ON HOW TO CONSTRUCT LEFT SEMIMODULES FROM THE RIGHT ONES

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Abstract. In the paper, various constructions of left semimodules from the right ones are investigated.

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1. Introduction

(Congruence-)simple semirings are studied, e.g., in [1], [2], [4], [5], [6], [7] and some of them are constructed as endomorphism semirings of commutative semigroups. That is, these semirings are characterized via semimodules. If S is a semiring and ${}_SM$ a left semimodule then the mapping $a \mapsto \lambda_a$, $\lambda_a(x) = ax$, $a \in S$, $x \in M$, is a (semiring) homomorphism of the semiring S into the full endomorphism semiring of the additive semigroup $M(+)$. This "canonical" homomorphism is injective if and only if the semimodule ${}_SM$ is faithful. Of course, if the semiring S is simple then the homomorphism is either constant or injective. Now, in order to get various more or less regular representations of (simple) semirings via endomorphisms, we have to find "nice" left semimodules (once we compose mappings from the right to the left). Unfortunately, it may happen that left semimodules of such kind are not easily available but, contrarywise, useful right semimodules are at hand. Therefore, we have to find a passage from the right side to the left side.

1. Preliminaries

A semiring is an algebraic structure possessing two associative binary operations (most frequently denoted as addition and multiplication) where the addition is commutative and the multiplication distributes over the addition from either side. Basic specimens are endomorphism semirings of commutative semigroups and basic pieces of information on semirings are available from [3].

A subset I of a semiring S is a left (right) ideal if $SI \cup (I + I) \subseteq I$ ($IS \cup (I + I) \subseteq I$). We put $\underline{R}(S) = \{ a \in S \mid Sa = \{a\} \}$. If $\underline{R}(S) \neq \emptyset$ then $\underline{R}(S)$ is additively idempotent and it is the smallest (right) ideal of S. If, moreover, the right S-semimodule $\underline{R}(S)$ is faithful then the semiring S is additively idempotent.

A semiring S is said to be (congruence-))simple if it has just two congruence relations. If S is simple and $|\underline{R}(S)| \geq 2$ then $\underline{R}(S)_S$ is faithful (see 7.1) and S is additively idempotent.

In the sequel, all semirings and all semimodules are assumed to be additively idempotent. If $M(+)$ is a semilattice then the basic order is defined on M by $x \leq y$ if and only if $x + y = y$. An element $w \in M$ is the greatest element in the ordered set $M(\leq)$ iff $x + w = w$ for every $x \in M$. That is, $w = o_M$ is the (only) additively absorbing element. The existence of such an element will be denoted by $o_M \in M$. On the other hand, $o_M \notin M$ means that M has no absorbing element. Symmetrically, w is the smallest element iff $w = 0_M$ is additively neutral. Again, $0_M \in M$ menas that such an element is present and $0_M \notin M$ means that not.

2. From the right to the left (a)

Let S be a non-trivial semiring and M $(= M_S(+, \cdot))$ a non-trivial right S-semimodule. For $x, y \in M$, let $P_{x,y} = \{ z \in M \mid z \leq x, z \leq y \}$. If this set is non-empty then it is a subsemilattice of $M(+)$. Anyway, we put $W_1 = \{ (x, y) \mid x, y \in M, P_{x,y} \neq \emptyset \}$ and $W_2 = \{ (x, y) \in W_1 \mid o_P \in P = P_{x,y} \}$. Clearly, the ordered set $M(\leq)$ is a lattice iff $W_2 = M \times M$.

Now, we define a (possibly partial) binary operation $*$ on M by $x * y = o_P$, $P = P_{x,y}$, for every pair $(x, y) \in W_2$. Some easy observations follow:

2.1 Lemma.

(i) *$(x, x) \in W_2$ and $x * x = x$ for every $x \in M$.*

(ii) *$x * y = y * x$ for every $(x, y) \in W_2$.*

(iii) *$(x, y) \in W_2$ and $x * y = x$ iff $x \leq y$.*

(iv) *$(x, x + y) \in W_2$ and $x * (x + y) = x$ for all $x, y \in M$.*

(v) *If $(x, y) \in W_2$ then $x + (x * y) = x$.*

(vi) *If $(x, y) \in W_2$, $(y, z) \in W_2$ and $(x * y, z) \in W_2$ then $(x, y * z) \in W_2$ and $x * (y * z) = (x * y) * z$.*

(vii) *If $o_M \in M$ then $(x, o_M) \in W_2$ and $x * o_M = x$ for every $x \in M$.*

(viii) *If $0_M \in M$ then $W_1 = M \times M$, $(x, 0_M) \in W_2$ and $x * 0_M = 0_M$ for every $x \in M$.*

For $a \in S$ and $x \in M$, let $Q_{x,a} = \{ y \in M \mid ya \leq x \}$. If this set is non-empty then it is a subsemilattice of $M(+)$. Anyway, we put $W_3 = \{ (x,a) \mid Q_{x,a} \neq \emptyset \}$ and $W_4 = \{ (x,a) \in W_3 \mid o_Q \in Q = Q_{x,a} \}$.

Now, we define a (possibly partial) left S-scalar multiplication on M by $a \circ x = o_Q$, $Q = Q_{x,a}$, $(x,a) \in W_4$. The following observations are quite easy to check:

2.2 Lemma.

(i) *$(xa,a) \in W_3$ for all $a \in S$ and $x \in M$.*

(ii) *If $(xa,a) \in W_4$ then $x \leq a \circ (xa)$ and $xa = (a \circ xa)a$.*

(iii) *If $o_M \in M$ then $(o_M, a) \in W_4$ for every $a \in S$ and $a \circ o_M = o_M$.*

(iv) *If $o_M \in M$ then $(o_M a, a) \in W_4$ for every $a \in S$ and $a \circ (o_M a) = o_M$.*

2.3 Lemma. *Let $a \in S$ and $x, y \in M$.*

(i) *If $(x,y) \in W_2$ then $Q_{x*y,a} = Q_{x,a} \cap Q_{y,a}$.*

(ii) *If $(x,y) \in W_2$ and $(x*y,a) \in W_3$ then $Q_{x,a} \cap Q_{y,a} \neq \emptyset$.*

(iii) *If $(x,a),(y,a) \in W_4$ and $(a \circ x, a \circ y) \in W_1$ then $Q_{x,a} \cap Q_{y,a} \neq \emptyset$.*

(iv) *If $Q_{x,a} \cap Q_{y,a} \neq \emptyset$ then $(x,y) \in W_1$ and $(x,a),(y,a) \in W_3$.*

(v) *If $Q_{x,a} \cap Q_{y,a} \neq \emptyset$ and $(x,y) \in W_2$ then $(x*y,a) \in W_3$.*

(vi) *If $(x,y) \in W_2$ and $(x,a),(y,a),(x*y,a) \in W_4$ then $(a \circ x, a \circ y) \in W_2$ and $a \circ (x*y) = (a \circ x) * (a \circ y)$.*

2.4 Lemma. *Let $a, b \in S$ and $x \in M$.*

(i) *$Q_{x,a+b} = Q_{x,a} \cap Q_{x,b}$.*

(ii) *If $(x,a),(x,b) \in W_4$ and $(a \circ x, b \circ x) \in W_1$ then $Q_{x,a+b} \neq \emptyset$.*

(iii) *If $Q_{x,a+b} \neq \emptyset$ then $(x,a),(x,b) \in W_3$.*

(iv) *If $(x,a),(x,b),(x,a+b) \in W_4$ then $(a \circ x, b \circ x) \in W_2$ and $(a+b) \circ x = (a \circ x) * (b \circ x)$.*

2.5 Lemma. *Let $a, b \in S$ and $x \in M$.*

(i) *If $(x,b) \in W_4$ then $Q_{b \circ x, a} = Q_{x,ab}$.*

(ii) *If $(x,b),(b \circ x, a) \in W_4$ then $(x,ab) \in W_4$ and $a \circ (b \circ x) = (ab) \circ x$.*

(iii) *If $(x,ab) \in W_3$ then $(x,b) \in W_3$.*

(iv) *If $(x,b),(x,ab) \in W_4$ then $(b \circ x, a) \in W_4$ and $a \circ (b \circ x) = (ab) \circ x$.*

(v) *If $(x,b) \notin W_3$ then $(x,ab) \notin W_3$.*

2.6 Remark. Assume that $W_3 = W_4$ and the (partial) operation $\circ$ can be completed to a full one (denoted again by $\circ$) in such a way that $a \circ (b \circ x) = (ab) \circ x$ for all $a, b \in S$ and $x \in M$. Furthermore, assume that for every $x \in M$ there is $a_x \in S$ with $M a_x = \{x\}$.

(i) We have $a_x \circ x = o_M \in M$ for every $x \in M$. Moreover, if the right semimodule M_S is faithful then $a_x \in \underline{R}(S)$.

(ii) Assume that the semiring S is simple. If M_S is not faithful then $xa = xb$ for all $a, b \in S$ and $x \in M$, and hence $xa = xa_x = x$ and $Ma_x = M$, a contradiction. Consequently, M_S is faithful.

(iii) Assume that $a_x \in \underline{R}(S)$ for every $x \in M$ (see (i) and (ii)). Then $b \circ o_M = b \circ (a_x \circ x) = (ba_x) \circ x = a_x \circ x = o_M$ for every $b \in S$. If $a \in \underline{R}(S)$, $x \in M$ and $y = a \circ x$ then $o_M = a_y \circ y = a_y(a \circ x) = (a_y a) \circ x = a \circ x$. Thus $\underline{R}(S) \circ M = \{o_M\}$.

(iv) Assume, finally, that the semiring S is simple and there is a binary operation $\boxplus$ defined on M such that $M(\boxplus, \circ)$ is a left S-semimodule. Combining (i), (ii) and (iii), we see that $\underline{R}(S) \circ M = \{o_M\}$, and hence the left S-semimodule ${}_S M = M(\boxplus, \circ)$ is not faithful. Since S is simple, it follows that $a \circ x = b \circ x$ for all $a, b \in S$ and $x \in M$. In particular, $a \circ x = a_x \circ x = o_M$, $S \circ M = \{o_M\}$ and $a \circ (xa) = o_M$. The latter equality implies $xa = o_M a$. Thus $xa = o_M a = xba$ for all $a, b \in$ and $x \in M$. Since M_S is faithful, $a = ba$ and, S being simple, $|S| = 2$.

2.7 Remark. Assume that $W_1 = W_2 \neq M \times M$, choose $\alpha \in M$ and put $x * y = \alpha$ for all $(x, y) \in (M \times M) \setminus W_1$. Now, the binary operation $*$ is defined on M and it is idempotent and commutative. If $u < \alpha$ and $(x, \alpha) \notin W_1$ then $(x * \alpha) * u = \alpha * u = u \neq \alpha = x * u = x * (\alpha * u)$. If $(x, y) \notin W_1$ then either $\alpha \not\leq x$ or $\alpha \not\leq y$. If $\alpha \not\leq x$ and $(x, \alpha) \in W_1$ then $(y * x) * x = \alpha * x \neq \alpha = y * x = y * (x * x)$. Now, it is easy to see that $M(*)$ is a semilattice if and only if α is a minimal element of the ordered set $M(\leq)$. Anyway, if α is minimal then $\alpha \not\leq v$ for at least one $v \in M$ (otherwise $\alpha = 0_M \in M$ and $W_1 = M \times M$) and $(v * \alpha) + v = \alpha + v \neq v$. It means that $M(+, *)$ is not a lattice.

2.8 Remark. If $0_M \in M$ then $W_1 = M \times M$ and $W_3 = \{\, (x, a) \mid 0_M a \leq x \,\}$.

3. From the right to the left (b)

The foregoing section is immediately continued. Here, we assume that $W_1 = W_2$ and $W_3 = W_4$. Choose and fix an element ω such that $\omega \notin S \cup M$, put $M^+ = M \cup \{\omega\}$ and extend the (partial) operations $*$ and $\circ$ defined on M in the following way: $x * u = z * \omega = \omega * z = \omega * \omega = \omega$ for all $x, y, z \in M$, $(x, y) \notin W_1$, and $a \circ u = b \circ \omega = \omega$ for all $a, b \in S$, $u \in M$, $(u, a) \notin W_3$. Furthermore, put $x + \omega = \omega + x = x$ for every $x \in M$, $\omega + \omega = \omega$ and $\omega a = \omega$ for every $a \in S$.

3.1 Proposition. *The algebraic structure $M^+(+, *)$ is a lattice.*

Proof. First, it follows from 2.1(i),(ii) that the binary structure $M^+(*)$ is both idempotent and commutative. If $\alpha, \beta, \gamma \in M^+$ and $\omega \in \{\alpha, \beta, \gamma\}$ then $(\alpha * \beta) * \gamma = \omega = \alpha * (\beta * \gamma)$ (use 2.1(vi) and the equality $W_1 = W_2$). We have proved that $M^+(*)$ is a semilattice. Using 2.1(iv),(v), we see that $M^+(+, *)$ is a lattice. ∎

3.2 Theorem. *The algebraic structure ${}_S M^+ = M^+(*, \circ)$ is a left S-semimodule.*

Proof. As we already know, $M^+(*)$ is a semilattice (see 3.1). It remains to show that $a \circ (\alpha * \beta) = (a \circ \alpha) * (a \circ \beta)$, $(a + b) \circ \alpha = (a \circ \alpha) * (b \circ \alpha)$ and $a \circ (b \circ \alpha) = (ab) \circ \alpha$

for all $a, b \in S$ and $\alpha, \beta \in M^+$. The case $\alpha = \omega$ (or $\beta = \omega$) is clear and we can assume $\alpha = x$, $\beta = y$, $x, y \in M$. Now, we distinguish the following cases:

(i) Let $(x, y) \in W_1$ and $(x * y, a) \in W_3$. By 2.3(i),(ii), we have $Q_{x*y,a} = Q_{x,a} \cap Q_{y,a} \neq \emptyset$ and, by 2.3(iv) we have $(x, a), (y, a) \in W_3$. Now, $(a \circ x, a \circ y) \in W_1$ and $a \circ (x * y) = (a \circ x) * (a \circ y)$ by 2.3(vi).

(ii) Let $(x, y) \in W_1$ and $(x * y, a) \notin W_3$. By 2.3(v), $Q_{x,a} \cap Q_{y,a} = \emptyset$, and hence $(a \circ x, a \circ y) \notin W_1$ follows from 2.3(ii). Thus $a \circ (x * y) = \omega = (a \circ x) * (a \circ y)$.

(iii) Let $(x, y) \notin W_1$. Then $a \circ (x * y) = a \circ \omega = \omega$. On the other hand, $Q_{x,a} \cap Q_{y,a} = \emptyset$ by 2.3(iv), and hence, due to 2.3(iii), $(a \circ x, a \circ y) \notin W_1$. Then $(a \circ x) * (a \circ y) = \omega$.

(iv) Let $(x, a + b) \in W_3$ and $(x, a), (x, b) \in W_3$. By 2.4(iv), $(a + b) \circ x = (a \circ x) * (b \circ x)$.

(v) Let $(x, a + b) \in W_3$ and $(x, a) \notin W_3$. Then $Q_{x,a} = \emptyset$, and hence $Q_{x,a+b} = Q_{x,a} \cap Q_{x,a} = \emptyset$, a contradiction.

(vi) Let $(x, a + b) \notin W_3$. Then $(a + b) \circ x = \omega$. We have $Q_{x,a+b} = \emptyset$, and therefore $(a \circ x, b \circ x) \notin W_1$ follows from 2.4(ii). Thus $(a \circ x) * (b \circ x) = \omega$.

(vii) Let $(x, b) \in W_3$ and $(b \circ x, a) \in W_3$. Then $a \circ (b \circ x) = (ab) \circ x$ by 2.5(ii).

(viii) Let $(x, b) \in W_3$ and $(b \circ x, a) \notin W_3$. Then $a \circ (b \circ x) = \omega$, $(x, ab) \notin W_3$ by 2.5(i) and $(ab) \circ x = \omega$.

(ix) Let $(x, b) \notin W_3$. Then $a \circ (b \circ x) = \omega = (ab) \circ x$ by 2.5(v). ∎

3.3 Proposition. *The algebraic structure $M_S^+ = M^+(+, \cdot)$ is a right S-semimodule.*

Proof. It is easy. ∎

3.4 Remark. Clearly, $(a \circ x)b \leq a \circ (xb)$ for all $a, b \in S$ and $x \in M^+$.

3.5 Lemma.

(i) $S \circ \omega = \{\omega\}$.

(ii) *If $o_M \in M$ then $S \circ o_M = \{o_M\}$.*

(iii) *If $0_M \in M$ then $S \circ 0_M = \{0_M\}$ iff $0_M S = \{0_M\}$ and $xa \neq 0_M$ for all $a \in S$ and $x \in M \setminus \{0_M\}$.*

(iv) *If $x \in M$ then $S \circ x = \{x\}$ iff $xS \leq x$ and $ya \not\leq x$ for all $a \in S$, $y \in M$, $y \not\leq x$.*

3.6 Lemma. *The following conditions are equivalent for $a, b \in S$:*

(i) *$a \circ x = b \circ x$ for every $x \in M$.*

(ii) *$xa = xb$ for every $x \in M$.*

3.7 Corollary. *The left semimodule ${}_SM^+$ is faithful if and only if the right semimodule M_S is so.*

3.8 Lemma. *Let $a \in S$ and $w \in M$ be such that $Ma \leq w$. Then $a \circ w = o_M \in M$.*

3.9 Proposition. *Assume that for every $x \in M$ there is $a_x \in S$ with $Ma_x = \{x\}$. Then:*

(i) $a_x \circ y = o_M \in M$ *for every* $y \in M^+$, $x \leq y$ *(or* $x * y = x$*).*

(ii) $a_x \circ z = \omega$ *for every* $z \in M^+$, $x \not\leq z$ *(or* $x * z \neq x$*).*

(iii) $a_x * M^+ = \{o_M, \omega\}$.

(iv) *The semimodule* ${}_SM^+$ *is simple.*

Proof. Only (iv) needs a short proof. Let $\varrho \neq \mathrm{id}$ be a congruence of ${}_SM^+$. We have $o_M \in M$ by 3.8 and if $(\omega, o_M) \in \varrho$ then $\varrho = M \times M$. If $(\omega, x) \in \varrho$ for some $x \in M$ then $(\omega, o_M) = (a_x \circ \omega, a_x \circ x) \in \varrho$. If $(x, y) \in \varrho$, where $x, y \in M$, $x \not\leq y$, then $(o_M, \omega) = (a_x \circ x, a_x \circ y) \in \varrho$. Thus $\varrho = M \times M$ anyway. ∎

The following assertions are easy.

3.10 Proposition. *Assume that* $0_M \in M$ *and put* $L^+ = M^+ \setminus \{0_M\}$*. Then* $W_1 = W_2 = M \times M$ *and* L^+ *is a subsemimodule of the left* S*-semimodule* ${}_SM^+$ *if and only if the following three conditions are satisfied:*

1. *For all* $x, y \in L = M \setminus \{o_M\}$ *there is at least one* $z \in L$ *with* $z \leq x$ *and* $z \leq y$*;*
2. *If* $a \in S$ *is such that* $0_M a = 0_M$ *then for every* $x \in L$ *there is at least one* $y \in L$ *with* $ya \leq x$*;*
3. *If* $a \in S$ *is such that* $0_M a \neq 0_M$ *then there is at least one* $v \in L$ *with* $0_M a = va$*.*

3.11 Lemma. *Assume that* $0_M \in M$ *and that the set* $L = M \setminus \{0_M\}$ *has the smallest element* w*. Let* $a, b \in S$ *be such that* $Ma = \{0_M\}$ *and* $Mb = \{w\}$*. Then:*

(i) $a \neq b$ *and* $a \circ x = b \circ x$ *for every* $x \in L^+ = L \cup \{w\}$*.*

(ii) *If* L^+ *is a subsemimodule of* ${}_SM^+$ *(see 3.10) then* L^+ *is not faithful.*

3.12 Remark.

(i) We have $P'_{x,y} = \{z \in M^+ \mid x \leq z, y \leq z\} = \{z \in M \mid x + y \leq z\}$, $P'_{x,\omega} = \{z \in M^+ \mid x \leq z, \omega \leq z\} = \{z \in M \mid x \leq z\}$ and $P'_{\omega,\omega} = \{z \in M^+ \mid \omega \leq z\} = M^+$ for all $x, y \in M$.

(ii) $Q'_{x,a} = \{y \in M^+ \mid x \leq a \circ y\} = \{y \in M \mid xa \leq y\}$ and $Q'_{\omega,a} = \{y \in M^+ \mid ya \leq \omega\} = \{\omega\}$ for all $a \in S$ and $x \in M$.

3.13 Remark. M is a subsemimodule of ${}_SM^+$ iff the following two conditions are satisfied:

1. For all $x, y \in M$ there is $z \in M$ with $z \leq x$ and $z \leq y$;
2. For all $a \in S$ and $x \in M$ there is $y \in M$ with $ya \leq x$.

If the condition (2) is true then the set Ma is downwards cofinal in $M(\leq)$ (cf. 3.9).

4. From the right to the left (c)

The second section is continued. We will assume here that $(W_1 =)\ W_2 = M \times M$ (cf. 2.6 and 2.7),

4.1 Proposition. *The algebraic structure $M(+,*)$ is a lattice.*

Proof. Use 2.1. ∎

Now, choose $\alpha \in M$ and put $W_3' = \{(x,a) \in W_3 \,|\, x \neq \alpha\}$. Furthermore, assume that $W_3 = W_4$ and put $a \triangle x = a \circ x$ for every pair $(x,a) \in W_3'$ and $b \triangle y = \alpha$ for every pair $(b,y) \in (S \times M) \setminus W_3'$.

4.2 Lemma. *$a \triangle (\alpha * x) = (a \triangle \alpha) * (a \triangle x)$ for all $a \in S$ and $x \in M$ iff the following two conditions are satisfied:*

1. *If $a \in S$ and $x \in M$ are such that $\alpha a \leq x$, $\alpha \not\leq x$ and $(\alpha,a) \in W_3$ then $(a \circ \alpha) * (a \circ x) = \alpha$;*
2. *If $a \in S$ and $x \in M$ are such that $x \neq \alpha$, $\alpha a \not\leq x$ and $(x,a) \in W_3$ then $(\alpha,a) \in W_3$, $\alpha \not\leq x$ and $(a \circ \alpha) * (a \circ x) = \alpha * (a \circ x)$.*

Proof. (i) Let $a \in S$ and $x \in M$ be such that $\alpha x \leq x$, $\alpha \not\leq x$ and $(\alpha,a) \in W_3$. We have $\alpha \in R_{x,a}$ and we get $a \triangle x = a \circ x$. Now, $(a \triangle \alpha) * (a \triangle x) = \alpha * (a \circ x) = \alpha$, since $\alpha \leq a \circ x$. On the other hand, $(\alpha,a),(x,a) \in W_3$, $\alpha * x \neq \alpha$, and hence $(\alpha * x, a) \in W_3$ and $a \circ (\alpha * x) = (a \circ \alpha) * (a \circ x)$ by 2.3(iii),(v). Of course, $a \circ (\alpha * x) = a \triangle (\alpha * x)$.

(ii) Let $a \in S$ and $x \in M \setminus \{\alpha\}$ be such that $\alpha a \not\leq x$ and $(x,a) \in W_3$. We have $(a \triangle \alpha) * (a \triangle x) = \alpha * (a \circ x) \neq \alpha$, since $\alpha a \not\leq x$. Now, if $a \triangle (\alpha * x) = (a \triangle \alpha) * (a \triangle x)$ then $a \triangle (\alpha * x) \neq \alpha$, and hence $a \triangle (\alpha * x) = a \circ (\alpha * x)$, $(\alpha * x, a) \in W_3'$. Consequently, $(\alpha,a) \in W_3$ and $a \circ (\alpha * x) = (a \circ \alpha) * (a \circ x)$ by 2.3(vi). If $\alpha \leq x$ then $\alpha * x = \alpha$ and $a \triangle (\alpha * x) = \alpha$.

(iii) Assume that both conditions (1) and (2) are satisfied. We wish to show that $y = z$, where $y = a \triangle (\alpha * x)$ and $z = (a \triangle \alpha) * (a \triangle x) = \alpha * (a \triangle x)$.

If $x = \alpha$ then $y = \alpha = z$. If $(x,a) \notin W_3$ then $(\alpha * x, a) \notin W_3$ and $y = \alpha = z$ again. Consequently, assume that $x \neq \alpha$ and $(x,a) \in W_3$, so that $(x,a) \in W_3'$ and $a \triangle x = a \circ x$, $z = \alpha * (a \circ x)$.

Let $\alpha a \leq x$. That is, $\alpha \leq a \circ x$ and $z = \alpha$. If $\alpha \leq x$ or $(\alpha * x, a) \notin W_3$ then $y = \alpha$. If $\alpha \not\leq x$ and $(\alpha * x, a) \in W_3$ then $(\alpha,a) \in W_3$, $\alpha * x \neq \alpha$ and $y = a \circ (\alpha * x) = (a \circ \alpha) * (a \circ x) = \alpha$ by (1) and 2.3(vi).

Let $\alpha a \not\leq x$. Then $\alpha \not\leq a \circ x$ and $z = \alpha * (a \circ x) \neq \alpha$. By (2), $(\alpha,a) \in W_3$, $\alpha \not\leq x$ and $z = \alpha * (a \circ x) = (a \circ \alpha) * (a \circ x)$. On the other hand, $(\alpha * x, a) \in W_3$ by 2.3(iii),(v) and $y = a \triangle (\alpha * x) = a \circ (\alpha * x) = (a \circ \alpha) * (a \circ x) = z$ by 2.3(vi). ∎

4.3 Lemma. *$a \triangle (x * y) = (a \triangle x) * (a \triangle y)$ for all $a \in S$ and $x, y \in M \setminus \{\alpha\}$ such that $x * y \in M \setminus \{\alpha\}$ iff the following condition is satisfied:*

1. *If $(z,b) \in W_3'$ is such that $(u,b) \notin W_3$ for some $u \in M \setminus \{\alpha\}$, $z * u \neq \alpha$, then $\alpha b \leq z$.*

Proof. (i) As concerns (1), we have $(z * u, b) \notin W_3$, and hence $\alpha = b \triangle (z * u)$. On the other hand, $(b \triangle z) * (b \triangle u) = (b \circ z) * \alpha$.

(ii) Assume that (1) is true and put $u = a \triangle (x * y)$ and $v = (a \triangle x) * (a \triangle y)$. If $(x,a) \notin W_3$ and $(y,a) \notin W_3$ then $(x * y, a) \notin W_3$ and we get $u = \alpha = v$. If $(x,a),(y,a) \in W_3$ then $(x * y, a) \in W_3$ (use 2.3) and $u = a \circ (x * y) = (a \circ x) *$

$(a \circ y) = v$ (use 2.3 again). On the other hand, if $(x, a) \in W_3$ and $(y, a) \notin W_3$ then $(x * y, a) \notin W_3$ and $u = \alpha$, $v = (a \circ x) * \alpha$. By (1), $\alpha a \leq x$, $\alpha \leq a \circ x$ and $v = \alpha$. ∎

4.4 Lemma. *$(a + b) \bigtriangleup x = (a \bigtriangleup x) * (b \bigtriangleup x)$ for all $a, b \in S$ and $x \in M$ iff the following condition is satisfied:*

1. *If $(x, a) \in W_3$, $x \neq \alpha$, and $(x, b) \notin W_3$ then $\alpha a \leq x$.*

Proof. (I) As concerns (1), we have $(x, a + b) \notin W_3$ by 2.4(i),(iii), and hence $(a + b) \bigtriangleup x = \alpha$. On the other hand, $(a \bigtriangleup x) * (b \bigtriangleup x) = (a \circ x) * \alpha$ and if $(a \circ x) * \alpha = \alpha$ then $\alpha a \leq x$.

(ii) Assume that (1) is true and put $y = (a+b) \bigtriangleup x$ and $z = (a \bigtriangleup x) * (b \bigtriangleup x)$. If $x = \alpha$ then $y = \alpha = z$, so that we will assume that $x \neq \alpha$. If $(x, a + b) \notin W_3$ then $y = \alpha$ and, by 2.4(ii), either $(x, a) \notin W_3$ or $(x, b) \notin W_3$. Assume the former case, the latter one being symmetric. If $(x, b) \in W_3$ then $z = \alpha * (b \circ x)$ $\alpha b \leq x$ by (1), $\alpha \leq b \circ x$ and $z = \alpha$. If $(x, b) \notin W_3$ then $z = \alpha * \alpha = \alpha$. Finally, let $(x, a + b) \in W_3$, so that $y = (a + b) \circ x$. By 2.4(i),(iii), we have $(x, a), (x, b) \in W_3$, and hence $z = (a \circ x) * (a \circ y) = (a + b) \circ x = y$ by 2.4(iv). ∎

4.5 Lemma. *$(ab) \bigtriangleup x = a \bigtriangleup (b \bigtriangleup x)$ for all $a, b \in S$ and $x \in M$ iff the following condition is satisfied:*

1. *If $(\alpha, c) \in W_3$, $(x, d) \in W_3$, $x \neq \alpha$ and $d \circ x = \alpha$ then $c \circ \alpha = \alpha$.*

Proof. (i) As concerns (1), we have $(x, cd) \in W_4$ and $c \circ \alpha = c \circ (d \circ x) = (cd) \circ x = (cd) \bigtriangleup x$ by 2.5(ii). On the other hand, $c \bigtriangleup (d \bigtriangleup x) = c \bigtriangleup (d \circ x) = c \bigtriangleup \alpha = \alpha$.

(ii) Assume that (1) is true. We wish to show that $y = z$, where $y = (ab) \bigtriangleup x$ and $z = a \bigtriangleup (b \bigtriangleup x)$. If $x = \alpha$ then $y = \alpha = z$, and hence we assume $x \neq \alpha$. If $(x, b) \notin W_3$ then $(x, ab) \notin W_3$ and $y = \alpha = z$ again. If $(x, b) \in W_3$ and $(b \circ x, a) \notin W_3$ then the same is true. If $(x, b) \in W_3$, $(b \circ x, a) \in W_3$ and $b \circ x \neq \alpha$ then $(x, ab) \in W_3$ and $y = (ab) \circ x = a \circ (b \circ x) = a \bigtriangleup (b \circ x) = a \bigtriangleup (b \bigtriangleup x) = z$ by 2.5(ii). Finally, if $(x, b), (b \circ x, a) \in W_3$ and $b \circ x = \alpha$ then $y = (ab) \circ x = a \circ (b \circ x) = a \circ \alpha = \alpha = a \bigtriangleup \alpha = a \bigtriangleup (b \circ x) = a \bigtriangleup (b \bigtriangleup x) = z$ by (1). ∎

4.6 Lemma. *Assume that 4.5(1) is true. Let $(x, a) \in W_3$, $x \neq \alpha = a \circ x$. Then:*

(i) *If $b \in S$ and $y \in M$ are such that $yb \leq \alpha$ then $y \leq \alpha$ and $ya \leq x$.*

(ii) *If $c \in S$ is such that $Mc \leq \alpha$ then $\alpha = o_M$ and $Ma \leq x$.*

Proof. It is easy. ∎

4.7 Lemma. *Let $a \in S$ and $x, y \in M \setminus \{\alpha\}$ be such that $x * y = \alpha$ and $a \bigtriangleup (x * y) = (a \bigtriangleup x) * (a \bigtriangleup y)$. Then:*

(i) *If $z \in M$ is such that $za \leq \alpha$ then $z \leq \alpha$ and $a \circ \alpha \leq \alpha$.*

(ii) *$va \not\leq \alpha$ for at least one $v \in M$.*

Proof. (i) First, $(a \triangle x) * (a \triangle y) = a \triangle (x * y) = a \triangle \alpha = \alpha$. Then $za \le \alpha < x, y$, so that $z \le a \circ x$ and $z \le a \circ y$. We conclude that $z \le (a \circ x) * (a \circ y) = (a \triangle x) * (a \triangle y) = \alpha$. If $Ma \le \alpha$ then $M \le \alpha$ by (i), and hence $o_M = \alpha < x$, a contradiction. ∎

In the remaining part of this section, assume that $\alpha = 0_M \in M$.

4.8 Lemma. *$a \triangle (x * y) = (a \triangle x) * (a \triangle y)$ for all $a \in S$ and $x, y \in M$ iff the following condition is satisfied:*

1. *If $a \in S$ and $x, y, w \in M \setminus \{0_M\}$ are such that $wa \le x$ and $wa \le y$ then $z \le x$ and $z \le y$ for at least one $z \in M \setminus \{0_M\}$.*

Proof. First, $0_M \le x$ for every $x \in M$, and hence the condition 4.2(1) is satisfied. Further, if $(x, a) \in W_3$ then $0_M a \le x$ and 4.2(2) is true as well. By 4.2, $a \triangle (x * y) = (a \triangle x) * (a \triangle y)$ whenever $0_M \in \{x, y\}$. Similarly, 4.3(1) is true and $a \triangle (x * y) = (a \triangle x) * (a \triangle y)$ whenever $x * y \ne 0_M$.

Now, let $x, y \in M \setminus \{0_M\}$ and $x * y = 0_M$. Then $a \triangle (x * y) = 0_M$ and, if $(x, a) \notin W_3$ or $(y, a) \notin W_3$ then $(a \triangle x) * (a \triangle y) = 0_M$. Assume, therefore, that $(x, a), (y, a) \in W_3$. Then $(a \triangle x) * (a \triangle y) = (a \circ x) * (a \circ y)$, and so $(a \triangle x) * (a \triangle y) = 0_M$ iff $wa \le x$ and $wa \le y$ implies $w = 0_M$. ∎

4.9 Lemma. *Assume that $wa_0 = 0_M$ for some $a_0 \in S$ and $w \in M$, $w \ne 0_M$. Then $a \triangle (x * y) = (a \triangle x) * (a \triangle y)$ for all $a \in S$ and $x, y \in M$ iff the following condition is satisfied:*

1. *For all $x, y \in M \setminus \{0_M\}$ there is $z \in M \setminus \{0_M\}$ with $z \le x$ and $z \le y$.*

Proof. See 4.8 (and (4.7). ∎

4.10 Lemma. *$(a + b) \triangle x = (a \triangle x) * (b \triangle x)$ for all $a, b \in S$ and $x \in M$.*

Proof. Apparently, the condition 4.4(1) is true. ∎

4.11 Lemma. *$(ab) \triangle x = a \triangle (b \triangle x)$ for all $a, b \in S$ and $x \in M$ iff the following condition is satisfied:*

1. *If $xa = 0_M$ for some $a \in S$ and $x \in M \setminus \{0_M\}$ then for all $b \in S$ and $y \in M \setminus \{0_M\}$ such that $0_M b \le y$ there is $z \in M \setminus \{0_M\}$ with $zb \le y$.*

Proof. Use 4.5, where $\alpha = 0_M$ (see also 4.6). ∎

4.12 Theorem. *The algebraic structure $M(*, \triangle)$ is a left S-semimodule if and only if the conditions* 4.8(1) *and* 4.11(1) *are satisfied.*

Proof. See 4.8, 4.10 and 4.11. ∎

4.13 Proposition. (cf. 3.10) *Assume that $w_0 a_0 = 0_M$ for some $a_0 \in S$ and $w_0 \in M \setminus \{0_M\}$. Then ${}_S M = M(*, \triangle)$ is a left S-semimodule if and only if the following three conditions are satisfied:*

1. *For all $x, y \in M \setminus \{0_M\}$ there is $z \in M \setminus \{0_M\}$ such that $z \le x$ and $z \le y$;*
2. *If $a \in S$ is such that $0_M a = 0_M$ then, for every $x \in M \setminus \{0_M\}$, there is at least one $y \in M \setminus \{0_M\}$ such that $ya \le x$;*

3. *If $a \in S$ is such that $0_M a \neq 0_M$ then there is at least one $v \in M \setminus \{0_M\}$ such that $0_M a = va$.*

Proof. See 4.9, 4.10 and 4.11 (clearly, (2) and (3) are equivalent to 4.11(1) under our assumptions). ∎

4.14 Proposition. *Assume that $xa \neq 0_M$ for all $a \in S$ and $x \in M \setminus \{0_M\}$ (i.e., $L = M \setminus \{0_M\}$ is a subsemimodule of M_S). Then $M(*, \triangle)$ is a left S-semimodule.*

Proof. See 4.8, 4.10 and 4.11. ∎

4.15 Lemma.

(i) $S \triangle 0_M = \{0_M\}$.

(ii) *If $o_M \in M$ then $S \triangle o_M = \{o_M\}$.*

(iii) *If $x \in M \setminus \{0_M\}$ then $S \triangle x = \{x\}$ iff $xS \leq x$ and $yz \not\leq x$ for all $a \in S$, $y \in M$, $y \not\leq x$.*

4.16 Lemma. *Let $a, b \in S$ be such that $a \triangle x = b \triangle x$ for every $x \in M$. If $y \in M \setminus \{0_M\}$ is such that $ya \neq 0_M \neq yb$ then $ya = yb$.*

Proof. We have $a \circ ya = a \triangle ya = b \triangle ya$ and $a \circ ya \geq y > 0_M$. Thus $b \triangle ya = b \circ ya \geq y$ and $yb \geq ya$. Symmetrically, $ya \geq yb$, so that $ya = yb$. ∎

4.17 Corollary. *Assume that ${}_SM$ is a semimodule and for all $a, b \in S$, $a \neq b$, there is $x \in M \setminus \{0_M\}$ such that $0_M \neq xa \neq xb \neq 0_M$. Then the left S-semimodule ${}_SM$ is faithful.*

4.18 Lemma. *Let $a, b \in S$ be such that $a \triangle x = b \triangle x$ for every $x \in M$. If $0_M a \neq 0_M \neq a \circ 0_M a$ and $0_M b \neq 0_M \neq b \circ 0_M b$ then $0_M a = 0_M b$.*

Proof. We can proceed similarly as in the proof of 4.16. ∎

4.19 Lemma. *Let $a \in S$ and $x \in M \setminus \{0_M\}$ be such that $Ma \leq x$. Then:*

(i) $a \triangle x = o_M \in M$.

(ii) *If $S \triangle x = \{o_M\}$ then $o_M S \leq x$.*

Proof. It is easy. ∎

4.20 Corollary. *Assume that the semiring S is simple, ${}_SM$ is a semimodule and there are $a, b \in S$ and $x \in M \setminus \{0_M\}$ such that $Ma \leq x$ and $o_M b \not\leq x$. Then the left S-semimodule ${}_SM$ is faithful.*

4.21 Lemma. *Let $x \in M \setminus \{0_M\}$ be such that $Ma_x = \{x\}$ for some $a_x \in S$. Then:*

(i) $a_x \triangle y = o_M \in M$ *for every $y \in M$, $x \leq y$ (or $x * y = x$).*

(ii) $a_x \triangle z = 0_M$ *for every $z \in M$, $x \not\leq z$ (or $x * z \neq x$).*

(iii) $a_x \triangle M = \{o_M, 0_M\}$.

4.22 Lemma. *Let $a \in S$ be such that $Ma = \{0_M\}$. Then:*

(i) $a \triangle x = o_M \in M$ *for every $x \in M \setminus \{0_M\}$.*

(ii) $a \triangle 0_M = 0_M$.

4.23 Proposition. *Assume that* ${}_SM = M(*, \triangle)$ *is a left* S*-semimodule and that for every* $x \in M \setminus \{0_M\}$ *there is* $a_x \in S$ *with* $Ma_x = \{x\}$*. Then the left semimodule* ${}_SM$ *is simple.*

Proof. We can proceed similarly as in the proof of 3.9(iv) (where ω is replaced by 0_M). ∎

4.24 Lemma. *Assume that the set* $L = M \setminus \{0_M\}$ *has the smallest element* w*. Let* $a, b \in S$ *be such that* $Ma = \{0_M\}$ *and* $Mb = \{w\}$*. Then:*

(i) $a \neq b$ *and* $a \triangle x = b \triangle x$ *for every* $x \in M$*.*

(ii) *If* ${}_SM$ *is a left* S*-semimodule then it is not faithful.*

4.25 Proposition. *Assume that* ${}_SM$ *is a faithful left* S*-semimodule and that for every* $x \in M$ *there is* $a_x \in S$ *with* $Ma_x = \{x\}$*. Then both* S *and* M *are infinite and the set* $L = M \setminus \{0_M\}$ *has no minimal element.*

Proof. Let $w \in L$ be minimal in L. We have $wa_0 = 0$ $(= 0_M)$ and it follows from 4.13(1) that w is the smallest element of L. Now, ${}_SM$ is not faithful due to 4.24(ii). ∎

4.26 Remark.

(i) Assume that the conditions 4.13(1),(2),(3) are satisfied. Then ${}_SM = M(*, \triangle)$ is a left S-semimodule (see 4.13, 4.14) and, by 3.10, the set $L^+ = (M \setminus \{0_M\}) \cup \{\omega\}$ is a subsemimodule of the left S-semimodule ${}_SM^+$. Now, it is easy to see that the mapping $x \mapsto x$, $x \in L = M \setminus \{0_M\}$ and $0_M \mapsto \omega$ is an isomorphism of the semimodule ${}_SM$ onto the semimodule ${}_SL^+$.

(ii) Assume that ${}_SM$ is a (left S-)semimodule. Then either all the three conditions 4.13(1),(2),(3) are true or $0_M \notin LS$ and $L = M \setminus \{0_M\}$ is a subsemimodule of the right semimodule M_S.

5. From the right to the left (d)

The second section is continued. Here, we assume that $0_M \in M$ (then $W_1 = M \times M$) and that $W_1 = W_2$ (then $W_2 = M \times M$ and $M(+, *)$ is a lattice – see 4.1). Furthermore, assume that $W_3' = \{\, (x, a) \in W_3 \mid x \neq 0_M \,\} = \{\, (x, a) \mid x \neq 0_M, 0_M a \leq x \,\} \subseteq W_4$. Similarly as in the fourth section, we put $a \triangle x = a \circ x$ for $(x, a) \in W_3'$ and $a \triangle x = 0_M$ for $(x, a) \in (S \times M) \setminus W_3'$.

5.1 Remark. Let $(x, a) \in W_3 \setminus W_3'$. Then $x = 0_M$, $Q = Q_{0_M,a} = \{y \mid ya = 0_M\} \neq \emptyset$ and we get $0_M \in Q$. If $Q = \{0_M\}$ then $o_Q = 0_M$ and $(0_M, a) \in W_4$. Of course, $Q \subseteq Q_{z,a}$ for every $z \in M$.

If $o_Q \notin Q$ (i.e., $(0_M, a) \notin W_4$) then $Q \neq Q_{u,a}$ for every $u \in M \setminus \{0_M\}$ and there is $v_u \in Q_{u,a}$ such that $0_M \neq v_u a \leq u$, $v_u \neq 0_M$. We have shown that for every $u \in L = M \setminus \{0_M\}$ there is $v \in L$ such that $va \leq u$, $va \in L$.

5.2 Lemma. $a \triangle (x * y) = (a \triangle x) * (a \triangle y)$ *for all* $a \in S$ *and* $x, y \in M$ *iff the condition* 4.8(1) *is satisfied.*

Proof. If $0_M \in \{x, y\}$ then $a \triangle (x * y) = a \triangle 0_M = 0_M = (a \triangle x) * (a \triangle y)$. Assume, therefore, that $x \neq 0_M \neq y$.

If $wa \leq x$ and $wa \leq y$ for some $w \in M \setminus \{0_M\}$ then $wa \leq x * y$, $w \leq a \triangle x$, $w \leq a \triangle y$ and $w \leq (a \triangle x) * (a \triangle y)$. Moreover, $(a \triangle x)a \leq x$, $(a \triangle y)a \leq y$, and hence $((a \triangle x) * (a \triangle y))a \leq x * y$. Then $(a \triangle x) * (a \triangle y) = a \triangle (x * y)$, provided that $x * y \neq 0_M$. On the other hand, if $x * y = 0_M$ then $a \triangle x) * y) = 0_M \neq (a \triangle x) * (a \triangle y)$.

If $wa \nleq x * y$ for every $w \in M \setminus \{0_M\}$ then $a \triangle (x * y) = 0_M = (a \triangle x) * (a \triangle y)$. ∎

5.3 Lemma. *$(a + b) \triangle x = (a \triangle x) * (b \triangle x)$ for all $a.b \in S$ and $x \in M$.*

Proof. Easy to check. ∎

5.4 Lemma. *$(ab) \triangle x = a \triangle (b \triangle x)$ for all $a, b \in S$ and $x \in M$ iff the condition* 4.11(1) *is satisfied.*

Proof. Easy to check. ∎

5.5 Theorem. *The algebraic structure ${}_SM = M(*, \triangle)$ is a left S-semimodule if and only if the conditions* 4.8(1) *and* 4.11(1) *are satisfied.*

Proof. Combine 5.2, 5.3 and 5.4. ∎

5.6 Proposition. *Assume that $w_0 a_0 = 0_M$ for some $a_0 \in S$ and $w_0 \in L = M \setminus \{0_M\}$ (equivalently, L is not a subsemimodule of M_S). Then ${}_SM = M(*, \triangle)$ is a left S-semimodule if and only if the conditions* 4.13(1),(2),(3) *are satisfied.*

Proof. See the proof of 4.13. ∎

5.7 Lemma.

(i) $S \triangle 0_M = \{0_M\}$.

(ii) *If $o_M \in M$ then $S \triangle o_M = \{o_M\}$.*

5.8 Proposition. *Assume that the semiring S is simple, ${}_SM$ is a semimodule and that there are $a, b \in S$ and $x \in M \setminus \{0_M\}$ such that $Ma \leq x$ and $o_M b \nleq x$. Then the left S-semimodule ${}_SM$ is faithful.*

Proof. See 4.19 and 4.20. ∎

5.9 Lemma. *Let $x \in M \setminus \{0_M\}$ be such that $Ma_x = \{x\}$ for some $a_x \in S$. Then:*

(i) *$a_x \triangle y = o_M \in M$ for every $y \in M$, $x \leq y$ (or $x * y = x$).*

(ii) *$a_x \triangle z = 0_M$ for every $z \in M$, $x \nleq z$) (or $x * z \neq x$).*

5.10 Lemma. *Let $a \in S$ be such that $Ma = \{0_M\}$. Then:*

(i) *$a \triangle x = o_M \in M$ for every $x \in M \setminus \{0_M\}$.*

(ii) *$a \triangle 0_M = 0_M$.*

5.11 Proposition. *Assume that ${}_SM = M(*, \triangle)$ is a left S-semimodule and that for every $x \in M \setminus \{0_M\}$ there is $a_x \in S$ with $Ma_x = \{x\}$. Then the left S-semimodule ${}_SM$ is simple.*

Proof. See the proof of 4.23. ∎

5.12 Proposition. *Assume that ${}_SM$ is a faithful left S-semimodule and that for every $x \in M$ there is $a_x \in S$ with $Ma_x = \{x\}$. Then both S and M are infinite and the set $M \setminus \{0_M\}$ has no minimal element.*

Proof. See the proof of 4.25. ∎

6. A few conditions

Let M $(= M(+))$ be a non-trivial semilattice and let $x \leq y$ iff $x + y = y$. Let $N = M \setminus \{o_M\}$ ($N = M$ iff $o_M \notin M$) and $N' = N \setminus \{o_N\}$ ($N' = N$ iff $o_N \notin N$). Consider the following conditions:

(C1) If $x_1 < x_2 < x_3 < \dots$ is an infinite strictly increasing sequence of elements from M then for every $x \in N'$ there is $i \geq 1$ with $x \leq x_i$;

(C2) If $x_1 < x_2 < x_3 < \dots$ is an infinite strictly increasing sequence of elements from M then for every $x \in N$ there is $i \geq 1$ with $x \leq x_i$;

(C3) If $x_1 < x_2 < x_3 < \dots$ is an infinite strictly increasing sequence of elements from M then for every $x \in M$ there is $i \geq 1$ with $x \leq x_i$;

(C4) There is no infinite strictly increasing sequence of elements from M.

Clearly, (C4) implies (C3), (C3) implies (C2) and (C2) implies (C1). If M is finite then (C4) is true. If $o_M \notin M$ then (C1), (C2) and (C3) are equivalent. If $o_N \notin N$ then (C1) and (C2) are equivalent. If $o_M \in M$ then (C3) and (C4) are equivalent. Finally, if $o_N \in N$ then (C2), (C3) and (C4) are equivalent.

6.1 Lemma. *Let M_S be a right S-semimodule satisfying* (C1). *Then $W_1 = W_2$.*

Proof. Let $x, y \in M$ and $P = P_{x,y}$. If $x \leq y$ or $y \leq x$ then $P \neq \emptyset$ and $o_P \in P$ trivially. Assume, therefore, that $x \not\leq y$ and $y \not\leq x$. Then $x, y \in N'$. Now, if $z_1 < z_2 < z_3 < \dots$ is an infinite strictly increasing sequence of elements from P then $x \leq z_i$ for some $i \geq 1$, so that $x = z_i$ and $z_i = z_{i+1}$, a contradiction. Consequently, P satisfies (C4) and for every $z \in P$ there is $v \in P$ such that $z \leq v$ and v is maximal in P. But $v + P \subseteq P$, and so $v = o_P \in P$. ∎

6.2 Lemma. *Let M_S be a right S-semimodule satisfying* (C1) *and let $a \in S$ and $x \in M$ be such that $Q = Q_{x,a} \neq \emptyset$ and $o_Q \in Q$. Then:*

(i) *$Q = N'$ (i.e., $N'a \leq x$).*

(ii) *The set N' has no maximal element.*

(iii) *$N' + N' = N'$ and $N + N = N$.*

(iv) *If $o_N \in N$ then M does not satisfy* (C2).

(v) *If $o_M \in M$ then M does not satisfy* (C3).

(vi) *M does not satisfy* (C4).

Proof. If $v \in Q$ is maximal in Q then $v + Q \subseteq Q$ implies $v = o_Q \in Q$, a contradiction. Therefore, the set Q has no maximal element and (C1) implies $N' \subseteq Q$. Of course, $o_M \notin Q$ (otherwise $o_M = o_Q$), and hence $o_N \notin Q$ either. Thus $Q = N'$ and the rest is clear. ∎

6.3 Lemma. *Let M_S be a right S-semimodule satisfying (C1) and $0_M \in M$. Let $a \in S$ be such that $0_M a = 0_M$ and $o_Q \notin Q = Q_{0_M,a}$. Then:*

(i) $N'a = \{0_M\}$.

(ii) *If* $o_M \in M$ *and* $o_N \in N$ *then* $Ma = \{0_M, o_N a, o_M a\}$.

(iii) *If* $o_M \in M$ *and* $o_N \notin N$ *then* $Na = \{0_M\}$ *and* $Ma = \{0_M, o_M a\}$.

(iv) *If* $o_M \notin M$ *then* $Ma = \{0_M\}$.

Proof. Use 6.2. ∎

6.4 Proposition. *Let* M_S *be a right* S*-semimodule satisfying* (C1) *and* $o_M \notin M$. *Then:*

(i) M *satisfies* (C3).

(ii) M *does not satisfy* (C4).

(iii) $W_1 = W_2$.

(iv) $W_3 = W_4$ *if and only if for all* $a \in S$ *and* $x \in M$ *there is at least one* $y \in M$ *with* $ya \nleq x$.

(v) *If* $0_M \in M$ *and* $W_3' \subseteq W_4$ *then* $W_3 = W_4$.

Proof. Combine 6.1 and 6.2. ∎

6.5 Proposition. *Let* M_S *be a right* S*-semimodule satisfying* (C1) *and such that* $o_M \in M$ *and* $o_N \notin N$. *Then:*

(i) M *satisfies* (C2).

(ii) M *satisfies* (C3) *if and only if* M *satisfies* (C4).

(iii) $W_1 = W_2$.

(iv) $W_3 = W_4$, *provided that either* $x + y = o_M$ *for some* $x, y \in N$ *or the set* N *has at least one maximal element.*

(v) *If* $0_M \in M$ *and* $W_3' \subseteq W_4$ *then* $W_3 = W_4$.

Proof. Combine 6.1 and 6.2. ∎

6.6 Proposition. *Let* M_S *be a right* S*-semimodule satisfying* (C1) *and such that* $o_M \in M$ *and* $o_N \in N$. *Then:*

(i) M *satisfies* (C2) *iff* M *satisfies* (C3) *iff* M *satisfies* (C4).

(ii) $W_1 = W_2$.

(iii) $W_3 = W_4$, *provided that either* $x + y = o_N$ *for some* $x, y \in N$ *or the set* N' *has at least one maximal element.*

(iv) *If* $0_M \in M$ *and* $W_3' \subseteq W_4$ *then* $W_3 = W_4$.

Proof. Combine 6.1 and 6.2. ∎

7. One example

In this section, let S be a semiring such that $|\underline{R}(S)| \geq 2$. Then $\underline{R}(S)_S$ is a non-trivial right S-semimodule and the semiring S is non-trivial. In fact, $\underline{R}(S)$ is the smallest (right) ideal of S.

7.1 Proposition. *If the semiring S is simple then the right semimodule $\underline{R}(S)_S$ is faithful.*

Proof. If $\underline{R}(S)_S$ is not faithful then, using the fact that S is simple, we see that $ab = ac$ for all $a \in \underline{R}(S)$ ans $b, c \in S$. In particular, if $b, c \in \underline{R}(S)$ then $b = bb = bc = c$, a contradiction. ∎

If $a, b \in \underline{R}(S)$ then $P_{a,b}=\{c \in \underline{R}(S) \mid a+c=a, b+c=b\}$. If $a \in S$ and $b \in \underline{R}(S)$ then $Q_{b,a}=\{\, c \in \underline{R}(S) \mid ca+b=b\}$. As we have already defined in the second part of this note, we put $W_1=\{(a,b) \mid P_{a,b} \neq \emptyset\}$, $W_2=\{(a,b) \in W_1 \mid o_P \in P = P_{a,b}\}$, $W_3 = \{(b,a) \mid Q_{b,a} \neq \emptyset\}$ and $W_4 = \{(ba) \in W_3 \mid o_Q \in Q = Q_{ba}\}$.

7.2 Assume that $W_1 = W_2$ and $W_3 = W_4$.

7.2.1 Theorem. *The algebraic structure $\frac{1}{S}\underline{R}(S) = \underline{R}(S)^+(*, \circ)$ is a simple left S-semimodule.*

Proof. See 3.2 and 3.9(iv). ∎

7.2.2 Proposition.

(i) *The left semimodule $\frac{1}{S}\underline{R}(S)$ is faithful if and only if the right semimodule $\underline{R}(S)_S$ is so.*

(ii) *If the semiring S is simple then the left semimodule $\frac{1}{S}\underline{R}(S)$ is faithful.*

Proof. See 3.7 and 7.1. ∎

7.2.3 Proposition. $o = o_{\underline{R}(S)} \in \underline{R}(S)$.

Proof. $a \circ a = o_{\underline{R}(S)}$ for every $a \in \underline{R}(S)$. ∎

7.2.4 Proposition.

(i) $S \circ \omega = \{\omega\}$.

(ii) $S \circ o = \{o\}$.

(ii) *If $a \in \underline{R}(S) \setminus \{o\}$ then $S \circ a \neq \{a\}$.*

Proof. See 3.5. ∎

7.2.5 Proposition. *Let $a \in \underline{R}(S)$. Then:*

(i) *$a \circ b = o$ for every $b \in \underline{R}(S)$ such that $a + b = b$.*

(ii) $a \circ a = o$.

(iii) *$a \circ c = \omega$ for every $c \in \underline{R}(S)$ such that $a + c \neq c$.*

(iv) $a \circ \omega = \omega$.

(v) $a \circ \underline{R}(S)^+ = \{o, \omega\}$.

Proof. See 3.9. ∎

7.2.6 Proposition. *Assume that $0 = 0_{\underline{R}(S)} \in \underline{R}(S)$ and put $L = \underline{R}(S) \setminus \{0\}$. Then:*

(i) $W_1 = W_2 = \underline{R}(S) \times \underline{R}(S)$.

(ii) $W_3 = W_4 = \{\, (b,a) \mid a \in S, b \in \underline{R}(S), 0a + b = b\,\}$.

(iii) $L^+ = L \cup \{\omega\}$ *is a subsemimodule of the left semimodule* ${}^1_S\underline{R}(S)$ *iff the following three conditions are satisfied:*

1. *For all* $a, b \in L$ *there is at least one* $c \in L$ *with* $c + a = a$ *and* $c + b = b$*;*
2. *If* $a \in S$ *is such that* $0a = 0$ *then for every* $b \in L$ *there is at least one* $c \in L$ *with* $ca + b = B$*;*
3. *If* $a \in S$ *is such that* $0a \neq 0$ *then there is at least one* $b \in L$ *with* $0a = ba$*.*

(iv) *If* L^+ *is a faithful semimodule then the set* L *has no minimal element.*

Proof. See 3.10 and 3.11. ∎

7.3 Assume that $0 = 0_{\underline{R}(S)} \in \underline{R}(S)$ and that $(W_1 =)\ W_2 = \underline{R}(S) \times \underline{R}(S)$. Furthermore, put $\alpha = 0$ and assume that $W_3' \subseteq W_4$ (see 7.2.6 and the fifth section of this note).

7.3.1 Theorem. *The algebraic structure* ${}^2_S\underline{R}(S) = \underline{R}(S)(*, \triangle)$ *is a left* S*-semimodule if and only if the three conditions* 7.2.6(iii)(1),(2),(3) *are satisfied.*

Proof. See 5.6 and 7.2.6(iii). ∎

In the rest of 7.3, assume that ${}^2_S\underline{R}(S)$ is a left semimodule (see 7.3.1).

7.3.2 Proposition. *The left semimodule* ${}^2_S\underline{R}(S)$ *is simple.*

Proof. See 4.23. ∎

7.3.3 Proposition. *The semimodule* ${}^2_S\underline{R}(S)$ *is faithful in each of the following two cases:*

(1) *For all* $a, b \in S$*,* $a \neq b$*, there is at least one* $c \in \underline{R}(S)$ *with* $c \neq 0 \neq ca \neq cb \neq 0$*.*

(2) $|\underline{R}(S)| \geq 3$ *and the semiring* S *is simple.*

Proof. See 4.18 and 4.20. ∎

7.3.4 Proposition. *If the left semimodule* ${}^2_S\underline{R}(S)$ *is faithful then* $\underline{R}(S)$ *is infinite and the set* $L = \underline{R}(S) \setminus \{0\}$ *has no minimal element.*

Proof. See 4.25. ∎

7.3.5 Proposition. *Let* $a \in \underline{R}(S)$*,* $a \neq 0$*. Then:*

(i) $a \triangle b = o = o_{\underline{R}(S)} \in \underline{R}(S)$ *for every* $b \in \underline{R}(S)$ *such that* $a + b = b$*.*

(ii) $a \triangle a = o$*.*

(iii) $a \triangle c = 0$ *for every* $c \in \underline{R}(S)$ *such that* $a + c \neq c$*.*

(iv) $a \triangle \underline{R}(S) = \{0, o\}$*.*

Proof. See 4.21. ∎

7.3.6 Proposition.

(i) $0 \triangle a = o$ *for every* $a \in \underline{R}(S) \setminus \{0\}$.

(ii) $0 \triangle 0 = 0$.

Proof. See 4.22. ■

7.3.7 Remark. By 7.2.6(iii), if the conditions (1),(2),(3) hold then the set $L^+ = L \cup \{\omega\}$, $L = \underline{R}(S) \setminus \{0\}$, is a subsemimodule of the left semimodule ${}^1_S\underline{R}(S)$. Now, the mapping $a \mapsto a$ for $a \in L$ and $0 \mapsto \omega$ is an isomorphism of ${}^2_S\underline{R}(S)$ onto ${}_SL^+$.

7.4 Proposition. *Assume that* $o_{\underline{R}(S)} \notin \underline{R}(S)$ *and that the right semimodule* $\underline{R}(S)_S$ *satisfies* (C1). *Then:*

(i) $\underline{R}(S)$ *satisfies* (C2) *and* (C3) *and does not satisfy* (C4).

(ii) $W_1 = W_2$.

(iii) $W_3 \neq W_4$.

Proof. See 6.4 (we have $(a, b) \in W_3 \setminus W_4$ for all $a, b \in \underline{R}(S)$, $a \leq b$). ■

7.5 Proposition. *Assume that* $o = o_{\underline{R}(S)} \in \underline{R}(S)$, $o_T \notin T$, $T = \underline{R}(S) \setminus \{o\}$ *and that* $\underline{R}(S)_S$ *satisfies* (C1). *Then:*

(i) $\underline{R}(S)$ *satisfies* (C2).

(ii) $W_1 = W_2$.

(iii) $W_3 = W_4$, *provided that either* $a + b = o$ *for some* $a, b \in T$ *or the set* T *has at least one maximal element.*

Proof. See 6.5. ■

7.6 Proposition. *Assume that* $o = o_{\underline{R}(S)} \in \underline{R}(S)$, $o_T \in T = \underline{R}(S) \setminus \{o_S\}$ *and that* $\underline{R}(S)_S$ *satisfies* (C1). *Then:*

(i) $W_1 = W_2$.

(ii) $W_3 = W_4$, *provided that either* $a + b = o_T$ *for some* $a, b \in T \setminus \{o\}$ *or the set* $T \setminus \{o_T\}$ *has at least one maximal element.*

Proof. See 6.6. ■

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