

A CHARACTERIZATION OF HIGHER DERIVATIONS ON BANACH ALGEBRAS

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Abstract. Let \mathcal{A} be a Banach algebra and let every module-valued derivation from \mathcal{A} to any Banach \mathcal{A} -bimodule be continuous. We show that if $\{d_m\}$ is a higher derivation from \mathcal{A} to a Banach algebra \mathcal{B} with continuous d_0 , then there exist a continuous left \mathcal{A} -module homomorphism $U : \mathfrak{B}(\mathcal{A}_1, \mathcal{B}) \rightarrow \mathcal{B}$ and a sequence $\{D_m\}$ of module-valued derivations from \mathcal{A} into $\mathfrak{B}(\mathcal{A}_1, \mathcal{B})$ such that $d_m = U \circ D_m$ ($m \geq 1$), and as a consequence $\{d_m\}$ is automatically continuous. We also obtain a partial result concerning innerness of higher derivations on W^* -algebras.

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1. Introduction

Let \mathcal{A} and \mathcal{B} be algebras. A family of linear mappings $\{d_m\}_{m=0}^k$ (k might be ∞) from \mathcal{A} into \mathcal{B} is called a *higher derivation of rank k* if

$$d_m(ab) = \sum_{j=0}^m d_j(a)d_{m-j}(b) \quad (a, b \in \mathcal{A}, \quad m = 0, 1, 2, \dots, k).$$

If there is no cause of ambiguity, a higher derivation will be simply denoted by $\{d_m\}$. It is obvious that for a higher derivation $\{d_m\}$, d_0 is a homomorphism and d_1 is a d_0 -derivation that is, $d_1(ab) = d_0(a)d_1(b) + d_1(a)d_0(b)$. A standard example

of a higher derivation of rank k is $\{\frac{D^m}{m!}\}_{m=0}^k$, where $D : \mathcal{A} \rightarrow \mathcal{A}$ is a derivation. A higher derivation $\{d_m\}$ is said to be continuous if d_m is continuous for all $m \geq 0$.

Higher derivations were introduced by Hasse and Schmidt [5] and algebraists sometimes call them Hasse-Schmidt derivations. The reader may find more about the algebraic properties of higher derivations in [1], [4], [5], [15], [18], [21], [19], [6], [12]. Loy [11], Jewell [9] and Villena [20] proved the automatic continuity of higher derivations in certain cases. In [7] and [8], the authors proved some results concerning higher derivations on JB^* -algebras and Banach algebras. If $\{d_m\}$ is a higher derivation from \mathcal{A} to \mathcal{A} such that d_0 is the identity map on \mathcal{A} , then d_1 is a derivation and $\{d_m\}$ is called a *strong* higher derivation. In [10] Jun and Lee proved the Singer-Wrerner theorem for strong higher derivations. Mirzavaziri in [13] gives a characterization of a strong higher derivation defined on an algebra.

Let $\{d_m\}$ be a higher derivation from a Banach algebra \mathcal{A} to a Banach algebra \mathcal{B} . Define

$$(1.1) \quad a.x = d_0(a)x, \quad x.a = xd_0(a) \quad (a \in \mathcal{A}, x \in \mathcal{B}).$$

Since d_0 is a homomorphism, \mathcal{B} is an \mathcal{A} -bimodule with respect to the mappings

$$(a, x) \rightarrow a.x, \quad (a, x) \rightarrow x.a, \quad \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{B}.$$

It is easy to see that \mathcal{B} is a Banach \mathcal{A} -bimodule provided that d_0 is continuous. In section 2 we give a characterization for higher derivations on certain Banach algebras. We show that if every module-valued derivation on a Banach algebra \mathcal{A} is continuous, then each higher derivation $\{d_m\}$ from \mathcal{A} to a Banach algebra \mathcal{B} with continuous d_0 , is of the form $d_m = U \circ D_m$ ($m \geq 1$), where U is a continuous left \mathcal{A} -module homomorphism and each D_m is a module-valued derivation. Therefore $\{d_m\}$ is continuous. As a consequence every higher derivation from a C^* -algebra, with continuous d_0 , is continuous. In section 3 we define an inner higher derivation. We show that if \mathcal{A} is a commutative W^* -subalgebra of a W^* -algebra \mathfrak{M} , then each strong higher derivation from \mathcal{A} to \mathfrak{M} is inner.

2. Characterization

Let \mathcal{A} be a Banach algebra and \mathcal{X} a Banach \mathcal{A} -bimodule. A linear map $S : \mathcal{A} \rightarrow \mathcal{X}$ is said to be left-intertwining if the map

$$b \mapsto aS(b) - S(ab), \quad \mathcal{A} \rightarrow \mathcal{X},$$

is continuous for each $a \in \mathcal{A}$, and right-intertwining if the map

$$a \mapsto S(a)b - S(ab), \quad \mathcal{A} \rightarrow \mathcal{X},$$

is continuous for all $b \in \mathcal{A}$. A linear map $S : \mathcal{A} \rightarrow \mathcal{X}$ is intertwining if it is both left- and right-intertwining. For more about these objects see [2, Section 2.7].

Remark 2.1

- (i) Let \mathcal{A} and \mathcal{B} be Banach algebras. Suppose that $\{d_m\}$ is a higher derivation from \mathcal{A} into \mathcal{B} for which d_0 is continuous. Consider \mathcal{B} as a Banach \mathcal{A} -bimodule as in (1.1). Then it is easy to see that for every integer $m \geq 1$, $d_m : \mathcal{A} \rightarrow \mathcal{B}$ is an intertwining map whenever d_0, \dots, d_{m-1} are continuous.
- (ii) Let \mathcal{A} be a Banach algebra and \mathcal{X} a Banach \mathcal{A} -bimodule. Consider $\mathcal{A}_1 = \mathbb{C} \oplus \mathcal{A}$ to be the Banach algebra unitization of \mathcal{A} . Even if \mathcal{A} is unital, $\mathcal{A}_1 \neq \mathcal{A}$ and \mathcal{A}_1 is a unital Banach algebra containing \mathcal{A} as a closed ideal. The identity $(1, 0)$ of \mathcal{A}_1 will be denoted by 1 . Set $\mathcal{F} = \mathfrak{B}(\mathcal{A}_1, \mathcal{X})$, the Banach space of all bounded linear operators from \mathcal{A}_1 to \mathcal{X} . For $a \in \mathcal{A}$ and $T \in \mathcal{F}$, define

$$(a.T)(b) = aT(b), \quad (T.a)(b) = T(ab) \quad (b \in \mathcal{A}_1).$$

Then \mathcal{F} is an \mathcal{A} -bimodule with respect to the maps

$$(a, T) \rightarrow a.T, \quad (a, T) \rightarrow T.a, \quad \mathcal{A} \times \mathcal{F} \rightarrow \mathcal{F}.$$

Now, the map

$$U : T \mapsto T(1), \quad \mathcal{F} \rightarrow \mathcal{X}$$

is a continuous linear operator and clearly

$$U(a.T) = aU(T) \quad (a \in \mathcal{A}, S \in \mathcal{F}),$$

so that U is a left \mathcal{A} -module homomorphism.

Dales and Villena in [3, Theorem 2.1] proved that \mathcal{F} is a Banach \mathcal{A} -bimodule. Also in the same theorem it has been shown that each left-intertwining map $S : \mathcal{A} \rightarrow \mathcal{X}$ is of the form $S = U \circ D$, where U is defined as above and $D : \mathcal{A} \rightarrow \mathcal{F} = \mathfrak{B}(\mathcal{A}_1, \mathcal{X})$ is a derivation defined by

$$D(a)(\beta, b) = S(\beta a + ab) - a.S(b) \quad (\beta \in \mathbb{C}, a, b \in \mathcal{A}).$$

Theorem 2.2 *Let \mathcal{A} be a Banach algebra for which every derivation from \mathcal{A} into an arbitrary Banach \mathcal{A} -bimodule is continuous. Suppose that $\{d_m\}$ is a higher derivation from \mathcal{A} to a Banach algebra \mathcal{B} with a continuous d_0 . Then there exists a sequence $\{D_m\}_{m \geq 1}$ of derivations from \mathcal{A} to $\mathfrak{B}(\mathcal{A}_1, \mathcal{B})$ such that $d_m = U \circ D_m$ ($m \geq 1$), where $U : \mathfrak{B}(\mathcal{A}_1, \mathcal{B}) \rightarrow \mathcal{B}$ is the continuous left \mathcal{A} -module homomorphism defined by $U(T) = T(1)$ for all $T \in \mathfrak{B}(\mathcal{A}_1, \mathcal{B})$. Moreover, $\{d_m\}$ is automatically continuous.*

Proof. By continuity of d_0 , \mathcal{B} is a Banach \mathcal{A} -bimodule with module operations defined in (1.1). Therefore d_1 is a module-valued derivation to a Banach \mathcal{A} -bimodule and also an intertwining by Remark 2.1 (i). Now, by Remark 2.1 (ii) there exists a derivation $D_1 : \mathcal{A} \rightarrow \mathfrak{B}(\mathcal{A}_1, \mathcal{B})$ such that $d_1 = U \circ D_1$, where

$U : \mathfrak{B}(\mathcal{A}_1, \mathcal{B}) \rightarrow \mathcal{B}$ is defined by $U(T) = T(1)$ ($T \in \mathfrak{B}(\mathcal{A}_1, \mathcal{B})$) which is a continuous left \mathcal{A} -module homomorphism. Continuity of d_1 is obvious by the assumption. Now by induction assume that for $i = 1, \dots, m-1$, $d_i = U \circ D_i$, where each D_i is a derivation from \mathcal{A} to $\mathfrak{B}(\mathcal{A}_1, \mathcal{B})$ which is continuous by the hypothesis and $U : \mathfrak{B}(\mathcal{A}_1, \mathcal{B}) \rightarrow \mathcal{B}$ is as before. Now we have d_0, d_1, \dots, d_{m-1} are continuous and hence d_m is an intertwining map and again by Remark 2.1 (ii) it is of the form $U \circ D_m$, where U is as before and $D_m : \mathcal{A} \rightarrow \mathfrak{B}(\mathcal{A}_1, \mathcal{B})$ is a derivation. Since by the assumption each D_m is continuous, the last assertion follows easily. ■

Corollary 2.3 *Every higher derivation $\{d_m\}$ from a C^* -algebra to a Banach algebra, with continuous d_0 , is continuous.*

Proof. Since every module-valued derivation from a C^* -algebra is continuous [14], then by Theorem 2.2 we have the result. ■

We recall that a derivation δ from a Banach algebra \mathcal{A} to a Banach \mathcal{A} -bimodule \mathcal{X} is said to be inner if there exists $x \in \mathcal{X}$ such that $\delta(a) = ax - xa$ ($a \in \mathcal{A}$). A Banach algebra \mathcal{A} for which every bounded module-valued derivation to an arbitrary Banach \mathcal{A} -bimodule is inner is called *super-amenable* [16].

Corollary 2.4 *Let \mathcal{A} be a super-amenable Banach algebra satisfying the hypothesis of Theorem 2.2. Then for every higher derivation $\{d_m\}$ from \mathcal{A} to a Banach algebra \mathcal{B} , with continuous d_0 , we have $d_m = U \circ \delta_m$ ($m \geq 1$), where each δ_m is an inner derivation from \mathcal{A} to $\mathfrak{B}(\mathcal{A}_1, \mathcal{B})$ and U is defined as in Remark 2.1 (ii).*

3. Inner higher derivations

We recall the definition of an inner higher derivation from [15].

Definition 3.1 Let \mathcal{A} and \mathcal{B} be Banach algebras and let $\{d_m\}$ be a higher derivation from \mathcal{A} into \mathcal{B} . Then $\{d_m\}$ is called *inner* if for each $m \in \mathbb{N}$, there are $u_1, \dots, u_m \in \mathcal{B}$ such that

$$d_m(a) = d_0(a)u_m - \sum_{i=0}^{m-1} u_{m-i}d_i(a) \quad (a \in \mathcal{A}, m \in \mathbb{N}).$$

Note that if d_0 is continuous, then the inner higher derivation $\{d_m\}$ is also continuous.

Example 3.2 If $\{d_m\}$ is a higher derivation from a unital Banach algebra \mathcal{A} to a Banach algebra \mathcal{B} such that $d_0(\mathcal{A})\mathcal{B} = 0$, then $\{d_m\}$ is inner. To see this, suppose $a \in \mathcal{A}$ and let e be the identity element of \mathcal{A} . Then

$$d_m(a) = d_m(ea) = \sum_{i=0}^m d_i(e)d_{m-i}(a) = d_1(e)d_{m-1}(a) + \dots + d_{m-1}(e)d_1(a) + d_m(e)d_0(a).$$

For $1 \leq i \leq m$ put $u_i = -d_i(e)$, then we have

$$d_m(a) = d_0(a)u_m - u_m d_0(a) - u_{m-1}d_1(a) - \dots - u_1 d_{m-1}(a).$$

Therefore $\{d_m\}$ is inner.

It is well known that every derivation from a W^* -algebra \mathfrak{M} to itself is inner [17, Theorem 2.5.3]. Also by [17, Corollary 2.5.4], if \mathcal{A} is a C^* -subalgebra of $\mathfrak{B}(\mathcal{H})$ where $\mathfrak{B}(\mathcal{H})$ is the C^* -algebra of all bounded linear operators on a Hilbert space \mathcal{H} , then every derivation $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is inner when we consider it as a derivation from \mathcal{A} to $\mathfrak{B}(\mathcal{H})$. More precisely, there exists $x \in \mathfrak{B}(\mathcal{H})$ such that $\delta(a) = ax - xa$ ($a \in \mathcal{A}$).

Proposition 3.3 *Let \mathfrak{M} be a commutative W^* -algebra and $\{d_m\}$ a strong higher derivation from \mathfrak{M} to \mathfrak{M} . Then each d_m ($m \geq 1$) is zero.*

Proof. By [17, Theorem 2.5.3] the result is obvious. ■

We are far from a proof of Sakai’s result [17, Theorem 2.5.3] for higher derivations, but we can prove some partial results concerning higher derivations on W^* -algebras.

We recall the well known Markov-Kakutani theorem.

Theorem 3.4 *Let K be a non-empty convex compact subset of a locally convex space and let \mathcal{S} be a commutative semigroup of continuous affine maps on K . Then \mathcal{S} has a fixed point.*

Theorem 3.5 *Let \mathfrak{M} be a W^* -algebra with identity element e and $\{d_m\}$ a strong higher derivation from \mathfrak{M} to \mathfrak{M} . Let \mathcal{A} be a commutative W^* -subalgebra of \mathfrak{M} containing e . Then for each $m \in \mathbb{N}$ there are $u_0 = e, u_1, \dots, u_m$ in \mathfrak{M} such that*

$$d_m(a) = au_m - \sum_{i=0}^{m-1} u_{m-i}d_i(a) \text{ for all } a \in \mathcal{A} \text{ and}$$

$$\|u_m\| \leq \|d_m\| + \|u_{m-1}\|\|d_1\| + \dots + \|u_1\|\|d_{m-1}\|.$$

Proof. By [17, Lemma 2.5.1] the result holds for $m = 1$. Now suppose that for each $j \in \{1, \dots, m - 1\}$ there exist $u_0 = e, u_1, \dots, u_j$ in \mathfrak{M} such that, $d_j(a) =$

$$au_j - \sum_{i=0}^j u_{j-i}d_i(a) \text{ for all } a \in \mathcal{A} \text{ and}$$

$$\|u_{m-1}\| \leq \|d_{m-1}\| + \|u_{m-2}\|\|d_1\| + \dots + \|u_1\|\|d_{m-2}\|.$$

Let \mathcal{A}^u be the group of all unitary elements in \mathcal{A} . Since each element of \mathcal{A} is a finite linear combination of elements in \mathcal{A}^u , so it is enough to show that the result holds for \mathcal{A}^u . For $a \in \mathcal{A}^u$, define $T_a(x) = [ax - d_m(a) - u_{m-1}d_1(a) - \dots - u_1d_{m-1}(a)]a^{-1}$ ($x \in \mathfrak{M}$). Then each T_a is an affine map. If $a, b \in \mathcal{A}^u$, then we have

$$\begin{aligned}
 T_a T_b(x) &= T_a[(bx - d_m(b) - u_{m-1}d_1(b) - \dots - u_1d_{m-1}(b))b^{-1}] \\
 &= (abxb^{-1} - ad_m(b)b^{-1} - au_{m-1}d_1(b)b^{-1} - \dots - au_1d_{m-1}(b)b^{-1} \\
 &\quad - d_m(a) - u_{m-1}d_1(a) - \dots - u_1d_{m-1}(a))a^{-1} \\
 &= (abx - d_m(ab) + d_m(a)b + d_{m-1}(a)d_1(b) + \dots + d_1(a)d_{m-1}(b))(ab)^{-1} \\
 &\quad - au_{m-1}d_1(b)(ab)^{-1} - au_{m-2}d_2(b)(ab)^{-1} - \dots - au_1d_{m-1}(b)(ab)^{-1} \\
 &\quad - d_m(a)a^{-1} - u_{m-1}d_1(a)a^{-1} - u_{m-2}d_2(a)a^{-1} - \dots - u_2d_{m-2}(a)a^{-1} - u_1d_{m-1}(a)a^{-1}.
 \end{aligned}$$

Now, by the induction hypothesis, we have

$$\begin{aligned}
 T_a T_b(x) &= abx(ab)^{-1} - d_m(ab)(ab)^{-1} - (au_{m-1} - d_{m-1}(a))d_1(b)(ab)^{-1} \\
 &\quad - \dots - (au_1 - d_1(a))d_{m-1}(b)(ab)^{-1} - u_{m-1}d_1(a)a^{-1} - u_{m-2}d_2(a)a^{-1} \\
 &\quad - \dots - u_2d_{m-2}(a)a^{-1} - u_1d_{m-1}(a)a^{-1} = abx(ab)^{-1} - d_m(ab)(ab)^{-1} \\
 &\quad - (au_{m-1} - au_{m-1} + u_{m-1}a + u_{m-2}d_1(a) + \dots + u_1d_{m-2}(a))d_1(b)(ab)^{-1} \\
 &\quad - \dots - (au_1 - u_1a + u_1a)d_{m-1}(b)(ab)^{-1} - u_{m-1}d_1(a)a^{-1} \\
 &\quad - u_{m-2}d_2(a)a^{-1} - \dots - u_2d_{m-2}(a)a^{-1} - u_1d_{m-1}(a)a^{-1} \\
 &= abx(ab)^{-1} - d_m(ab)(ab)^{-1} - u_{m-1}(ad_1(b) + d_1(a)b)(ab)^{-1} \\
 &\quad - u_{m-2}(d_1(a)d_1(b) + d_2(a)b + ad_2(b))(ab)^{-1} \\
 &\quad - \dots - u_1(ad_{m-1}(b) + d_1(a)d_{m-2}(b) + \dots + d_{m-2}(a)d_1(b) + d_{m-1}(a)b)(ab)^{-1} \\
 &= (abx - d_m(ab) - u_{m-1}d_1(ab) - u_{m-2}d_2(ab) - \dots - u_1d_{m-1}(ab))(ab)^{-1} \\
 &= T_{ab}(x) = T_{ba}(x) = T_b T_a(x).
 \end{aligned}$$

Let σ denote the weak operator topology on \mathfrak{M} and let K_m be the σ -closed convex hull of $\{T_a(0) : a \in \mathcal{A}^u\}$. Since each T_a is σ -continuous and $T_a T_b(x) = T_{ab}(x)$ ($a, b \in \mathcal{A}^u, x \in \mathfrak{M}$), it is easy to show that $T_a(K_m) \subseteq K_m$. On the other hand

$$\begin{aligned}
 \|T_a(0)\| &= \|(-d_m(a) - u_{m-1}d_1(a) - \dots - u_1d_{m-1}(a))a^{-1}\| \\
 &\leq \|d_m\| + \|u_{m-1}\|\|d_1\| + \dots + \|u_1\|\|d_{m-1}\|,
 \end{aligned}$$

and it follows that

$$\sup_{x \in K_m} \|x\| \leq \|d_m\| + \|u_{m-1}\|\|d_1\| + \dots + \|u_1\|\|d_{m-1}\|.$$

Therefore, K_m is σ -compact and $\{T_a : a \in \mathcal{A}^u\}$ is a commutative semigroup of σ -continuous affine maps on K_m . Thus by Theorem 3.4 there is an element $u_m \in \mathfrak{M}$ such that

$$T_a(u_m) = u_m \quad (a \in \mathcal{A}^u).$$

Therefore

$$d_m(a) = au_m - \sum_{i=0}^{m-1} u_{m-i}d_i(a) \quad (a \in \mathcal{A}^u),$$

and clearly

$$\|u_m\| \leq \|d_m\| + \|u_{m-1}\|\|d_1\| + \dots + \|u_1\|\|d_{m-1}\|.$$

■

Corollary 3.6 *If \mathcal{A} is a commutative W^* -subalgebra of a W^* -algebra \mathfrak{M} containing the identity element, then each strong higher derivation from \mathcal{A} to \mathfrak{M} is inner.*

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