left[B_0+\frac{B_1}{1!}\lambda\ln(a)\frac{d}{d\xi}+\frac{B_2}{2!}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^2+\frac{B_3}{3!}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^3+\cdots\right]$$

or

$$\frac{\lambda\ln(a)}{a^{\lambda\frac{d}{d\xi}}-1}=D_\xi^{-1}\sum_{n=0}^{\infty}\frac{B_n}{n!}(\lambda\ln(a))^n\left(\frac{d}{d\xi}\right)^n, \tag{3.4}$$

where

$$B_0=1,\ B_1=-\frac{1}{2},\ B_2=\frac{1}{6},\ B_3=-1,... \tag{3.5}$$

are Bernoulli numbers (see [9]; p.300(9)) and D_ξ^{-1} is the inverse of the derivative operator. Since $g(x)$ has a Taylor expansion ($g(x)=\sum_{m=0}^{\infty}b_mx^m$), we get from equations (3.3), (3.4)

$$f(e^{\xi}) = D_{\xi}^{-1}\sum_{n=0}^{\infty}\frac{B_n}{n!}(\lambda\ln(a))^n\left(\frac{d}{d\xi}\right)^n\left(\sum_{m=0}^{\infty}b_m e^{m\xi}\right)$$

$$= D_{\xi}^{-1}\left[B_0 + \frac{B_1}{1!}\lambda\ln(a)\frac{d}{d\xi} + \frac{B_2}{2!}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^2 + \frac{B_3}{3!}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^3 + \cdots\right]\sum_{m=0}^{\infty}b_m e^{m\xi}$$

$$= D_{\xi}^{-1}\left[1 - \frac{1}{2}\lambda\ln(a)\frac{d}{d\xi} + \frac{1}{12}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^2 - \frac{1}{6}\left(\lambda\ln(a)\frac{d}{d\xi}\right)^3 + \cdots\right]\sum_{m=0}^{\infty}b_m e^{m\xi}$$

$$= D_{\xi}^{-1}\Bigg[b_0 + \sum_{m=1}^{\infty}b_m e^{m\xi} - \frac{1}{2}\lambda\ln(a)\sum_{m=0}^{\infty}b_m m e^{m\xi}$$

$$+\frac{1}{12}(\lambda\ln(a))^2\sum_{m=0}^{\infty}b_m(m)^2e^{m\xi} - \frac{1}{6}(\lambda\ln(a))^3\sum_{m=0}^{\infty}b_m(m)^3e^{m\xi} + \cdots\Bigg]$$

$$= \Bigg[b_0\xi + \sum_{m=1}^{\infty}\frac{b_m}{m}e^{m\xi} - \frac{1}{2}\lambda\ln(a)\sum_{m=0}^{\infty}b_m e^{m\xi}$$

$$+\frac{1}{12}(\lambda\ln(a))^2\sum_{m=0}^{\infty}b_m(m)e^{m\xi} - \frac{1}{6}(\lambda\ln(a))^3\sum_{m=0}^{\infty}b_m(m)^2e^{m\xi} + \cdots\Bigg]$$

$$= b_0\xi + \left(1 - \frac{\lambda\ln(a)}{2} + \frac{(\lambda\ln(a))^2}{12} - \frac{(\lambda\ln(a))^3}{6} + \cdots\right)b_1e^{\xi}$$

$$+\left(1 - \frac{2\lambda\ln(a)}{2} + \frac{(2\lambda\ln(a))^2}{12} - \frac{(2\lambda\ln(a))^3}{6} + \cdots\right)\frac{b_2}{2}e^{2\xi} + \cdots$$

$$= b_0\xi + b_1e^{\xi}\left[1 + \left(\frac{\lambda\ln(a)}{2!} + \frac{(\lambda\ln(a))^2}{3!} + \cdots\right)\right]^{-1}$$

$$+b_2e^{\xi}\left[1 + \left(\frac{2\lambda\ln(a)}{2!} + \frac{(2\lambda\ln(a))^2}{3!} + \cdots\right)\right]^{-1} + \cdots$$

$$= b_0\xi + \frac{b_1e^{\xi}}{(1 + \frac{\lambda\ln(a)}{2!} + \frac{(\lambda\ln(a))^2}{3!} + \cdots)} + \frac{b_2e^{\xi}}{2[1 + (\frac{2\lambda\ln(a)}{2!} + \frac{(2\lambda\ln(a))^2}{3!} + \cdots)]} + \cdots$$

$$= b_0\xi + \frac{\lambda\ln(a)b_1e^{\xi}}{\frac{(\lambda\ln(a)}{1!} + \frac{(\lambda\ln(a))^2}{2!} + \frac{(\lambda\ln(a))^3}{3!} + \cdots} + \frac{\lambda\ln(a)b_2e^{\xi}}{\frac{2\lambda\ln(a)}{1!} + \frac{2(\lambda\ln(a))^2}{2!} + \frac{(2\lambda\ln(a))^3}{3!} + \cdots} + \cdots$$

$$= b_0\xi + \frac{\lambda\ln(a)b_1e^{\xi}}{e^{\lambda\ln(a)} - 1} + \frac{\lambda\ln(a)b_2e^{2\xi}}{e^{2\lambda\ln(a)} - 1} + \frac{\lambda\ln(a)b_3e^{3\xi}}{e^{3\lambda\ln(a)} - 1} + \cdots$$

or

$$f(e^{\xi}) = \sum_{m=1}^{\infty}b_m\frac{\lambda\ln(a)e^{m\xi}}{e^{\lambda m\ln(a)} - 1} + b_0\xi,$$

or

$$f(e^{\xi}) = \sum_{m=1}^{\infty} b_m \frac{\lambda \ln(a) e^{m\xi}}{a^{\lambda m} - 1} + b_0 \xi. \tag{3.6}$$

Going back to the original variable, we get

$$f(x) = \sum_{m=1}^{\infty} b_m \frac{\lambda \ln(a) x^m}{a^{\lambda m} - 1} + b_0 \ln(x). \tag{3.7}$$

The series on the right hand side of equation (3.6) provides the solution of our problem.

Taking another example, $g(x) = \sin(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$, we find

$$\begin{aligned}
f(e^{\xi}) &= D_{\xi}^{-1} \sum_{n=0}^{\infty} \frac{B_n (\lambda \ln(a))^n}{n!} \left(\frac{d}{d\xi}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} \\
&= D_{\xi}^{-1} \left[B_0 + \frac{B_1}{1!} \lambda \ln(a) \frac{d}{d\xi} + \frac{B_2}{2!} \left(\lambda \ln(a) \frac{d}{d\xi}\right)^2 \right. \\
&\qquad \left. + \frac{B_3}{3!} \left(\lambda \ln(a) \frac{d}{d\xi}\right)^3 + \cdots \right] \cdot \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} \\
&= D_{\xi}^{-1} \left[1 - \frac{1}{2} \lambda \ln(a) \frac{d}{d\xi} + \frac{1}{12} \left(\lambda \ln(a) \frac{d}{d\xi}\right)^2 \right. \\
&\qquad \left. - \frac{1}{6} \left(\lambda \ln(a) \frac{d}{d\xi}\right)^3 + \cdots \right] \cdot \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} \\
&= D_{\xi}^{-1} \left[\sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} - \frac{1}{2} \left(\lambda \ln(a) \frac{d}{d\xi}\right) \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} \right. \\
&\quad \left. + \frac{1}{12} \left(\lambda \ln(a) \frac{d}{d\xi}\right)^2 \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} - \frac{1}{6} \left(\lambda \ln(a) \frac{d}{d\xi}\right)^3 \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)!} \cdots \right] \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)(2m+1)!} e^{(2m+1)\xi} - \frac{1}{2} \lambda \ln(a) \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} e^{(2m+1)\xi} \\
&\quad + \frac{1}{12} (\lambda \ln(a))^2 \sum_{m=0}^{\infty} \frac{(-1)^m (2m+1)}{(2m+1)!} e^{(2m+1)\xi} \\
&\qquad - \frac{1}{6} (\lambda \ln(a))^3 \cdot \sum_{m=0}^{\infty} \frac{(-1)^m (2m+1)^2}{(2m+1)!} e^{(2m+1)\xi} \cdots
\end{aligned}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)(2m+1)!} \left[1 - \frac{\lambda \ln(a)(2m+1)}{2} \right.$$

$$\left. + \frac{(\lambda \ln(a))^2 (2m+1)^2}{12} - \frac{(\lambda \ln(a))^3 (2m+1)^3}{6} + \cdots \right]$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m e^{(2m+1)\xi}}{(2m+1)(2m+1)!} \left[1 + \frac{\lambda \ln(a)(2m+1)}{2!} + \frac{(\lambda \ln(a))^2 (2m+1)^2}{3!} + \cdots \right]^{-1}$$

or

$$f(e^{\xi}) = \sum \frac{\lambda \ln(a)}{a^{\lambda(2m+1)} - 1} \frac{(-1)^m}{(2m+1)!} e^{(2m+1)\xi}$$

or

$$f(x) = \sum \frac{\lambda \ln(a)}{a^{\lambda(2m+1)} - 1} \frac{(-1)^m}{(2m+1)!} x^{2m+1}. \tag{3.8}$$

It is essentially the series defining $g(x)$, provided that b_m is replaced by

$$\frac{b_m \lambda \ln a}{a^{\lambda m} - 1} \, .$$

If, e.g., we take $g(x) = \cos(x)$, we find

$$f(x) = \sum \frac{\lambda \ln(a)}{a^{\lambda(2m)} - 1} \frac{(-1)^m}{(2m)!} x^{2m}, \tag{3.8$'$}$$

and for $g(x) = e^{x^q}$, we get

$$f(x) = \sum_{m=1}^{\infty} \frac{\lambda \ln(a)}{a^{\lambda q m} - 1} \frac{x^{qm}}{m!} + \ln(x). \tag{3.9}$$

We can, therefore, conclude that the primitive of a Jackson derivative can be constructed according to the above-quoted recipe.

This method can also be generalized and the concept of Jackson derivative extended to other forms of exponential operators. In this case we consider equation of the type

$$\frac{f(x)\cos(\lambda \ln(a)) + \sqrt{1-x^2}\sin(\lambda \ln(a))}{\lambda \ln(a)} \;=\; g(x), \tag{3.10}$$

with the assistance of equation (2.2), we write equation (3.10) as follows:

$$\frac{a^{\lambda\sqrt{1-x^2}} f(x) - f(x)}{\lambda \ln(a)} = g(x) \quad \text{or} \quad \frac{(a^{\lambda\sqrt{1-x^2}} - 1)}{\lambda \ln(a)} f(x) = g(x), \tag{3.11}$$

by assuming $g(x)$ is an odd function there taking $x = \sin\xi$, we have

$$\frac{d}{d\xi} = \frac{d}{dx}\frac{dx}{d\xi} = \cos\xi\frac{d}{dx},$$

$$\frac{(a^{\lambda\frac{d}{d\xi}} - 1)}{\lambda\ln(a)}f(\sin\xi) = g(\sin\xi), \quad \text{or} \tag{3.12}$$

$$f(\sin\xi) = \frac{\lambda\ln(a)}{a^{\lambda\frac{d}{d\xi}} - 1}g(\sin\xi),$$

let us find out the expansion of the first factor of the r.h.s. of equation (3.12) with the help of equation (3.4), we have

$$f(\sin\xi) = D_\xi^{-1}\sum_{n=0}^{\infty}\frac{B_n}{n!}(\lambda\ln(a))^n\left(\frac{d}{d\xi}\right)^n g(\sin\xi), \tag{3.13}$$

since $g(x)$ in an odd and analytic function, then $g(\sin\xi)$, can be expanded by Taylor expansion such as $g(\sin\xi) = \sum_{m=0}^{\infty} b_{2m+1}(\sin\xi)^{2m+1}$, we have from (3.13),

$$f(\sin\xi) = D_\xi^{-1}\sum_{n=0}^{\infty}\frac{B_n}{n!}(\lambda\ln(a))^n\left(\frac{d}{d\xi}\right)^n\sum_{m=0}^{\infty} b_{2m+1}(\sin\xi)^{2m+1}$$

$$= D_\xi^{-1}\sum_{n=0}^{\infty}\frac{B_n}{n!}(\lambda\ln(a))^n\left(\frac{d}{d\xi}\right)^n\sum_{m=0}^{\infty} b_{2m+1}\left[\frac{e^{i\xi} - e^{-i\xi}}{2i}\right]^{2m+1}$$

$$= D_\xi^{-1}\sum_{n=0}^{\infty}\frac{B_n}{n!}(\lambda\ln(a))^n\left(\frac{d}{d\xi}\right)^n\sum_{m=0}^{\infty}\frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}}(-e^{-i\xi})^{2m+1}[1 - e^{2i\xi}]^{2m+1}$$

$$= D_\xi^{-1}\left[B_0 + \frac{B_1(\lambda\ln(a))}{1!}\left(\frac{d}{d\xi}\right) + \frac{B_2(\lambda\ln(a))^2}{2!}\left(\frac{d}{d\xi}\right)^2 + \frac{B_3(\lambda\ln(a))^3}{3!}\left(\frac{d}{d\xi}\right)^3\cdots\right]$$

$$\cdot\sum_{m=0}^{\infty}\frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}}(-e^{-i\xi})^{2m+1}\sum_{s=0}^{2m+1}\binom{2m+1}{s}(-1)^{2m+1-s}e^{[2(2m+1-s)]i\xi}.$$

Substituting the values of Bernoulli's numbers from equation (3.5), we have

$$(3.14) = D_\xi^{-1}\left[1 - \frac{(\lambda\ln(a))}{2}\left(\frac{d}{d\xi}\right) + \frac{(\lambda\ln(a))^2}{12}\left(\frac{d}{d\xi}\right)^2 - \frac{(\lambda\ln(a))^3}{6}\left(\frac{d}{d\xi}\right)^3\cdots\right]$$

$$\cdot\sum_{m=1}^{\infty}\frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}}\sum_{s=0}^{2m+1}\binom{2m+1}{s}(-1)^s e^{i[2(m-s)+1]\xi}$$

$$= \sum_{m=1}^{\infty}\frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}}D_\xi^{-1}\left[\sum_{s=0}^{2m+1}\binom{2m+1}{s}(-1)^s e^{i[2(m-s)+1]\xi} - \frac{(\lambda\ln(a))}{2}\left(\frac{d}{d\xi}\right)\right.$$

$$\cdot \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s e^{i[2(m-s)+1]\xi} + \frac{(\lambda \ln(a))^2}{12} \left(\frac{d}{d\xi}\right)^2$$

$$\left.\cdot \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s e^{i[2(m-s)+1]\xi} - \cdots\right]$$

$$= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} D_{\xi}^{-1} \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s e^{i[2(m-s)+1]\xi}\right.$$

$$-\frac{(\lambda \ln(a))}{2} \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s [i\{2(m-s)+1\}\xi] e^{i[2(m-s)+1]\xi}$$

$$\left.+\frac{(\lambda \ln(a))^2}{12} \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s [i\{2(m-s)+1\}\xi]^2 e^{i[2(m-s)+1]\xi} - \cdots\right]$$

$$= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s} \frac{(-1)^s e^{i[2(m-s)+1]\xi}}{i[2(m-s)+1]}\right.$$

$$-\frac{(\lambda \ln(a))}{2} \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s e^{i[2(m-s)+1]\xi}$$

$$\left.+\frac{(\lambda \ln(a))^2}{12} \sum_{s=0}^{2m+1} \binom{2m+1}{s} (-1)^s [i\{2(m-s)+1\}\xi] e^{i[2(m-s)+1]\xi} - \cdots\right]$$

$$= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s} \frac{(-1)^s e^{i[2(m-s)+1]\xi}}{i[2(m-s)+1]}\right.$$

$$\left\{1 - \frac{(\lambda \ln(a))}{2}[i\{2(m-s)+1\}] + \frac{(\lambda \ln(a))^2}{12}[i\{2(m-s)+1\}]^2\right.$$

$$\left.\left.-\frac{(\lambda \ln(a))^3}{6}[i\{2(m-s)+1\}]^3 + \cdots\right\}\right]$$

$$= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s}\right.$$

$$\left.\cdot \frac{(-1)^s e^{i[2(m-s)+1]\xi}}{i[2(m-s)+1]\left\{1+\frac{1}{2!}[i\{2(m-s)+1\}\lambda \ln(a)]+\frac{1}{3!}[i\{2(m-s)+1\}\lambda \ln(a)]^2+\cdots\right\}}\right]$$

$$= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s}\right.$$

$$\left.\cdot \frac{(-1)^s \lambda \ln(a) (e^{i\xi})^{2(m-s)+1}}{[i\{2(m-s)+1\}\lambda \ln(a)]+\frac{1}{2!}[i\{2(m-s)+1\}\lambda \ln(a)]^2+\frac{1}{3!}[i\{2(m-s)+1\}\lambda \ln(a)]^3+\cdots}\right]$$

$$= \sum_{m=1}^{\infty} \frac{b_{2m+1}(-i)^{2m+1}}{2^{2m+1}} \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s} \frac{(-1)^s \lambda \ln(a) (\cos\xi + i\sin\xi)^{2(m-s)+1}}{e^{i[2(m-s)+1]\lambda \ln(a)} - 1}\right]$$

or

$$\begin{aligned} f(\sin\xi) &= \sum_{m=1}^{\infty} \frac{b_{2m+1}\lambda\ln(a)}{2^{2m+1}}(-i)^{2m+1} \\ &\cdot \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s} \frac{(-1)^s(\sqrt{1-\sin\xi^2}+i\sin\xi)^{2(m-s)+1}}{e^{i[2(m-s)+1]\lambda\ln(a)}-1}\right]. \end{aligned}$$

Finally,

$$\begin{aligned} f(x) &= \sum_{m=1}^{\infty} \frac{b_{2m+1}\lambda\ln(a)}{2^{2m+1}}(-i)^{2m+1} \\ &\cdot \left[\sum_{s=0}^{2m+1} \binom{2m+1}{s} \frac{(-1)^s(\sqrt{1-x^2}+ix)^{2(m-s)+1}}{a^{i[2(m-s)+1]\lambda}-1}\right] \end{aligned} \tag{3.15}$$

It is interesting to note that, in this case too, the criterion to evaluate the primitive of the Jackson derivative, associated with the operator (2.2), can easily be inferred.

Let us note that the procedure we have discussed can also be extended to the cases involving the generalized Gauss transform. In fact the solution of

$$\frac{a^{\lambda(x\frac{d}{dx})^2}-1}{\lambda\ln(a)}f(x) = g(x) \tag{3.16}$$

or in other words, we have

$$\frac{\lambda\ln(a)}{a^{\lambda(x\frac{d}{dx})^2}-1}g(x) = f(x), \tag{3.17}$$

let us suppose $x = e^{\xi}$, then

$$\frac{d}{d\xi} = \frac{d}{dx}\frac{dx}{d\xi} = e^{\xi}\frac{d}{dx} = x\frac{d}{dx}\cdot$$

Now, from equation (3.17), we have

$$\frac{\lambda\ln(a)}{a^{\lambda(\frac{d}{d\xi})^2}-1}g(e^{\xi}) = f(e^{\xi}). \tag{3.18}$$

Further, after following the steps as we followed in getting the result (3.4), we obtain the expansion of first factor of l.h.s. as

$$\frac{\lambda\ln(a)}{a^{\lambda(\frac{d}{d\xi})^2}-1} = D_{\xi}^{-2}\sum_{n=0}^{\infty}\frac{B_n}{n!}(\lambda\ln(a))^n\left(\frac{d}{d\xi}\right)^{2n}. \tag{3.19}$$

Now by the virtue of the analyticity of $g(x)$, we expand $g(x)$, by Taylor series, i.e.,

$$f(e^{\xi}) = D_{\xi}^{-2} \sum_{n=0}^{\infty} \frac{B_n}{n!} (\lambda \ln(a))^n \left(\frac{d}{d\xi}\right)^{2n} \sum_{m=0}^{\infty} b_m e^{mx},$$

proceeding of the steps as proceeded in finding the result (3.7), we obtain

$$f(e^{\xi}) = \sum_{m=1}^{\infty} b_m \frac{\lambda \ln(a)}{e^{\lambda \ln(a) m^2} - 1} e^{m\xi} + b_0 \xi$$

or in other words, if we take $b_0 = 0$, then

$$f(x) = \sum_{m=1}^{\infty} b_m \frac{\lambda \ln(a)}{a^{\lambda m^2} - 1} x^m. \tag{3.20}$$

Further comments on the results of this section will be discussed in the forthcoming concluding section.

3(a). Particular case: If we substitute $a = e$, in equations (3.1), (3.2), (3.3), (3.4), (3.7), (3.8), (3.9), (3.10), (3.15), (3.16) and (3.20), then we obtain the equations (31), (32), (33), (34), (36), (37), (38), (39), (40), (41) and (42) on page numbers 657-658 due to Dattoli et al. [1].

4. Remarks

In the previous section we have considered linear difference equations, a (trivial) non-linear example, similar to "Riccati" equation, is given blow

$$f(ax) - f(x) + b^{\ln(x)} f(ax) f(x) = 0, \tag{4.1}$$

which can be solved using the auxiliary function $g(x) = \dfrac{1}{f(x)}$ and thus getting

$$\frac{1}{f(x)} - \frac{1}{f(ax)} + b^{\ln(x)} = 0, \qquad \text{or}$$
$$g(ax) - g(x) = b^{\ln(x)}. \tag{4.2}$$

From operator (1.5), we have

$$a^{x\frac{d}{dx}} g(x) - g(x) = b^{\ln(x)}, \tag{4.3}$$

or, in other words, we write the above equation (4.3) as

$$\begin{aligned} g(x) &= \frac{b^{\ln(x)}}{a^{x\frac{d}{dx}}-1} \\ &= -[1-a^{x\frac{d}{dx}}]^{-1}b^{\ln(x)} \\ &= -[1+a^{x\frac{d}{dx}}+a^{2x\frac{d}{dx}}+a^{3x\frac{d}{dx}}+\cdots]b^{\ln(x)} \\ &= -[b^{\ln(x)}+a^{x\frac{d}{dx}}b^{\ln(x)}+a^{2x\frac{d}{dx}}b^{\ln(x)}+a^{3x\frac{d}{dx}}b^{\ln(x)}+\cdots] \\ &= -[b^{\ln(x)}+b^{\ln(a)}b^{\ln(x)}+b^{2\ln(a)}b^{\ln(x)}+b^{3\ln(a)}b^{\ln(x)}+\cdots] \\ &= -[1+b^{\ln(a)}+b^{2\ln(a)}+b^{3\ln(a)}+\cdots]b^{\ln(x)} \\ &= -[\frac{1}{1-b^{\ln(a)}}]b^{\ln(x)} \\ &= \frac{b^{\ln(x)}}{b^{\ln(a)}-1} \end{aligned}$$

thus finding as a particular solution

$$f(x)=\frac{1}{g(x)}=\frac{b^{\ln(a)}-1}{b^{\ln(x)}}. \tag{4.4}$$

Moreover, in general, equations of the type

$$\sum_{\alpha=0}^{N} b_\alpha f(F^{-1}(\alpha\ln(a)+F(x)))=\epsilon[f(x)]^n, \tag{4.5}$$

standard perturbation methods can be used. At the lowest order in $\epsilon(f\cong f_0+\epsilon f_1)$, we find

$$\sum_{\alpha=0}^{N} b_\alpha f_0(F^{-1}(\alpha\ln(a)+F(x)))=0 \tag{4.6}$$

and

$$\sum_{\alpha=0}^{N} b_\alpha f_1(F^{-1}(\alpha\ln(a)+F(x)))=R^{n\int^x\frac{d\xi}{q(\xi)}}, \tag{4.7}$$

where $R^{\ln(a)}$ is one of the roots of the characteristic equation associated with (4.5). The first-order contribution f_1 can therefore be evaluated by using equation (2.15), which should be modified as follows:

$$\begin{aligned} f(x) &= \frac{C(\int^x\frac{d\xi}{q(\xi)})b^{\int^x\frac{d\xi}{q(\xi)}-1}}{\Psi'(b^{\ln(a)})}, \\ \Psi'(b^{\ln(a)}) &= \left[\frac{d}{dR}\Psi(R^{\ln(a)})\right]_{R=b}, \end{aligned} \tag{4.8}$$

if $b^{\ln(a)}$ is a simple root of the characteristic equation.

4(a). Particular case: The replacement of a with e in the equations (4.1), (4.4) and (4.8) reduce to the equations (43), (45) and (49) of Dattoli et al. [1].

Let us now go back to the problem of treating exponential operators of the type

$$\widehat{A}_{m,\lambda} = a^{\lambda(q(x)\frac{d}{dx})^m}. \tag{4.9}$$

We have seen that, for $m = 2$ and $\lambda > 0$, they can be viewed as generalized Gauss transform. Before discussing the problem more deeply, we recall the following important relation [2]:

$$\begin{cases} a^{\lambda(\frac{d}{dx})^m} x^n = H_n^{(m)}(x, \lambda \ln(a)), \\ \\ H_n^{(m)}(x, \lambda \ln(a)) = n! \displaystyle\sum_{r=0}^{[\frac{n}{m}]} \frac{(\lambda \ln(a))^r x^{n-mr}}{r!(n-mr)!}. \end{cases} \tag{4.10}$$

which holds for negative or positive λ and $H_n^{(m)}(x, \lambda \ln(a))$ are Kampé de Feriét polynomials, and satisfy the identity

$$\frac{\partial}{\partial \lambda} H_n^{(m)}(x, \lambda \ln(a)) = \ln(a) \left(\frac{\partial}{\partial x} \right)^m H_n^{(m)}(x, \lambda \ln(a)). \tag{4.11}$$

According to equation (4.8) we also find

$$\begin{aligned} a^{\lambda(\frac{d}{dx})^m} g(x) &= a^{\lambda(\frac{d}{dx})^m} \sum_{n=0}^{\infty} b_n x^n \\ &= \sum_{n=0}^{\infty} b_n H_n^{(m)}(x, \lambda \ln(a)). \end{aligned} \tag{4.12}$$

Therefore, it is easy to realize that

$$A_{m,\lambda} x = \sum_{n=0}^{\infty} \phi_n H_n^{(m)}(F(x), \lambda \ln(a)), \tag{4.13}$$

where we have assumed that the function F^{-1} can be expanded in power series

$$F^{-1}(\zeta) = \sum_{n=0}^{\infty} \phi_n \zeta^n. \tag{4.14}$$

It is clear that equation (4.12) can be further handled to extend the action of operators (4.8) to any function $g(x)$. It is worth considering the possibility of extending the definition of operators (4.8) to the case of not necessarily integer m. In the case of $m = \frac{1}{2}$, equation (4.9) should be replaced by

$$\begin{cases} a^{\lambda(\frac{d}{dx})^{\frac{1}{2}}} x^n = H_n^{(\frac{1}{2})}(x, \lambda \ln(a)), \\ \\ H_n^{(\frac{1}{2})}(x, \lambda \ln(a)) = n! \displaystyle\sum_{r=0}^{2n} \frac{(\lambda \ln(a))^r x^{n-\frac{r}{2}}}{r!\Gamma(n-\frac{r}{2}+1)}. \end{cases} \tag{4.15}$$

It is evident that in this case $H_n^{(\frac{1}{2})}(x, \lambda \ln(a))$ is a relation analogous to (4.10) holds, namely

$$\frac{\partial^2}{\partial \lambda^2} H_n^{(\frac{1}{2})}(x, \lambda \ln(a)) = (\ln(a))^2 \left(\frac{\partial}{\partial x}\right) H_n^{(\frac{1}{2})}(x, \lambda \ln(a)). \tag{4.16}$$

or involving semi derivatives [8]

$$\frac{\partial}{\partial \lambda} H_n^{(\frac{1}{2})}(x, \lambda \ln(a)) = \ln(a) \left(\frac{\partial}{\partial x}\right)^{\frac{1}{2}} H_n^{(\frac{1}{2})}(x, \lambda \ln(a)). \tag{4.16$'$}$$

This definition can be extended to any $m = \dfrac{1}{p}$ (p, integer).

4(b). Particular case: Equations (4.9), (4.10), (4.11), (4.12), (4.13), (4.15) and (4.16) lead to Dattoli's et al. [1] equations (50), (51), (52), (53), (54), (56) and (57).

Concluding remark. It is hope that for the values other than e some more use of the generalized exponential operators can be obtained.

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