TOWARD A NEW ALGORITHM FOR SYSTEMS OF FRACTIONAL DIFFERENTIAL-ALGEBRAIC EQUATIONS

H.M. Jaradat[1]
M. Zurigat
Safwan Al-Shara'

Department of Mathematics
Al al-Bayt University
Jordan

Qutaibeh Katatbeh

Department of Mathematics and Statistics
Jordan University of Science & Technology
Jordan

Abstract. This paper is concerned with the development of an efficient algorithm for the analytic solutions of systems of fractional differential -algebraic equations (FDAE). The proposed algorithm is an elegant combination of the Laplace transform method (LTM) with the homotopy analysis method (HAM). The biggest advantage of the Laplace homotopy analysis method (LHAM) over the existing standard analytical techniques is that it overcomes the difficulty arising in calculating complicated terms. Numerical examples are examined to highlight the significant features of this method.

Keywords: analytic solution; Laplace transform; HAM; fractional differential-algebraic equations.

Introduction

The fractional calculus has a long history from 30 September 1695, when the derivative of order $\alpha = 1/2$ has been described by Leibniz [18], [21]. The theory of derivatives and integrals of non-integer order goes back to Leibniz, Liouville, Grünwald, Letnikov and Riemann.There are many interesting books about fractional calculus and fractional differential equations [18], [21], [6], [22]. Differential equations of fractional order have been found to be effective to describe some physical phenomena such as rheology, damping laws, fluid flow and so on [15],

[1]Corresponding author. E-mail: husseinjaradat@yahoo.com; Tel:+962777719675; Fax: +962(5)3903349.

[11]. Recently, many important mathematical models can be expressed in terms of systems of algebraic differential equations of fractional order.

Derivatives of non-integer order can be defined in different ways, e.g. Riemann–Liouville, Grünwald–Letnikow, Caputo and Generalized Functions Approach [21]. In this paper we focus attention on Caputo's definition which turns out to be more useful in real-life applications since it can be coupled with initial conditions having a clear physical meaning.

DAEs arise in the mathematical modeling of a wide variety of problems from engineering and science such as in multibody and flexible body mechanics, electrical circuit design, optimal control, incompressible fluids, molecular dynamics, chemical kinetics (quasi steady state and partial equilibrium approximations), and chemical process control. The index concept plays an important role in the analysis of DAEs. The index is a measure of the degree of difficulty in the numerical solution. In general, the higher the index is, the more difficult it is to solve the DAE. While many different concepts exist to assign an index to a DAE such as the differentiation index [3], [5], the perturbation index [4], the tractability index [23], and the nilpotency index [4]. In the case of linear DAEs with constant coefficients, all these indices are equal. In order to transform a DAE into an alternative form easier to solve, some index reduction methods have been developed [10]. These methods introduce additional variables, which leads to a drawback that the resulting DAE is a larger system than the original one.

There are only a few techniques for the solution of fractional differential -algebraic equations, since it is relatively a new subject in mathematics. The solution of fractional differential equations is much involved. In general, there exists no method that yields exact solutions for fractional differential equations. Only approximate solutions can be derived using linearization or perturbation method. In recent years, much research has been focused on the numerical solution of systems of ordinary differential equations and algebraic differential equations. Some numerical methods have been developed, such as implicit Runge Kutta method [1], Pade approximation method [12], [7], homotopy perturbation method (HPM) [16], [20], Adomian decomposition method (ADM) [8], [9], variation iteration method (VIM) [19], [17] and homotopy analysis method HAM [25].

The ADM and VIM are limited in that the former has complicated algorithms in calculating Adomian polynomials for nonlinear problems, and the later has an inherent inaccuracy in identifying the Lagrange multiplier for fractional operators, which is necessary for constructing variational iteration formula. The HPM is indeed a special case of the HAM [13], [14], [2]. However, mostly, the results given by HPM converge to the corresponding numerical solutions in a rather small region. Although the HAM provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary parameters h, we face the difficulty in calculating complicated integrals that arise when dealing with strongly nonlinear problems.

Therefore, in this work we will introduce a new alternative procedure to eliminate these disadvantages in solving FDAE. The newly developed technique by no means depends on complicated tools from any field. This can be the most important advantage over the other methods. It is worth mentioning that the proposed algorithm is an elegant combination of Laplace transform method and the homotopy analysis method. Some FDAE are examined to illustrate the effectiveness, accuracy and convenience of this method. and in all cases, the presented technique performed excellently.

1. Preliminaries and notations

In this section, let us recall essentials of fractional calculus. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration. For the purpose of this paper the Caputo's definition of fractional differentiation will be used, taking the advantage of Gaputo's approach that the initial conditions for fractional differential equations with Caputo's derivatives take on the traditional form as for integer-order differential equations.

Definition 1.1 Caputo's definition of the fractional-order derivative is defined as

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau,$$

where $n-1<\alpha\leq n$, $n\in\mathbb{N}$, α is the order of the derivative.

For the Caputo's derivative we have:

$$\begin{aligned} D^{\alpha}C &= 0, \quad C \text{ is constant,} \\ D^{\alpha}t^{\beta} &= \begin{cases} 0 & \beta\leq\alpha-1, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\,t^{\beta-\alpha} & \beta>\alpha-1. \end{cases} \end{aligned}$$

Caputo's fractional differentiation is a linear operation and if $f(\tau)$ is continuous in $[a,t]$ and $g(\tau)$ has $n+1$ continuous derivatives in $[a,t]$, it satisfies the so-called Leibnitz rule:

$$D^{\alpha}(f(t)g(t)) = \sum_{k=0}^{\infty}\binom{\alpha}{k}g^{(k)}(t)D^{\alpha-k}f(t)$$

For establishing our results, we also necessarily introduce the following Riemann–Liouville fractional integral operator.

Definition 1.2 A real function $f(x)$, $x>0$, is said to be in the space C_{μ}, $\mu\in R$ if there exists a real number $p>\mu$ such that $f(x)=x^{p}f_1(x)$,where $f_1(x)\in C[0,\infty)$. Clearly $C_{\mu}\subset C_{\beta}$ if $\beta\leq\mu$.

Definition 1.3 The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau$$

We mention only some properties of the operator J^α:
For $f \in C_\mu$, $\mu, \gamma \geq -1$, $\alpha, \beta \geq 0$:

1. $J^0(t) = f(t)$,
2. $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t) = J^\beta J^\alpha f(t)$,
3. $J^\alpha t^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}, \ \alpha > 0, \ \gamma > -1, \ t > 0.$

Also, we need here two of its basic properties. If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, and $f \in C_\mu^m$, $\mu \geq -1$, then

$$D^\alpha J^\alpha f(t) = f(t), \qquad J^\alpha D^\alpha f(t) = f(t) - \sum_{i=0}^{m-1} f^{(i)}(0^+) \frac{t^i}{i!}, \quad t > 0.$$

For more information on the mathematical properties of fractional derivatives and integrals, one can consult [22].

Lemma 1.4 *If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, and $f \in C_\mu^m$, $\mu \geq -1$, then the Laplace transform of the fractional derivative $D^\alpha f(t)$ is*

$$\pounds(D^\alpha f(t)) = s^\alpha F(s) - \sum_{i=1}^{m-1} f^{(i)}(0^+) s^{\alpha-i-1}, t \geq 0$$

Here $\pounds(f(t)) = F(s)$; for more details, see [26].

2. Laplace Homotopy analysis method

In this section, we employ the Laplace homotopy analysis method to the discussed problem. To show the basic idea, let us consider the FDAEs

$$D^{\alpha_i} u_i(t) = f_i(t, u_1, u_2, ..., u_n, u_1', u_2', ..., u_n') \tag{2.1a}$$
$$0 = g(t, u_1, u_2, ..., u_n), \ i = 1, 2, ..., n-1, \ 0 < \alpha_i \leq 1, \tag{2.1b}$$

subject to the initial conditions

$$u_i(0) = a_i, \ i = 1, 2, ..., n$$

where f_i are known analytical functions.

Applying the Laplace transform to both sides of (2.1a) and using the linearity of Laplace transforms, we get

$$\begin{aligned} \pounds[D^{\alpha_i}u_i(t)] &= \pounds[f_i(t,u_1,u_2,...,u_n,u_1',u_2',...,u_n')],\ i=1,2,...,n-1,\ 0<\alpha_i\le 1,\\ 0 &= g(t,u_1,u_2,...,u_n) \end{aligned}$$

Using Lemma 1.4, and applying the formulas of the Laplace transform, we get

$$\begin{aligned} U_i(s) &= \frac{a_i}{s}+\frac{1}{s^{\alpha_i}}\pounds[f_i(t,u_1,u_2,...,u_n,u_1',u_2',...,u_n')],\\ 0 &= g(t,u_1,u_2,...,u_n),\ i=1,2,...,n-1,\ 0<\alpha_i\le 1, \end{aligned} \tag{2.2}$$

where $U_i(s)=\pounds(u_i(t))$.

The so-called zeroth-order deformation equations of equations (2.2) are

$$\begin{aligned} &(1-q)L_i[\Phi_i(s,q)-U_{i,0}(s)] = qh_i[\Phi_i(s,q)-\frac{a_i}{s}\\ &\quad -\frac{1}{s^{\alpha_i}}\pounds[f_i(t,\phi_1(t;q),...,\phi_n(t;q),\frac{\partial}{\partial t}\phi_1(t;q),...,\frac{\partial}{\partial t}\phi_n(t;q))],\\ &\qquad\qquad i=1,2,...,n-1,\\ &(1-q)L_n[\phi_n(t;q)-u_{n,0}(t)] = -qh_n g(t,\phi_1(t;q),...,\phi_n(t;q)), \end{aligned} \tag{2.3}$$

where $q\in[0,1]$ is an embedding parameter, when $q=0$ and $q=1$, we have

$$\begin{aligned} &\Phi_i(s,0)=U_{i,0}(s),\ \Phi_i(s,1)=U_i(s),\ i=1,2,...,n-1,\\ &\phi_n(t;0)=u_{n,0}(t),\ \phi_n(t;1)=u_n(t). \end{aligned}$$

Expanding $\Phi_i(s,q)$, $i=1,2,...,n-1$ and $\phi_n(t;q)$ in Taylor series with respect to q, we get

$$\begin{aligned} \Phi_i(s;q) &= U_{i,0}(s)+\sum_{m=1}^{\infty}U_{i,m}(s)q^m,\ i=1,2,...,n-1,\\ \phi_n(t;q) &= u_{n,0}(t)+\sum_{m=1}^{\infty}u_{n,m}(t)q^m, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} U_{i,m}(s) &= \frac{1}{m!}\frac{\partial^m\Phi_i(s;q)}{\partial q^m}|_{q=0},\ i=1,2,...,n-1,\\ u_{n,m}(t) &= \frac{1}{m!}\frac{\partial^m\phi_n(t;q)}{\partial q^m}|_{q=0}. \end{aligned}$$

If the initial guesses, the auxiliary linear operator $L_i, i=1,2,...,n$ and the nonzero auxiliary parameter h_i are properly chosen so that the power series (2.4) converges at $q=1$.

Then, we have under these assumptions the solution series

$$U_i(s) = \Phi_i(s;1) = U_{i,0}(s) + \sum_{m=1}^{\infty} U_{i,m}(s),\ i = 1, 2, ..., n-1,$$
$$u_n(t) = \phi_n(t;1) = u_{n,0}(t) + \sum_{m=1}^{\infty} u_{i,m}(t)$$

For brevity, define the vector

$$\overrightarrow{U}_{i,m}(s) = \{U_{i,0}(s), U_{i,1}(s), U_{i,2}(s), ..., U_{i,m}(s)\},\ i = 1, 2, ..., n-1,$$
$$\overrightarrow{u}_{n,m}(t) = \{u_{n,0}(t), u_{n,1}(t), u_{m,2}(t), ..., u_{n,m}(t)\},$$

Differentiating the zero-order deformation equation (2.3) m times with respective to q and then dividing by $m!$ and finally setting $q = 0$, we have the so-called high-order deformation equation

$$\begin{aligned} U_{i,m}(s) &= \chi_m U_{i,m-1}(s) + h_i \Re_{i,m}(\overrightarrow{U}_{i,m-1}(s)),\ i = 1, 2, ..., n-1, \\ u_{n,m}(t) &= \chi_m u_{n,m-1}(t) + h_n \Re_{n,m}(\overrightarrow{u}_{n,m-1}(t)) \end{aligned} \tag{2.5}$$

where

$$\Re_{i,m}(\overrightarrow{U}_{i,m-1}(s)) = U_{i,m-1}(s) - \frac{1}{s^{\alpha_i}}\left[\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}(\pounds[f_i(t, \phi_1(t;q), ..., \phi_n(t;q), \frac{\partial}{\partial t}\phi_1(t;q), ..., \frac{\partial}{\partial t}\phi_n(t;q))])|_{q=0}\right] - \frac{a_i}{s}(1-\chi_m),$$

$$\Re_{n,m}(\overrightarrow{u}_{n,m-1}(t)) = \frac{-1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}[g(t, \phi_1(t;q), ..., \phi_n(t;q))]|_{q=0},\ i = 1, 2, ..., n-1,$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

Finally, applying the inverse Laplace transforms of (2.5), then we have a power series solution

$$u_i(t) = \sum_{m=0}^{\infty} u_{i,m}(t), i = 1, 2, ..., n. \tag{2.6}$$

Note that we have great freedom to choose the value of the auxiliary parameter h_i. Mathematically, the value of $u_i(t)$ at any finite order of approximation is dependent upon the auxiliary parameter h_i, because the zeroth and high order deformation equations contain h_i. Let R_{h_i} denote the set of all values of h_i which ensure the convergence of the LHAM series solution (2.6) of $u_i(t)$. Let h_i be the variable of the horizontal axis and the limit of the series solution (2.6) of $u_i(t)$ be the variable of vertical axis. Plot the curve $u_i(t) \sim h_i$, where $u_i(t)$ denotes the limit of the series (2.6). Because the limit of all convergent series solutions

(2.6) is the same for a given a, there exists a horizontal line segment above the region $h \in R_{h_i}$. So, by plotting the curve $u_i(t) \sim h_i$ at a high enough order approximation, one can find an approximation of the set R_{h_i} [13].

3. Applications

In this part, we introduce some applications on LHAM to solve differential-algebraic equations with fractional derivatives.

Example 1. Consider the following fractional differential-algebraic equations

$$\begin{aligned} D^\alpha u(t) &= -2tu^2(t) + u(t)v(t) - 1,\ 0 < \alpha \leq 1, \\ v(t) &= u(t)v(t) + t^2,\ u(0) = v(0) = 1, \end{aligned} \tag{3.1}$$

for $\alpha = 1$, the exact solution $u(t) = \dfrac{1}{t^2+1}, v(t) = t^2 + 1$.

To derive the solution, take the Laplace transform of both sides of (3.1), we get

$$\pounds[D^\alpha u(t)] = -2\pounds(tu^2(t)) + \pounds(u(t)v(t)) - \frac{1}{s},$$
$$v(t) = u(t)v(t) + t^2,$$

so

$$s^\alpha U(s) - s^{\alpha-1} = -2\pounds(tu^2(t)) + \pounds(u(t)v(t)) - \frac{1}{s},$$
$$v(t) = u(t)v(t) + t^2,$$

or

$$U(s) = -\frac{2}{s^\alpha}\pounds(tu^2(t)) + \frac{1}{s^\alpha}\pounds(u(t)v(t)) + \left(\frac{1}{s} - \frac{1}{s^{\alpha+1}}\right),$$
$$v(t) = u(t)v(t) + t^2.$$

Hence, the mth-order deformation equations can be given by

$$U_m(s) = \chi_m U_{m-1}(s) + h_1 \Re_{1,m}(\overrightarrow{U}_{m-1}(s)), \tag{3.2}$$
$$v_m(t) = \chi_m v_{m-1}(t) + h_2 \Re_{2,m}(\overrightarrow{v}_{m-1}(t)),\ m = 1, 2, 3, ... \tag{3.3}$$

subject to the initial condition

$$u_m(0) = v_m(0) = 0, m = 1, 2, 3, ...$$

where

$$\begin{aligned} \Re_{1,m}(\overrightarrow{U}_{m-1}(s)) = U_{m-1}(s) &+ \frac{2}{s^\alpha}\pounds\left(t\sum_{i=0}^{m-1} u_i(t)u_{m-i-1}(t)\right) \\ &- \frac{1}{s^\alpha}\pounds\left(\sum_{i=0}^{m-1} u_i(t)v_{m-i-1}(t)\right) - \left(\frac{1}{s} - \frac{1}{s^{\alpha+1}}\right)(1-\chi_m), \end{aligned}$$

$$\Re_{2,m}(\overrightarrow{v}_{m-1}(t)) = v_{m-1}(t) - \sum_{i=0}^{m-1} u_i(t)v_{m-i-1}(t)) - t^2(1-\chi_m),\ m = 1, 2, 3, ...$$

According to the initial condition in (3.1), we can choose the initial guess of $U(s)$ and $v(t)$ as follows:

$$U_0(s) = \frac{1}{s},\ v_0(t) = 1$$

Using the mth-order deformation equations (3.2), (3.3)we can find that

$$\begin{aligned}
U_0(s) &= \frac{1}{s}, \\
u_0(t) &= 1 \\
U_1(s) &= \frac{2h_1}{s^{\alpha+2}}, \\
u_1(t) &= \frac{2h_1}{\Gamma(2+\alpha)} t^{\alpha+1} \\
U_2(s) &= \frac{2h_1}{s^{\alpha+2}} + \frac{2h_1^2}{s^{\alpha+2}} + \frac{8h_1^2(2+\alpha)}{s^{2\alpha+3}} + \frac{2h_1h_2}{s^{\alpha+3}} - \frac{2h_1^2}{s^{2\alpha+2}}, \\
u_2(t) &= \frac{2h_1}{\Gamma(2+\alpha)} t^{\alpha+1} + \frac{2h_1^2}{\Gamma(2+\alpha)} t^{\alpha+1} + \frac{8h_1^2(2+\alpha)}{\Gamma(3+2\alpha)} t^{2\alpha+2} + \frac{2h_1h_2}{\Gamma(3+\alpha)} t^{\alpha+2} \\
&- \frac{2h_1^2}{\Gamma(2+2\alpha)} t^{2\alpha+1} \\
&\vdots \\
v_0(t) &= 1 \\
v_1(t) &= -h_2 t^2 \\
v_2(t) &= -h_2 t^2 - \frac{2h_1h_2}{\Gamma(2+\alpha)} t^{\alpha+1} \\
&\vdots
\end{aligned}$$

Then, using a mathematical software the solution, we successfully obtain

$$u(t) = \sum_{m=0}^{\infty} u_m(t),\ v(t) = \sum_{m=0}^{\infty} v_m(t).$$

The convergence of these series is strongly depends upon the values of the auxiliary parameters h_i. In order to find the range of admissible values of h_i, the $h_i - curves$ are plotted in Fig.1 for the 8th-order of approximation. We can see that the range for values of h_1, h_2 are in the range $-1.3 \leq h_1,\ h_2 \leq -0.4$. Using $h_1 = h_2 = -1$, $\alpha = 1$ we get

$$\begin{aligned}
&u_0(t) = 1, u_1(t) = -t^2, u_2(t) = t^4, u_3(t) = -t^6, u_4(t) = t^8, ... \\
&v_0(t) = 1, v_1(t) = t^2, v_2(t) = 0, v_2(t) = 0, v_2(t) = 0, ...
\end{aligned}$$

Proceeding in the same manner, we get that the solution $u(t)$ and $v(t)$ is given in series form by

$$u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t) = 1 - t^2 + t^4 - t^6 + t^8 - t^{10} + \cdots$$
$$v(t) = v_0(t) + \sum_{m=1}^{\infty} v_m(t) = 1 + t^2$$

and so in closed form

$$u(t) = \frac{1}{t^2 + 1}, v(t) = t^2 + 1$$

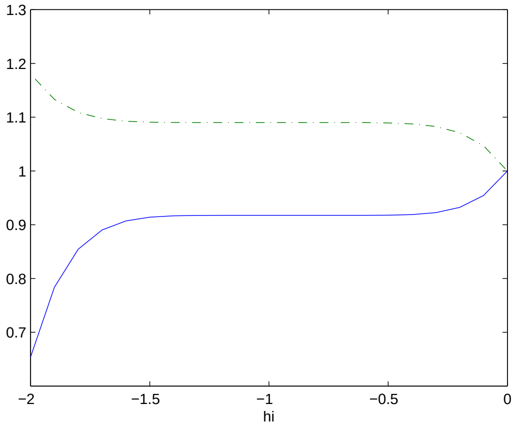

Fig.1. The h_i-curve of 8th-order approximation for $u(0.3)$ and $v(0.3)$, $\alpha = 1$ of Example 1

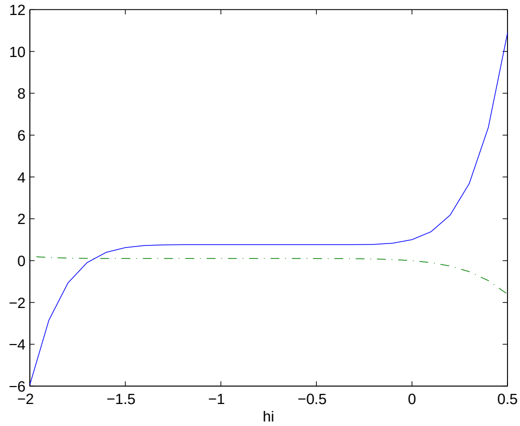

Fig.2. The h_i-curve of 8th-order approximation for $u(0.1)$ and $v(0.1)$, $\alpha = 0.5$ of Example 2

Example 2. Consider the following system of fractional differential-algebraic equations

$$D^{\alpha}u(t) = tv'(t) - u(t) + (1 + t)v(t), 0 < \alpha \leq 1, \quad (3.4)$$
$$v(t) = \sin(t), u(0) = 1, v(0) = 0,$$

for $\alpha = 1$, the exact solution $u(t) = e^{-t} + \sin(t), v(t) = \sin(t)$.

Take the Laplace transform of both sides of (3.4) we get

$$U(s) = \frac{1}{s^{\alpha}}\pounds(tv'(t)) - \frac{1}{s^{\alpha}}U(s) + \frac{1}{s}V(s) + \frac{1}{s^{\alpha}}\pounds(tv(t)) + \frac{1}{s},$$
$$v(t) = \sin(t),$$

Hence, the mth-order deformation equations can be given by

$$U_m(s) = \chi_m U_{m-1}(s) + h_1 \Re_{1,m}(\overrightarrow{U}_{m-1}(s)), \quad (3.5)$$
$$v_m(t) = \chi_m v_{m-1}(t) + h_2 \Re_{2,m}(\overrightarrow{v}_{m-1}(t)), m = 1, 2, 3, ... \quad (3.6)$$

subject to the initial condition

$$u_m(0) = v_m(0) = 0.$$

where

$$\Re_{1,m}(\overrightarrow{U}_{m-1}(s)) = U_{m-1}(s) - \frac{1}{s^\alpha}\pounds(tv'_{m-1}) + \frac{1}{s^\alpha}U_{m-1}(s) - \frac{1}{s}V_{m-1}(s)$$
$$-\frac{1}{s^\alpha}\pounds(tv_{m-1}(t)) + \frac{1}{s}(1-\chi_m),$$
$$\Re_{2,m}(\overrightarrow{v}_{m-1}(t)) = v_{m-1}(t) - \sin(t)(1-\chi_m), \ m = 1,2,3,...$$

According to the initial condition in (3.4), we can choose the initial guess of $U(s)$ and $v(t)$ as follows:

$$U_0(s) = \frac{1}{s}, \ v_0(t) = 0$$

In order to find range of admissible values of h_i, the h_i-curve is plotted for 8th-order approximation when see Fig. 2. We can see that the range of values for h_i is between $-1.3 \le h_i \le -0.4$, using the mth-order deformation equations (3.5), (3.6)and $h_1 = h_2 = -1$, we can find that

(3.7)
$$\begin{aligned} u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t) &= 1 + \frac{1}{\Gamma(1+\alpha)}t^\alpha + \frac{2}{\Gamma(2+\alpha)}t^{\alpha+1} \\ &+ \left[\frac{-2}{\Gamma(1+2\alpha)} + \frac{3.4^{-\alpha}\sqrt{\pi}}{\Gamma(1+\alpha)\Gamma(\frac{1}{2}+\alpha)}\right]t^{2\alpha} \\ &+ \frac{2}{\Gamma(3+\alpha)}t^{\alpha+2} - \frac{2}{\Gamma(2+2\alpha)}t^{2\alpha+1} - \frac{1}{\Gamma(1+3\alpha)}t^{3\alpha} + \cdots \\ v(t) = v_0(t) + \sum_{m=1}^{\infty} v_m(t) &= t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{720} + \cdots \end{aligned}$$

Setting $\alpha = 1$ in (3.7), we obtain the following series solution

$$\begin{aligned} u(t) &= u_0(t) + \sum_{m=1}^{\infty} u_m(t) = 1 - t + \frac{3t^2}{2} - \frac{t^3}{3!} - \frac{t^4}{8} - \frac{t^5}{5!} + \frac{7t^6}{720} - \frac{t^7}{7!} - \frac{t^8}{5760} \\ &-\frac{t^9}{9!} + \frac{11t^{10}}{3628800} - \frac{t^{11}}{11!} + \cdots \\ v(t) &= v_0(t) + \sum_{m=1}^{\infty} v_m(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \frac{t^{11}}{11!} + \cdots \end{aligned}$$

which is the exact solution for system (3.4), which better than solutions given by Celik, Bayram and Yeloglu [8] using Adomian decomposition method and than solutions given by Zurigat , Momani, and Alawneh [25] using HAM. However, the results given by the Adomian decomposition method converge to the corresponding numerical solutions in a rather small region. But in this method, the Laplace homotopy analysis method gives a greater region of convergence with the exact solution.

Fig. 3 shows the LHAM approximate solutions for various values of α which have the same trajectories.

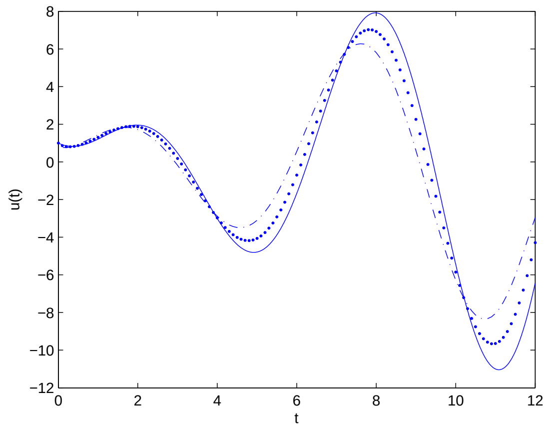

Fig.3. Plots of solution of system (10) when $h_1 = h_2 = -1$.
Solid line: $\alpha = 1$, dotted line: $\alpha = 0.8$, dash dotted line: $\alpha = 0.5$

Example 3. Consider the following system of fractional differential-algebraic equations

$$\begin{aligned} D^{\alpha}u_1(t) &= u_1(t) - u_2(t)v(t) + \sin(t) + t\cos(t), 0 < \alpha \le 1, \\ D^{\alpha}u_2(t) &= tv(t) + u_1^2(t) + \sec^2(t) - t^2(\sin^2(t) + \cos(t)) \\ v(t) &= u_1(t) + t(\cos(t) - \sin(t)), u_1(0) = u_1(0) = v(0) = 0, \end{aligned} \tag{3.8}$$

for $\alpha = 1$, the exact solution $u_1(t) = t\sin(t), u_2(t) = \tan(t), v(t) = t\cos(t)$.

To derive the solution, we take the Laplace transform of both sides of (3.8) and we get

$$\begin{aligned} s^{\alpha}U_1(s) &= U_1(s) - \pounds(u_2(t)v(t)) + \pounds(\sin(t) + t\cos(t)), \\ s^{\alpha}U_2(s) &= \pounds(tv(t)) + \pounds(u_1^2(t)) + \pounds(\sec^2(t) - t^2(\sin^2(t) + \cos(t))), \\ v(t) &= u_1(t) + t(\cos(t) - \sin(t)), \end{aligned}$$

or

$$\begin{aligned} U_1(s) &= \frac{1}{s^{\alpha}}U_1(s) - \frac{1}{s^{\alpha}}\pounds(u_2(t)v(t)) + \frac{1}{s^{\alpha}}\pounds(\sin(t) + t\cos(t)), \\ U_2(s) &= \frac{1}{s^{\alpha}}\pounds(tv(t)) + \frac{1}{s^{\alpha}}\pounds(u_1^2(t)) + \frac{1}{s^{\alpha}}\pounds(\sec^2(t) - t^2(\sin^2(t) + \cos(t))), \\ v(t) &= u_1(t) + t(\cos(t) - \sin(t)). \end{aligned}$$

Hence, the mth-order deformation equations can be given by

$$\begin{aligned} U_{i,m}(s) &= \chi_m U_{i,m-1}(s) + h_i \Re_{i,m}(\overrightarrow{U}_{m-1}(s)),\ i = 1, 2, \\ v_m(t) &= \chi_m v_{m-1}(t) + h_3 \Re_{3,m}(\overrightarrow{v}_{m-1}(t)),\ m = 1, 2, 3, ... \end{aligned} \tag{3.9}$$

subject to the initial condition

$$U_{i,m}(0) = v_m(0) = 0, \; i = 1, 2,$$

where

$$\begin{aligned}
\Re_{1,m}(\overrightarrow{U}_{1,m-1}(s)) &= U_{1,m-1}(s) - \frac{1}{s^\alpha} U_{1,m-1}(s) \\
&+ \frac{1}{s^\alpha} \pounds \left(\sum_{i=0}^{m-1} u_{2,i}(t) v_{m-i-1}(t) \right) \\
&- \frac{1}{s^\alpha} \pounds (\sin(t) + t\cos(t))(1 - \chi_m), \\
\Re_{2,m}(\overrightarrow{U}_{2,m-1}(s)) &= U_{2,m-1}(s) - \frac{1}{s^\alpha} \pounds (t v_{m-1}(t)) \\
&- \frac{1}{s^\alpha} \pounds \left(\sum_{i=0}^{m-1} u_{1,i}(t) u_{1,m-i-1}(t) \right) \\
&- \frac{1}{s^\alpha} \pounds (\sec^2(t) - t^2(\sin^2(t) + \cos(t)))(1 - \chi_m), \\
\Re_{3,m}(\overrightarrow{v}_{m-1}(t)) &= \; v_{m-1}(t) - u_{1,m-1}(t) - t(\cos(t) - \sin(t))(1 - \chi_m), \\
& \qquad m = 1, 2, 3, ...
\end{aligned}$$

According to the initial condition in (3.8), we can choose the initial guess of $U(s)$ and $v(t)$ as follows:

$$U_{1,0}(s) = 0, U_{2,0}(s) = 0, v_0(t) = 0$$

The proper values of h_1, h_2, h_3 found from the h_i-curve shown in Fig.4, it is clear that the series of $u_i(t), v(t)$ convergent when $-1.4 \le h_i \le -0.7, i = 1, 2, 3$, if we set $\alpha = 1$, $h_1 = h_2 = h_3 = -1$ in (3.9), then we obtain the following series solution

$$\begin{aligned}
u_1(t) &= u_{1,0}(t) + \sum_{m=1}^{\infty} u_{1,m}(t) = t^2 - \frac{t^4}{6} + \frac{t^6}{120} - \frac{t^8}{5040} + \frac{t^{10}}{362880} + \cdots \\
u_2(t) &= u_{2,0}(t) + \sum_{m=1}^{\infty} u_m(t) = t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{17t^7}{315} + \frac{62t^9}{2835} + \cdots \\
v(t) &= v_0(t) + \sum_{m=1}^{\infty} v_m(t) = t - \frac{t^3}{2} + \frac{t^5}{24} - \frac{t^7}{720} + \frac{t^9}{40320} + \cdots
\end{aligned}$$

which the same solutions given by F. Soltanian, S.M. Karbassi, M.M. Hosseini [24] using He's variational iteration method.

Figs. 5, 6 and 7 shows the LHAM approximate solutions for various values of α which have the same trajectories.

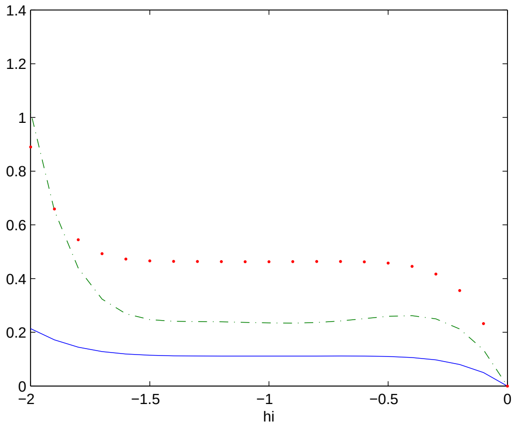

Fig.4. The h_i-curve of 8th-order approximation for $u_1(0.1), u_2(0.4)$ and $v(0.3), \alpha = 0.75$ of Example 3

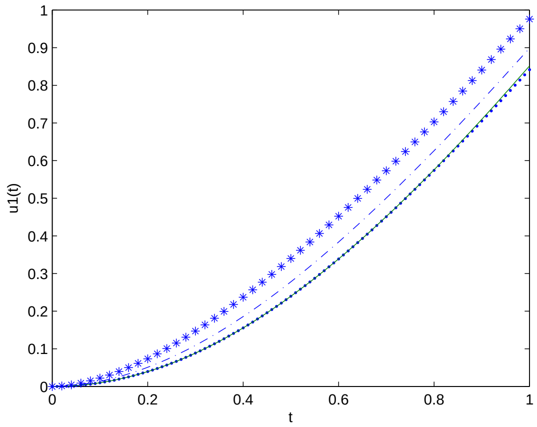

Fig.5. Plots of solution of system (14) when $h_1 = h_2 = -1$. dotted line: exact solution when $\alpha = 1$, solid line: $\alpha = 1$, dash dotted line: $\alpha = 0.9$, star line: $\alpha = 0.75$

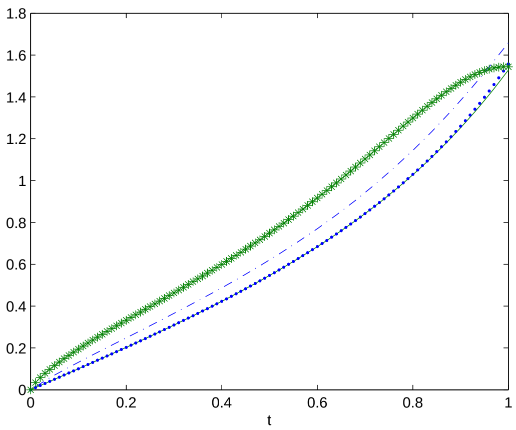

Fig.6. Plots of solution of system (14) when $h_1 = h_2 = -1$. dotted-line: exact solution when $\alpha = 1$, solid line: $\alpha = 1$, dash dotted line: $\alpha = 0.9$, star line: $\alpha = 0.75$

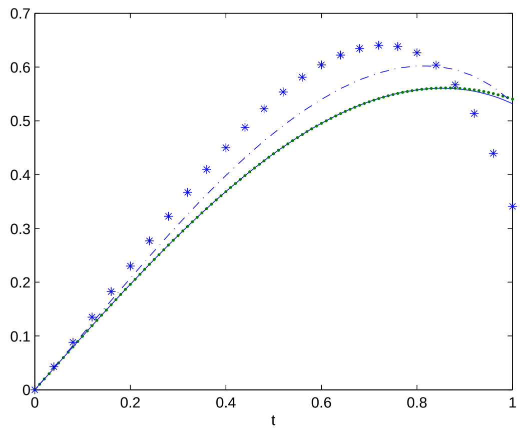

Fig.7. Plots of solution of system (14) when $h_1 = h_2 = -1$. dotted-line: exact solution when $\alpha = 1$, solid line: $\alpha = 1$, dash dotted line: $\alpha = 0.9$, star line: $\alpha = 0.75$

4. Conclusion

A combined form of the Laplace transform method with Homotopy analysis method is effectively used to handle linear and nonlinear fractional differential-algebraic equations. The main advantage of the method is its fast convergence to the solution. Moreover, it avoids the volume of calculations that reguired by other existing analytical methods. In practice, the utilization of the method is straightforward if some symbolic software as Matlab is used to implement the calculations. The new method leads to higher accuracy and simplicity, and in all cases the solutions obtained are easily programmable approximants to the analytic solutions of the original problems with the accuracy required. The proposed scheme can be applied for other nonlinear equations.

References

[1] Ascher, U.M., Petzold, L.R., *Projected implicit Runge Kutta methods for differential algebraic equations*, SIAM J. Numer. Anal., 28 (1991), 1097-1120.

[2] Awawdeh, F., Jaradat, H.M., Alsayyed, O., *Solving system of DAEs by homotopy analysis method*, Chaos, Solitons Fractals, 42 (2009), 1422-1427.

[3] Brenan, K.E., Campbell, S.L., Petzold, L.R., *Numerical Solution of Initial-Value Problem in Differential-Algebraic Equations*, SIAM, Philadelphia, 2nd edition., 1996.

[4] Bujakiewicz, P., *Maximum Weighted Matching for High Index Differential Algebraic Equations*, Doctor's dissertation, Delft University of Technology., 1994.

[5] Campbell, S.L., Gear, C.W., *The index of general nonlinear DAEs*, Numerische Mathematik, 72 (1995), 173-196.

[6] Caputo, M., *Linear models of dissipation whose Q is almost frequency independent*. Part II. J Roy Austral Soc., 13 (1967), 529–539.

[7] Çelik, E., Bayram, M., *The numerical solution of physical problems modeled as a systems of differential algebraic equations* (DAEs), J. Franklin Inst., 342 (1) (2005), 1-6.

[8] Celik, E., Bayram, M., Yeloglu, T., *Solution of differential algebraic equations (DAE's) by Adomian decomposition method*, Internat. J. Pure Appl. Math. Sci., 3 (1) (2006), 93-100.

[9] Daftardar-Gejji, V., Jafari, H., *Adomian decomposition: A tool for solving a system of fractional differential equations*, J. Math. Anal. Appl., 301 (2) (2005), 508-518.

[10] Gear, C.W., *Differential-algebraic equation index transfomations*, SIAM Joumal on Scientific and Statistical Computing, 9 (1988), 39-47.

[11] Gorenflo, R., Mainardi, F., *Fractional calculus: integral and differential equations of fractional order*, in Fractals and Fractional Calculus in Continuum Mechanics, vol. 378 of CISM Courses and Lectures, pp. 223–276, Springer, Vienna, Austria, 1997.

[12] Guzel, N., Bayram, M., *On the numerical solution of differential algebraic equations with index* 3, Appl. Math. Comput., 175 (2) (2006), 1320-1331.

[13] Liao, S.J., *On the homotopy analysis method for nonlinear problems*, Appl. Math. Comput., 147 (2004), 499–513.

[14] Liao, S.J., *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, Chapman & Hall/CRC Press, Boca Raton, 2003.

[15] Mainardi, F., *Fractional calculus: some basic problems in continuum and statistical mechanics*, in Fractals and Fractional Calculus in Continuum Mechanics, vol. 378 of CISM Courses and Lectures, pp. 291–348, Springer, Vienna, Austria, 1997.

[16] Momani, S., Odibat, Z., *Numerical approach to differential equations of fractional order*, J. Comput. Appl. Math. 207 (2007), 96-110.

[17] Momani, S., Odibat, Z., *Analytical approach to linear fractional partial differential equations arising in fluid mechanics*, Phys. Lett., A, 355 (2006), 271-279.

[18] Munkhammar, J. D., *Fractional calculus and the Taylor–Riemann series*, Undergraduate Math. Journal, 6 (2005), 1-19.

[19] Odibat, Z., Momani, S., *Application of variation iteration method to nonlinear differential equations of fractional order*, Int. J. Nonlin. Sci. Numer. Simulat., 1 (7) (2006), 15-27.

[20] Odibat, Z., Momani, S., *Modified homotopy perturbation method: Application to quadratic Riccati differential equation of fractional order*, Chaos, Solitons Fractals, 36 (1) (2006), 167-174.

[21] Podlubny, I., *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Academic Press, New York, 1999.

[22] Samko, S.G., Kilbas, A.A., Marichev, O.I., *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, London, 1993.

[23] Schwarz, D. E., Tischendorf, C., *Structural analysis of electric circuits and consequences for MNA*, Intemational Joumal of Circuit Theory and Applications, 28 (2000), 131-162.

[24] SOLTANIAN, F., KARBASSI, S.M., HOSSEINI, M.M., *Application of He's variational iteration method for solution of differential-algebraic equations*, Chaos, Solitons Fractals, 41 (2009), 436-445.

[25] ZURIGAT, M., MOMANIB, S., ALAWNEHA, A., *Analytical approximate solutions of systems of fractional algebraic differential equations by homotopy analysis method*, Computers and Mathematics with Applications, 59 (2010), 1227-1235.

[26] ZURIGAT, M., *Analytical Approximate Solutions of Fractional Differential Equations by Laplace Adomian Decomposition Method*, Fasc. Matematica, tom XVII, 2 (2010), 173-184.

Accepted: 25.01.2012