# SEMIGROUP DISTANCES OF FINITE GROUPOIDS

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**Abstract.** The simplest cases of non-associative groupoids are presented by groupoids (so called SH-groupoids) having just one non-associative (ordered) triple of elements. In this paper, only SH-groupoids having the simplest possible non-associative triple $(a, a, a)$ are investigated. For each positive integer $n$ finite SH-groupoids $E_n(\cdot)$ generated by at most two elements are constructed and their semigroup distances are described. It is proved that there are finite non-associative groupoids having their semigroup distance equal just to the arbitrary given positive integer $n$.

**Keywords:** groupoid, non-associative triple, distance of groupoids, semigroup distance.
**1991 Mathematics Subject Classification:** 20N05.

## Introduction

Groupoids having only one non-associative (ordered) triple were studied at first by G. Szász in [10] and [11], and by P. Hájek in [2] and [3]. Therefore they are called Szász-Hájek groupoids and shortly denoted as SH-groupoids. Sets of non-associative triples and semigroup distances of some finite groupoids were also investigated in [1], [4] and [5]. The structure of SH-groupoids was studied later by T. Kepka and M. Trch, and the main properties of SH-groupoids were described in [6], [7], [8] and [9] for each of possible non-associative triples. Further, it was proved in [6] that the semigroup distance of a minimal SG-groupoid having the only non-associative triple $(a, a, a)$ is equal to the number 1 or to the number 2. In [13], a minimal SH-groupoid with the semigroup distance equal to 3 is constructed. This paper is an immediate continuation of [12], [14] and [15].

## 1. Preliminaries

A groupoid $G(\cdot)$ is called *$\sigma$-stratified groupoid* if there exists a mapping $\sigma$ of the set $G$ to the set of positive integers such that $\sigma(x \cdot y) = \sigma(x) + \sigma(y)$ for each couple $x, y \in G$. The mapping $\sigma$ is called a mapping *stratifying* the underlying *set $G$*.

For each positive integer $n$, the corresponding subset $L_nG = \{x \in G | \sigma(x) = n\}$ is called *$n$-th layer* of the underlying set $G$. Of course, we have

$$G = L_1(G) \cup L_2(G) \cup \cdots \cup L_n(G) \cup L_{n+1}(G) \ldots .$$

Furthermore, if $G(\cdot)$ is a $\sigma$-stratified groupoid then each subgroupoid $H(\cdot)$ of the groupoid $G(\cdot)$ is also a $\sigma$-stratified groupoid.

If $G(\cdot)$ is a $\sigma$-stratified groupoid and $\kappa$ is a congruence on $G(\cdot)$ then the congruence $\kappa$ is called *$\sigma$-stratified congruence* if $\sigma(x) = \sigma(y)$ for each $(x, y) \in \kappa$. In this case the corresponding groupoid $G/\kappa(\cdot)$ is a $\sigma$-stratified groupoid, too.

Let $G(\cdot)$ be a $\sigma$-stratified groupoid. Further, let $k$ be an arbitrary given positive integer and, finally, let $t \in G$ be an arbitrary chosen element having $\sigma(t) \geq k$. Then there exists a groupoid $G^k(\circ)$ such that $x \circ y = x \cdot y$ for every $x, y \in G$ having $\sigma(x) + \sigma(y) < k$ and $x \circ y = t$ whenever $\sigma(x) + \sigma(y) \geq k$. The corresponding groupoid $G^k(\circ)$ is called *restriction of the $k$-th order* of the groupoid $G(\cdot)$. If the groupoid $G(\cdot)$ is finitely generated then the corresponding groupoid $G^k(\circ)$ is a finite groupoid which can be seen as a $\sigma$-stratified groupoid up to the $k$-layer.

If $G(\cdot)$ is a non-associative $\sigma$-stratified groupoid, then the set $M$ containing positive integers $\sigma(x) + \sigma(y) + \sigma(z)$ for every $x, y, z \in G$ such that $x \cdot yz \neq xy \cdot z$ is non-empty. This set contains the least element $m$ having the same property and the corresponding restriction of the $(m + 1)$-th order is a non-associative groupoid. The corresponding groupoid will be called *fundamental restriction* of the non-associative $\sigma$-stratified groupoid and it will be denoted as $\overline{G}(\cdot)$.

The simplest type of non-associative groupoids are groupoids containing just one non-associative ordered triple of elements. A groupoid $G(\cdot)$ is called an *SH-groupoid of the type* $(a, a, a)$ if there exists an element $a \in G$ such that the ordered triple $(a, a, a)$ is the only non-associative triple of the groupoid $G(\cdot)$. An SH-groupoid $G(\cdot)$ of the type $(a, a, a)$ is called *minimal SH-groupoid* if the groupoid $G(\cdot)$ is generated by the one-element set $\{a\}$.

Let $H(\cdot)$ be a subgroupoid of an SH-groupoid $G(\cdot)$ having the non-associative triple $(a, a, a)$. Then either $(a, a, a) \in H^3$ and $H(\cdot)$ is an SH-groupoid having the non-associative triple $(a, a, a)$, or $H(\cdot)$ is a semigroup in the opposite case.

Let $\kappa$ be a congruence on SH-groupoid $G(\cdot)$. If $(a, a, a)$ is the corresponding non-associative triple then either $(a \cdot aa, aa \cdot a) \in \kappa$ and then the corresponding groupoid $G(\cdot)/\kappa$ is a semigroup, or $(a \cdot aa, aa \cdot a) \notin \kappa$ and then the corresponding groupoid $G(\cdot)/\kappa$ is an SH-groupoid of the same type $(a, a, a)$.

The following main theorem concerning SH-groupoids was proved by G. Szász:

**1.1. Theorem.** *Let $G(\cdot)$ be an SH-groupoid. If $(a, b, c)$ is the only non-associative triple of $G(\cdot)$ and let $x, y \in G$ be such that $x \cdot y \in \{a, b, c\}$. Then $x \cdot y \in \{x, y\}$.*

Let $G(\diamond)$ and $G(*)$ be a couple of groupoids having the same underlying set $G$. Then $\mathrm{dist}(G(\diamond), G(*))$ denotes $\mathrm{card}\{(x,y) \in G^2 \mid x \diamond y \neq x * y\}$.

Let $G(\cdot)$ be a groupoid. Let $\mathrm{sdist}(G(\cdot))$ be the minimum of cardinal numbers $\mathrm{dist}(G(\cdot), G(*))$, where $G(*)$ runs through the set of all semigroups having the underlying set $G$. The number $\mathrm{sdist}(G(\cdot))$ is called *semigroup distance* of the groupoid $G(\cdot)$.

Let $G(\cdot)$ be a groupoid and let $G(\star)$ be a semigroup having the same underlying set $G$. If $\mathrm{sdist}(G(\cdot)) = \mathrm{dist}(G(\cdot), G(\star))$ then the semigroup $G(\star)$ will be further denoted as *nearest semigroup* of the groupoid $G(\cdot)$.

## 2. Stratified minimal SH-groupoids of the type $(a,a,a)$

In this part, let $F(\cdot)$ denote the absolutely free groupoid generated by a one-element set $\{a\}$. Let $\lambda(u)$ denote the length of the term $u$ for each element $u \in F$. It is obvious that $\lambda(x \cdot y) = \lambda(x) + \lambda(y)$ for every $x, y \in F$. Thus, the groupoid $F(\cdot)$ is a $\lambda$-stratified groupoid.

If $G(\cdot)$ is an arbitrary minimal SH-groupoid of the type $(a,a,a)$ then there exists a congruence $\kappa$ on $F(\cdot)$ such that $G(\cdot)$ is an isomorphic image of the groupoid $F/\kappa(\cdot)$. Of course, the corresponding congruence $\kappa$ satisfies the following two conditions: $(a \cdot aa, aa \cdot a) \notin \kappa$ and $(x \cdot yz, xy \cdot z) \in \kappa$ for every $x, y, z \in G$, $(x,y,z) \neq (a,a,a)$.

Suppose at first that there exists at least one stratified SH-groupoud $G(\cdot)$ of the type $(a,a,a)$. If $\sigma$ is the corresponding stratifying mapping of the underlying set $G$ then the inequality $\sigma(a \cdot a) = 2 \cdot \sigma(a) > \sigma(a)$ has to be valid.

**2.1. Construction.** Suppose that the elements $a, b, c, d, f, g$ and, further, $a^4, a^5, ..., a^n, a^{n+1}, ...$ are pairwise different.

(i) Put $V = \{a, b, c, d, a^4, a^5, \dots, a^k, a^{k+1}, \dots\}$. Define a mapping $\lambda$ of the set $V$ to the set of positive integers such that $\lambda(a) = 1$, $\lambda(b) = 2$, $\lambda(c) = 3 = \lambda(d)$ and let $\lambda(a^k) = k$ for each positive integer $k \geq 4$.

Define, further, a binary operation $x \cdot y$ on the set $V$ such that $a \cdot a = b$, $a \cdot b = c$, $b \cdot a = d$ and put $x \cdot y = a^k$ whenever $\lambda(x) + \lambda(y) = k \geq 4$. Then $V(\cdot)$ becomes a groupoid and it is easy to check that $V(\cdot)$ is a $\lambda$-stratified groupoid.

(ii) Put $W = \{a, b, c, d, f, g, a^5, a^6, \dots, a^m, a^{m+1}, \dots\}$ and define a mapping $\lambda$ of the set $W$ to the set of positive integers such that $\lambda(a) = 1$, $\lambda(b) = 2$, $\lambda(c) = 3 = \lambda(d)$, $\lambda(f) = 4 = \lambda(g)$ and let $\lambda(a^m) = m$ for each positive integer $m \geq 5$.

Define, further, a binary operation $x \cdot y$ on the set $W$ such that $a \cdot a = b$, $a \cdot b = c$, $b \cdot a = d$, $f = a \cdot c = b \cdot b = d \cdot a$, $a \cdot d = g = c \cdot a$ and, further, put $x \cdot y = a^m$ whenever $\lambda(x) + \lambda(y) = m \geq 5$. Then $W(\cdot)$ becomes a groupoid and it is easy to check that $W(\cdot)$ is again a $\lambda$-stratified groupoid.

**2.2. Lemma.** *The groupoids* $V(\cdot)$ *and* $W(\cdot)$ *are the only two different* $\lambda$*-stratified minimal SH-groupoids of the type* $(a, a, a)$*.*

**Proof.** It is obvious that both groupoids $V(\cdot)$, $W(\cdot)$ are generated by the one-element set $\{a\}$ and $a \cdot aa \neq aa \cdot a$. From the definition it follows immediately that $\lambda(x \cdot y) \neq 1$. Thus, $x \cdot y \neq a$ for every $x, y \in V$. The same is also true for every $x, y \in W$.

(i) At first consider the groupoid $V(\cdot)$. It is easy to check that $x \cdot yz = xy \cdot z$ whenever $(x, y, z) \neq (a, a, a)$. From the construction it follows that $\lambda(x \cdot yz) = \lambda(xy \cdot z) \geq 4$. But each layer $L_k(V)$ contains only one element for each $k \geq 4$. Thus, $V(\cdot)$ is a $\lambda$-stratified minimal SH-groupoid of the type $(a, a, a)$. It is easy to see that $\mathrm{sdist}(V(\cdot)) = 1$. In fact, if we put $a \star b = d \neq c = a \cdot b$ and $x \star y = x \cdot y$ in the remaining cases then the groupoid $V(\star)$ is a semigroup. Therefore the SH-groupoid $V(\cdot)$ of the type $(a, a, a)$ will be further called *minimal SH-groupoid of the first kind.*

(ii) Consider, further, the groupoid $W(\cdot)$, in which $a(a \cdot aa) = aa \cdot aa = (aa \cdot a)a$, $a(aa \cdot a) = (a \cdot aa)a$ and, thus, $f = a \cdot c = b \cdot b = d \cdot a$, $a \cdot d = g = c \cdot a$. We have $\lambda(x \cdot yz) = \lambda(xy \cdot z) \geq 5$ in the remaining cases. But each layer $L_k(W)$ contains only one element for each $k \geq 5$ and, thus, $x \cdot yz = xy \cdot z$ if $(x, y, z) \neq (a, a, a)$. Therefore, $W(\cdot)$ is the other $\lambda$-stratified minimal SH-groupoid of the type $(a, a, a)$. Now, it is easy to check that $\mathrm{sdist}(W(\cdot)) = 2$. In fact, if we put $c \star a = f \neq g = c \cdot a$, and $b \star a = c \neq d = b \cdot a$ and $x \star y = x \cdot y$ in the remaining cases then the groupoid $W(\star)$ is a semigroup. Therefore, the SH-groupoid $W(\cdot)$ of the type $(a, a, a)$ will be further called *minimal SH-groupoid of the second kind.*

(iii) Finally, let $\kappa$ be an arbitrary $\lambda$-stratified congruence on $F(\cdot)$. It follows from $\lambda(x \cdot y) > 1$ that $x \cdot y \neq a$ for every $x, y \in F/\kappa$. Therefore, $a \cdot f = a \cdot bb = a \cdot ac = b \cdot c = ba \cdot b = (ba \cdot a) \cdot a = f \cdot a$. Similarly, we get $f \cdot a = bb \cdot a = ac \cdot a = a \cdot ca = a \cdot g = g \cdot a = ad \cdot a = a \cdot da = a \cdot f$. Thus, the layer $L_5(F/\kappa)$ contains only one element $a^5$ and the rest is clear. ∎

**2.3. Lemma.** *Let* $k$ *be an arbitrary given positive integer such that* $k \geq 5$*. Consider the* $k$*-th restriction* $V^k(\cdot)$ *for the SH-groupoid* $V(\cdot)$*. Then the underlying set of the restricted groupoid* $V^k(\cdot)$ *is finite and the groupoid* $V^k(\cdot)$ *is an SH-groupoid of the type* $(a, a, a)$ *having* $\mathrm{sdist}(\mathrm{V}^k(\cdot)) = 1$*.*

**Proof.** It follows from the construction that the underlying set $V^k$ of the groupoid $V^k(\cdot)$ is finite and it contains just $k + 1$ different elements. Further, define on $V^k$ a new binary operation $\star$ such that $b \star a = c \neq d = b \cdot a$ and $x \star y = x \cdot y$ in the remaining cases. It is easy to see that $V^k(\star)$ is a semigroup and the rest is clear. ∎

**2.4. Definition.** Let $V(\cdot)$ be the minimal SH-groupoid of the first kind constructed in 2.1 and let $k = 4$. The corresponding 4-th restriction of the SH-groupoid $V(\cdot)$ of the type $(a, a, a)$ will be further called *fundamental restriction of the first kind* of the minimal SH-groupoid $V(\cdot)$.

**2.5. Example.** The following five-element groupoid $F(\cdot)$ is just the fundamental restriction of the minimal SH-groupoid $V(\cdot)$ and $\mathrm{sdist(F(\cdot))} = 1$.

| $F$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $b$ | $c$ | $e$ | $e$ | $e$ |
| $b$ | $d$ | $e$ | $e$ | $e$ | $e$ |
| $c$ | $e$ | $e$ | $e$ | $e$ | $e$ |
| $d$ | $e$ | $e$ | $e$ | $e$ | $e$ |
| $e$ | $e$ | $e$ | $e$ | $e$ | $e$ |

It is easy to check that the groupoid $F(\cdot)$ is also the fundamental restriction of the SH-groupoid $W(\cdot)$.

**2.6. Lemma.** *Let $k$ be an arbitrary given positive integer such that $k \geq 5$. Consider the $k$-th restriction $W^k(\cdot)$ for the SH-groupoid $W(\cdot)$. Then the underlying set of the restricted groupoid $W^k(\cdot)$ is finite and the groupoid $W^k(\cdot)$ is an SH-groupoid of the type $(a, a, a)$ having* $\mathrm{sdist(W^k(\cdot))} = 2$.

**Proof.** It is obvious and similar to the proof of Lemma 2.3. ∎

**2.7. Definition.** Let $W(\cdot)$ be the minimal SH-groupoid of the second kind constructed in 2.1 and let $k = 5$. Then the corresponding 5-th restriction of the SH-groupoid $W(\cdot)$ of the type $(a, a, a)$ will be further called *fundamental restriction of the second kind* of the minimal SH-groupoid $W(\cdot)$.

**2.8. Example.** The following seven-element groupoid $H(\cdot)$ is the fundamental restriction of the minimal SH-groupoid $W(\cdot)$ of the second kind and $\mathrm{sdist(H(\cdot))} = 2$.

| $G$ | $a$ | $b$ | $c$ | $d$ | $f$ | $g$ | $h$ |
|---|---|---|---|---|---|---|---|
| $a$ | $b$ | $c$ | $f$ | $g$ | $h$ | $h$ | $h$ |
| $b$ | $d$ | $f$ | $h$ | $g$ | $h$ | $h$ | $h$ |
| $c$ | $g$ | $h$ | $h$ | $h$ | $h$ | $h$ | $h$ |
| $d$ | $f$ | $h$ | $h$ | $h$ | $h$ | $h$ | $h$ |
| $f$ | $h$ | $h$ | $h$ | $h$ | $h$ | $h$ | $h$ |
| $g$ | $h$ | $h$ | $h$ | $h$ | $h$ | $h$ | $h$ |
| $h$ | $h$ | $h$ | $h$ | $h$ | $h$ | $h$ | $h$ |

## 3. Stratified SH-groupoids generated by two-element set

In this part, let $F(\cdot)$ denote an absolutely free groupoid generated by a two-element set $\{a, p\}$ and let $\lambda(u)$ denote the length of an arbitrary element $u \in G$.

**3.1. Construction.** Denote by $\kappa$ the least congruence on $F(\cdot)$ such that $a \cdot aa \neq aa \cdot a$, $aa \cdot aa = a(aa \cdot a)$ and $x \cdot yz = xy \cdot z$ for every $x, y, z \in G$ whenever $(a, a, a) \neq (x, y, z)$. Consider the groupoid $F/\kappa(\cdot)$. It is obvious that $\lambda(x \cdot y) = \lambda(x) + \lambda(y) \neq 1$ and, thus, $x \cdot y \neq a$ for every $x, y \in F$.

It follows from this that $(tu{\cdot}v){\cdot}w = tu{\cdot}vw = t{\cdot}(u{\cdot}vw)$ for every $t, u, v, w \in F$. Therefore, $x_1 \cdot x_2 \dots x_n = x_1x_2 \cdot x_3 \dots x_n = \cdots = x_1x_2 \dots x_{n-1} \cdot x_n$, for each positive integer $n \geq 5$ and $x_1, x_2, \dots x_n \in F$.

Denote, for the simplicity, the groupoid $F/\kappa(\cdot)$ as $E(\cdot)$. The groupoid $E(\cdot)$ contains a proper subgroupoid generated by the one-element set $\{a\}$. It is obvious that this is just the $\lambda$-stratified SH-groupoid $V(\cdot)$ constructed in 2.1.

Similarly, $E(\cdot)$ contains a proper subgroupoid $P(\cdot)$ generated by the one-element set $\{p\}$. It follows immediately from the construction that it is an infinite semigroup generated by the one-element set $\{p\}$. Further, it is easy to see that for each positive integer $m \geq 2$ and each $m$-element ordered set $(x_1, x_2, \dots, x_m) \in \{a, p\}^m$ such that $(a, a, \dots, a) \neq (x_1, x_2, \dots, x_m) \neq (p, p, \dots, p)$ the underlying set $E$ contains just $2^m - 2$ different elements $x_1x_2 \dots x_m$ of the length $m$. Therefore, for each positive integer $n \geq 4$ the corresponding layer $L_n(E)$ contains $2^n$ elements. Further, it obvious that $card(L_3(E)) = 9$, $card(L_2(E)) = 4$ and $card(L_1(E)) = 2$.

**3.2. Lemma.** *Let $E(\cdot)$ be the groupoid constructed in* 3.1 *Then*

(i) *$E(\cdot)$ is the $\lambda$-stratified groupoid;*

(ii) *$E(\cdot)$ is a primitive extension of the SH-groupoid $V(\cdot)$, see* [14];

(iii) *$E(\cdot)$ is an SH-groupoid of the type $(a, a, a)$;*

(iv) $\mathrm{sdist}(E(\cdot)) = 1$.

**Proof.** It follows immediately from Construction 3.1. ∎

**3.3. Construction.** Let $E(\cdot)$ be the $\lambda$-stratified SH-groupoid constructed in 3.1. For each positive integer $n \geq 2$ and $(a, a, \dots, a) \neq (x_1, x_2, \dots, x_m) \neq (p, p, \dots, p)$, consider the element $u = x_1x_2 \dots x_n$ of the length $\lambda(u) = n$. Let $\alpha(u)$ denote the number of all placements of the element $a$ in the corresponding ordered set $(x_1, x_2, \dots, x_n)$. Similarly, let $\pi(u)$ denote the number of all placements of the element $p$ in the corresponding ordered set $(x_1, x_2, \dots, x_n)$.

For arbitrary given positive integers $k, m$, define a mapping $\sigma$ of the set $E$ to the set of positive integers such that $\sigma(u) = k \cdot \alpha(u) + m \cdot \pi(u)$. Especially, we have $\sigma(a) = k$, $\sigma(b) = 2k$, $\sigma(c) = 3k = \sigma(d)$ and $\sigma(a^n) = n \cdot k$ for each $n \geq 4$. Finally, we have $\sigma(p^n) = n \cdot m$ for each positive integer $n$. It is easy to check that the mapping $\sigma$ is also a mapping stratifying the underlying set $E$. Of course, some of the corresponding $\sigma$-layers $L_n(E)$ could be empty subsets of the set $E$.

## 4. Reduced stratified SH-groupoids of type $(a, a, a)$

In this part, of the paper consider the groupoid $E(\cdot)$ constructed in 3.1. Let $n$ be an arbitrary given positive integer. Suppose, at first, that there is at least one stratified groupoid $E_n(\cdot)$ satisfying the condition $a^2 = p^n$. If $\sigma$ is a mapping stratifying the underlying set $E$ then $2 \cdot \sigma(a) = n \cdot \sigma(p)$.

**4.1. Construction.** Let $n = 2k$ where $k$ is an arbitrary positive integer and let $E(\cdot)$ be the groupoid constructed in 3.1. Put $\sigma(a) = k$, $\sigma(p) = 1$ and, further, let $\sigma(u) = k \cdot \alpha(u) + \pi(u)$ in the remaining cases. Now, consider the least congruence $\kappa$ on $E$ containing the ordered pair $(b, p^{2k})$. Then $\sigma(b) = \sigma(p^{2k})$ and, thus, $\kappa$ is a $\sigma$-stratified congruence on the groupoid $E(\cdot)$. Denote, for the simplicity, the corresponding groupoid $E/\kappa(\cdot)$ as $E_{2k}(\cdot)$.

**4.2. Construction.** Let $n = 2k + 1$ where $k$ is an arbitrary positive integer and let $E(\cdot)$ be the groupoid constructed in 3.1. Put $\sigma(a) = 2k + 1$, $\sigma(p) = 2$ and, further, let $\sigma(u) = (2k + 1) \cdot \alpha(u) + 2 \cdot \pi(u)$ in the remaining cases. Now, consider the least congruence $\kappa$ on $E(\cdot)$ containing the ordered pair $(b, p^{2k+1})$. Then $\sigma(b) = \sigma(p^{2k+1})$ and, thus, $\kappa$ is a $\sigma$-stratified congruence on the groupoid $E(\cdot)$. Denote, for the simplicity, the corresponding groupoid $E/\kappa(\cdot)$ as $E_{2k+1}(\cdot)$.

**4.3. Lemma.** *Let $n \geq 2$ and let $E_n(\cdot)$ be the groupoid constructed in* 4.1, 4.2. *Then*

(i) *$E_n(\cdot)$ is a groupoid generated by the two-element set $\{a, p\}$;*

(ii) *$E_n(\cdot)$ is a $\sigma$-stratified SH-groupoid containing the non-associative triple $(a, a, a)$;*

(iii) *the equation $x \cdot y = b$ is solvable in $E_n(\cdot)$ and it has just $n$ different solutions;*

(iv) *the equation $x \cdot y = c$ is solvable in $E_n(\cdot)$ and it has just $n$ different solutions;*

(v) *the equation $x \cdot y = d$ is solvable in $E_n(\cdot)$ and it has just $n$ different solutions.*

**Proof.** (i) It is obvious that $c = a \cdot b = a \cdot aa \neq aa \cdot a = b \cdot a = d$. Further, it follows from the construction that $x \cdot yz = xy \cdot z$ whenever $x, y, z \in H_n$ are such that $(a, a, a) \neq (x, y, z)$.

(ii) The mapping $\sigma$ is a mapping stratifying the underlying set $E_n$. Thus, $E_n(\cdot)$ is a $\sigma$-stratified SH-groupoid of the type $(a, a, a)$.

(iii) It is obvious that $b = a \cdot a = p \cdot p^{n-1} = p^2 \cdot p^{n-2} = \cdots = p^{n-1} \cdot p$. Thus, the equation $x \cdot y = b$ has just only $n$ solutions $(a, a)$,$(p, p^{n-1})$, $\ldots$,$(p^{n-1}, p)$.

(iv) Similarly, $c = a \cdot b = ap \cdot p^{n-1} = ap^2 \cdot p^{n-2} = \cdots = ap^{n-1} \cdot p$. Thus, the equation $x \cdot y = c$ has just only $n$ solutions $(a, b)$,$(ap, p^{n-1})$, $\ldots$,$(ap^{n-1}, p)$.

(v) It is obvious that $d = b \cdot a = p \cdot p^{n-1}a = p^2 \cdot p^{n-2}a = \cdots = p^{n-1} \cdot pa$. The equation $x \cdot y = d$ has just only $n$ solutions $(b, a)$,$(p, p^{n-1}a)$, $\ldots$,$(p^{n-1}, pa)$. ∎

**4.4. Lemma.** *Let $n \geq 2$ be a positive integer. Then* $\operatorname{sdist}(E_n(\cdot)) \leq n$.

**Proof.** Define on the set $E_n$ a new binary operation $\circ$ such that $b \circ a = c \neq d = b \cdot a$ and $p^k \circ p^{n-k}a = c \neq d = p^k \cdot p^{n-k}a$ for each $1 \leq k < n$ and, further, let $x \circ y = x \cdot y$ in the remaining cases. Then $a \circ (a \circ a) = (a \circ a) \circ a$. It is possible to check that $x \circ (y \circ z) = (x \circ y) \circ z$ whenever $x, y, z \in E_n$ are such that $(x, y, z) \neq (a, a, a)$. Thus, $E_n(\circ)$ is a semigroup and the rest is clear. ∎

**4.5. Lemma.** *Let* $E_n(\cdot)$ *be the groupoid constructed in* 4.1, 4.2. *Further, let* $E_n(\star)$ *be an arbitrary semigroup having* $\mathrm{sdist}(\mathrm{E_n}(\cdot)) = \mathrm{dist}(E_n(\cdot)), (E_n(\star))$. *Then*

(i) *either* $d = a \star b \neq a \cdot b = c$, *or* $c = b \star a \neq b \cdot a = d$;

(ii) *if* $b \star a \neq b \cdot a$ *and* $x \cdot y = b$, *then* $c = b \star a = x \star (y \star a) \neq x \cdot (y \cdot a)$;

(iii) *if* $a \star b \neq a \cdot b$ *and* $x \cdot y = b$, *then* $d = a \star b = (a \star x) \star y \neq (a \cdot x) \cdot y$.

**Proof.** (i) The subgroupoid $V(\cdot)$ generated by the one-element set $\{a\}$ is a minimal SH-groupoid of the type $(a, a, a)$ and the equation $x \cdot y = b$ is solvable in $V(\cdot)$ only if $(x, y) = (a, a)$. Therefore, either $d = a \star b \neq a \cdot b = c$, or $c = b \star a \neq b \cdot a = d$.

(ii) Now, if $c = b \star a$ then also $c = p^k \star (p^{n-k} \star a) \neq p^k \cdot (p^{n-k} \cdot a) = d$ for each $1 \leq k < n$. Therefore, $\mathrm{dist}(E_n(\star), E_n(\cdot)) \geq n$ and the rest is clear.

(iii) The other case $d = a \star b$ is similar to (ii). ∎

**4.6. Lemma.** *Let* $E_n(\cdot)$ *be the groupoid constructed in* 4.1, 4.2. *Then*

$$\mathrm{sdist}(\mathrm{E_n}(\cdot)) = n.$$

**Proof.** It follows from proof of the previous lemma. ∎

## 5. Fundamental restrictions of reduced stratified SH-groupoids

In this part, consider the reduced stratified SH-groupoids $E_n(\cdot)$ constructed in 4.1 and 4.2. For each positive integer $m$ there are finite groupoids $F_{(n,m)}(\cdot)$ constructed as a $m$-th restriction of the reduced stratified SH-groupoid $E_n(\cdot)$.

Put, further, $m = 3 \cdot \sigma(a) + 1$ and consider the fundamental restriction of the reduced $\sigma$-stratified SH-groupoid $E_n(\cdot)$.

Especially, for each even positive integer $n = 2k$ let $\sigma$ be a mapping stratifying the underlying set $E_{2k}$ such that $\sigma(a) = k$ and $\sigma(p) = 1$. Then we have $m = 3k + 1$. Denote the corresponding fundamental restriction of the SH-groupoid $E_{2k}(\cdot)$ as $\overline{E}_{2k}(\cdot)$.

Similarly, for an arbitrary given odd positive integer $n = 2k + 1$, $n \geq 3$, let $\sigma$ be a mapping stratifying the underlying set $E_{2k+1}$ such that $\sigma(a) = 2k + 1$ and $\sigma(p) = 2$. Then $m = 6k + 4$. Denote the corresponding fundamental restriction of the SH-groupoid $E_{2k+1}(\cdot)$ as $\overline{E}_{2k+1}(\cdot)$.

It is obvious that the fundamental restrictions $\overline{E}_{2k}(\cdot)$ and $\overline{E}_{2k+1}(\cdot)$ are finite non-associative SH-groupoids. In this section, it will be proved that

$$\mathrm{sdist}(\overline{\mathrm{E}}_{\mathrm{2k}}(\cdot)) = 2k \text{ and } \mathrm{sdist}(\overline{\mathrm{E}}_{\mathrm{2k+1}}(\cdot)) = 2k + 1.$$

**5.1 Lemma.** *For each positive integer* $n \geq 2$ *we have* $\mathrm{sdist}(\overline{\mathrm{E}}_{\mathrm{n}}(\cdot)) \leq n$.

**Proof.** The proof is similar to the proof of Lemma 4.4. ∎

**5.2 Proposition.** *For each positive integer $k$ we have* $\mathrm{sdist}(\overline{\mathrm{E}}_{2k+1}(\cdot)) = 2k+1$.

**Proof.** Consider the fundamental restriction $\overline{E}_{2k+1}(\cdot)$ and suppose that there exists a semigroup $\overline{E}_{2k+1}(\nabla)$ having $\mathrm{dist}(\overline{\mathrm{E}}_{2k+1}(\cdot), \overline{\mathrm{E}}_{2k+1}(\nabla)) < 2k+1$. In this case we have $a^2 = p^{2k+1}$ and the corresponding stratifying mapping $\sigma$ satisfies the condition $\sigma(a) = 2k+1$ and $\sigma(p) = 2$.

Denote by $D_p$ the set describing "different products" and containing just only all ordered pairs $(x, y)$ satisfying the condition $x \cdot y \neq x \star y$. An unknown semigroup $\overline{E}_{2k+1}(\nabla)$ will be investigated step by step. For each reconstructing step denote by $K_p$ the set of so far known pairs $(x, y)$ having $x \cdot y \neq x \star y$ and let $U_p$ denote the set of the remaining (still unknown) ordered pairs from the set $D_p$.

It is obvious that at least one of the following three inequalities has to be valid: $a\nabla a \neq a \cdot a$, $a\nabla b \neq a \cdot b$, $b\nabla a \neq b \cdot a$. Thus, the set $K_p$ contains at least one element at the starting step. Therefore, the set $U_p$ can contain at most $2k-1$ elements at the same time.

Finally, consider the following $2k$-element set $I_u$ of "further investigated" ordered pairs $(p, u), (p^2, u), \ldots, (p^k, u), (u, p^k), (u, p^{k-1}), \ldots, (u, p)$ for $u \in U_p$. It is obvious that there is at least one positive integer $1 \leq i \leq k$ such that $p^i\nabla u = p^i \cdot u$ or $u\nabla p^i = u \cdot p^i$.

(i) In the first part of the proof suppose that there exists an element $u \in \overline{E}_{2k+1}$ such that $u = a\nabla a \neq b = a \cdot a$. Let, for example, $u\nabla p^i = u \cdot p^i$. If $a\nabla p^i = a \cdot p^i = r_i$ and $a\nabla r_i = a \cdot r_i$ then $\sigma(u\nabla p^i) = \sigma(u) + \sigma(p^i) = 2i + \sigma(u)$. But we also have $\sigma(a\nabla a\nabla p^i) = \sigma(a) + \sigma(a\nabla p^i) = \sigma(a) + \sigma(a \cdot p^i) = 2\sigma(a) + \sigma(p^i) = 2\sigma(a) + 2i$. It follows from this that $\sigma(u) = 2\sigma(a)$. But the corresponding layer contains only one element and this is just only the element $b = a \cdot a$, a contradiction. Therefore, at least one of the following two conditions has to be valid: $a\nabla p^i \neq a \cdot p^i = r_i$ or $a\nabla r_i \neq a \cdot r_i$. Thus, after this step, the set $K_p$ contains at least two different elements and the set $U_p$ can contain at most $2k-2$ elements, now.

It follows from this that $p^i\nabla u = p^i \cdot u$ or in the set $I_u$ there has to be another positive integer $j$ that $p^j\nabla u = p^j \cdot u$ or $u\nabla p^j = u \cdot p^j$. Suppose, for example, that $p^j\nabla u = p^j \cdot u$. Now, if $p^j\nabla a = p^j \cdot a = t_j$ and $t_j\nabla a = t_j \cdot a$ then we obtain again $\sigma(p^j\nabla u) = \sigma(u) + \sigma(p^j) = 2j + \sigma(u)$.

But we also have $\sigma(p^j\nabla a\nabla a) = \sigma(p^j \cdot a\nabla a) = \sigma(t_j\nabla a) = \sigma(t_j \cdot a) = \sigma(p^j \cdot b)$ $= 2j + 2\sigma(a)$. Therefore, we get $\sigma(u) = \sigma(b)$. But the corresponding layer contains only one element and, thus, $u = b$, a contradiction. It follows from this again that $p^j\nabla a \neq p^j \cdot a = t_j$ or $t_j\nabla a \neq t_j \cdot a$. That means, that after this step the set $K_p$ contains at least three different elements and the set $U_p$ can contain at most $2k-3$ elements, now.

It is possible to continue in this way till the set $U_p$ is empty. But then the set $I_u$ still contains another positive integer $m$ such that it is valid $u\nabla p^m = u \cdot p^m$ or $p^m\nabla u = p^m \cdot u$. In this case we get $a\nabla p^m = a \cdot p^m = r_m$, $a\nabla r_m = a \cdot r_m$ and, also, $p^m\nabla a = p^m \cdot a = t_m$, $t_m\nabla a = t_m \cdot a$. It is easy to see that from this we get a contradiction. Thus, $a\nabla a = a \cdot a$ in any case.

(ii) In this part of the proof suppose, that there exists an element $v \in \overline{E}_{2k+1}$ such that $v = a\nabla b \neq c = a \cdot b$ and let $a\nabla a = a \cdot a$. Of course, the set $D_p$ contains at most $2k$ elements and the set $U_p$ can contain at most $2k-1$ elements, now. Consider the following set $I_p = \{(p, p^{2k}), (p^2, p^{2k-1}), \ldots, (p^{2k}, p)\}$. It is obvious that there exists a positive integer $i$ such that $p^i \nabla p^{2k+1-i} = p^{2k+1} = b = a \cdot a$. But then $v = a\nabla b = a\nabla p^{2k+1} = a\nabla p^i \nabla p^{2k+1-i}$.

Further, if $a\nabla p^i = a \cdot p^i = q_i$ and $q_i \nabla p^{2k+1-i} = q_i \cdot p^{2k+1-i}$, then $v = (a\nabla p^i)\nabla p^{2k+1-i} = q_i \cdot p^{2k+1-i} = (ap^i) \cdot p^{2k+1-i} = a \cdot p^{2k+1} = a \cdot b = c$. This is a contradiction. Thus, at least one of the following two inequalities has to be valid: $a\nabla p^i \neq a \cdot p^i$ or $q_i \nabla p^{2k+1-i} \neq q_i \cdot p^{2k+1-i}$. Therefore, the set $K_p$ contains at least two different elements and the set $U_p$ can contain at most $2k-2$ elements, now. It follows from this that the set $I_u$ contains another positive integer $j \neq m$ such that again $p^j \nabla p^{2k+1-j} = p^{2k+1} = b = a \cdot a$. Now, similarly as above, both equalities $a\nabla p^j = a \cdot p^j = q_j$ and $q_j \nabla p^{2k+1-j} = q_j \cdot p^{2k+1-j}$ cannot be valid at the same time. The process of the reconstruction of the expected semigroup can continue step by step in this way till the time when the set $U_p$ is empty. But, the set $I_u$ contains again another positive integer $m$ such that $p^m \nabla p^{2k+1-m} = p^{2k+1} = b = a \cdot a$. It is obvious that we get immediately another contradiction. Thus, $a\nabla b = a \cdot b$.

(iii) Finally, it is clear that the case $w = b\nabla a \neq d = b \cdot a$ and $a\nabla a = a \cdot a$ is similar to (ii). Thus, there is no semigroup $\overline{E}_{2k+1}(\nabla))$ having $\mathrm{dist}(\overline{\mathrm{E}}_{2\mathrm{k}+1}(\cdot), \overline{\mathrm{E}}_{2\mathrm{k}+1}(\nabla)) < 2k+1$ and the proof is finished. ■

**5.3 Proposition.** *For each positive integer $k$ we have* $\mathrm{sdist}(\overline{\mathrm{E}}_{2\mathrm{k}}(\cdot)) = 2k$.

**Proof.** Consider the fundamental restriction $\overline{E}_{2k}(\cdot)$ and suppose that there exists a semigroup $\overline{E}_{2k}(\nabla)$ having $\mathrm{dist}(\overline{\mathrm{E}}_{2\mathrm{k}}(\cdot), \overline{\mathrm{E}}_{2\mathrm{k}}(\nabla)) < 2k$. In this case we get $a^2 = p^{2k}$ and, thus, the corresponding stratifying mapping $\sigma$ satisfies the condition $\sigma(a) = k$ and $\sigma(p) = 1$.

Denote by $D_p$ the set containing just only all ordered pairs $(x, y)$ satisfying the condition $x \cdot y \neq x\nabla y$. An unknown semigroup $\overline{E}_{2k}(\nabla)$ will be investigated step by step during the proof. For each reconstructing step denote by $K_p$ the set of all so far known pairs $(x, y)$ having $x \cdot y \neq x\nabla y$ and denote by $U_p$ the set of the remaining (still unknown) ordered pairs from the set $D_p$.

It is obvious that at least one of the following three inequalities has to be valid: $a\nabla a \neq a \cdot a$, $a\nabla b \neq a \cdot b$, $b\nabla a \neq b \cdot a$. The set $K_p$ contains at least one element in the beginning and, thus, the set $U_p$ can contain at most $2k-2$ elements at the same time.

(i) In the first part of the proof suppose that there exists an element $u \in \overline{E}_{2k}$ such that $u = a\nabla a \neq b = a \cdot a$ and $\sigma(u) \neq 2\sigma(a)$. Now consider the $2k$-element set $I_u$ containing the ordered pairs $(p, u), (p^2, u), \ldots, (p^k, u), (u, p^k), (u, p^{k-1}), \ldots, (u, p)$. It is obvious that there has to be a positive integer $1 \leq i \leq k$ such that $p^i \nabla u = p^i \cdot u$ or $u\nabla p^i = u \cdot p^i$. Let, for example, $p^i \nabla u = p^i \cdot u$. Further, suppose that $p^i \nabla a = p^i \cdot a = t_i$ and $t_i \nabla a = t_i \cdot a$. Then $\sigma(p^i \nabla u) = \sigma(p^i \cdot u) = i + \sigma(u) \neq i + 2\sigma(a)$. But we also have $\sigma(p^i \nabla u) = \sigma((p^i \nabla a)\nabla a) = \sigma((p^i \cdot a)\nabla a) = \sigma(t_i \nabla a) = \sigma(t_i \cdot a) = \sigma(p^i \cdot (aa)) = i + 2\sigma(a)$, which is a contradiction. Thus, at least one of the following two conditions $p^i \nabla a \neq p^i \cdot a = t_i$, $t_i \nabla a \neq t_i \cdot a$ holds. Now the set $K_p$ contains at least two elements and, thus, the set $U_p$ can contain at most $2k-3$ elements. It is easy to see that $u\nabla p^i = u \cdot p^i$ or there is another positive integer

$j \neq i$ such that $u\nabla p^j = u \cdot p^j$ or $p^j \nabla u = p^j \cdot u$. Let, for example, $u\nabla p^j = u \cdot p^j$. Similarly as above, it is easy to check that $a\nabla p^j \neq a \cdot p^j = r_j$ or $a\nabla r_j \neq a \cdot r_j$.

Of course, it is possible to continue in this way till the time when the set $U_p$ is empty. In this case there exists again another positive integer $m$ such that $p^m\nabla u = p^m \cdot u$ or $u\nabla p^m = u \cdot p^m$. But now we get $p^m\nabla a \neq p^m \cdot a = t_m$, $t_m\nabla a \neq t_m \cdot a$ and $a\nabla p^j = a \cdot p^j = r_j$, $a\nabla r_j = a \cdot r_j$. It is clear that from this we get a contradiction again. Thus, we have either $\sigma(u) = 2\sigma(a)$ or $a\nabla a = a \cdot a$.

(ii) In this part of the proof suppose that there is an element $u = a\nabla a \neq a \cdot a$ having $\sigma(u) = 2\sigma(a)$. Then there exists such a positive integer $m$ that $u = p^m a p^{k-m}$. The set $D_p$ contains at most $2k-1$ elements including the ordered pair $(a, a)$. The set $K_p$ contains now one element and, thus, the set $U_p$ can contain at most $2k-2$ other elements.

The set of the ordered pairs $(p, u), (p^2, u), \dots, (p^k, u), (u, p^k), \dots, (u, p)$ contains at least one ordered pair $(p^i, u)$ or $(u, p^i)$ such that $p^i\nabla u = p^i \cdot u$ or $u\nabla p^i = u \cdot p^i$. Let, for example, $p^i\nabla u = p^i \cdot u$ and suppose, furthermore, that $p^i\nabla a = p^i \cdot a = t_i$ and $t_i\nabla a = t_i \cdot a$. Then we get $p^i\nabla u = p^i\nabla(a\nabla a) = (p^i\nabla a)\nabla a = p^i a \cdot a = p^i \cdot aa = p^i b = p^i \cdot p^{2k}$. But we also have $p^i\nabla u = p^i \cdot p^m a p^{k-m}$ and $p^{2k+i} \neq p^{i+m} a p^{k-m}$ in $\overline{E}_{2k}$. Thus, at least one of the following inequalities has to be valid: $p^i\nabla a \neq p^i a = t_i$ or $t_i\nabla a \neq t_i \cdot a$. Then the set $K_p$ contains at least two different elements and, thus, the set $U_p$ can contain at most $2k-3$ elements.

Therefore $u\nabla p^i = u \cdot p^i$ or there has to be another positive integer $j \neq i$ such that $u\nabla p^j = u \cdot p^j$ or $p^j\nabla u = p^j \cdot u$. Let, for example, $u\nabla p^j = up^j$. Now, suppose, furthermore, that $a\nabla p^j = ap^j = r_j$ and $a\nabla r_j = a \cdot r_j = a \cdot ap^j$. It is easy to check that, similarly as above, we get $a\nabla p^j \neq ap^j$ or $a\nabla r_j \neq a \cdot r_j$. It is possible to continue in this way till the set $U_p$ is empty and, then, we obtain again a contradiction. It is proved now that $a\nabla a = b = a \cdot a$.

(iii) This part of the proof is similar to (ii) of the proof of the previous proposition. Let $b = a\nabla a = a \cdot a = p^{2k}$ and put $v = a\nabla b \neq a \cdot b = c$. Then $a\nabla b = a\nabla a\nabla a = b\nabla a$. The set $K_p$ contains one element and, thus, the set $U_p$ can contain at most $2k-2$ elements. Now, consider $2k-1$ of the following ordered pairs $(p, p^{2k-1}), (p^2, p^{2k-2}), \dots, (p^{2k-1}, p)$. It is obvious that there is a positive integer $i$ such that $p^i\nabla p^{2k-i} = p^i \cdot p^{2k-i} = p^{2k} = b$. But then we get $v = a\nabla p^{2k} = a\nabla(p^i\nabla p^{2k-i})$.

Now, suppose, that $a\nabla p^i = ap^i = r_i$ and $r_i\nabla p^{2k-i} = r_i \cdot p^{2k-i} = ap^i \cdot p^{2k-i} = a \cdot p^{2k} = a \cdot b = c$. In this case we get $v = a\nabla p^{2k} = a\nabla(p^i\nabla p^{2k-i}) = (ap^i)\nabla p^{2k-i} = r_i \cdot p^{2k-i} = (ap^i) \cdot p^{2k-i} = a \cdot (p^i p^{2k-i}) = a \cdot p^{2k} = a \cdot b = c$, a contradiction. Thus $a\nabla p^i \neq ap^i$ or $r_i\nabla p^{2k-i} \neq r_i \cdot p^{2k-i}$. But then the set $K_p$ contains at least two different elements and, thus, the set $U_p$ can contain at most $2k-3$ elements. Therefore, there exists another positive integer $j \neq i$ such that $p^j\nabla p^{2k-j} = p^j \cdot p^{2k-j} = p^{2k} = b$. It is obvious that this process can continue again in the same way till the the set $U_p$ is empty. But then there has to be another positive integer $m$ such that $p^m\nabla p^{2k-m} = p^m \cdot p^{2k-m} = p^{2k} = b$ and we again obtain a contradiction.

(iv) In the remaining part let $b = a\nabla a = a \cdot a = p^{2k}$ and put $w = b\nabla a \neq b \cdot a = d$. It is clear that this case is similar to (iii). We can again, step by step, obtain a system of contradictions and, finally, it is clear that there is no semigroup having the corresponding property.

It follows immediately from this that $\mathrm{sdist}(\overline{\mathrm{E}}_{2\mathrm{k}}(\cdot)) \geq 2k$ and the rest is clear. ∎

**5.4 Theorem.** *For an arbitrary given positive integer $n$ there exists a finite non-associative groupoid $G(\cdot)$ generated by the two-element set $\{a, p\}$ satisfying the condition $a \neq x \cdot y \neq p$ and having* $\mathrm{sdist}(\mathrm{G}(\cdot)) = n$.

**Proof.** It follows immediately from the construction and from 5.2 and 5.3. ∎

**5.5 Example.** It is easy to check that the following groupoid $H(\cdot)$ is just the least minimal SH-groupoid of the type $(a, a, a)$ and of the second kind having $\mathrm{sdist}(\mathrm{H}(\cdot)) = 2$, see Lemma 2.2.

| $H$ | $a$ | $b$ | $c$ | $d$ | $f$ | $g$ |
|---|---|---|---|---|---|---|
| $a$ | $b$ | $c$ | $f$ | $g$ | $f$ | $f$ |
| $b$ | $d$ | $f$ | $f$ | $f$ | $f$ | $f$ |
| $c$ | $g$ | $f$ | $f$ | $f$ | $f$ | $f$ |
| $d$ | $f$ | $f$ | $f$ | $f$ | $f$ | $f$ |
| $f$ | $f$ | $f$ | $f$ | $f$ | $f$ | $f$ |
| $g$ | $f$ | $f$ | $f$ | $f$ | $f$ | $f$ |

**5.6 Example.** The following table is just the table of the 11-element SH-groupoid $\overline{E}_2(\cdot)$ having $\mathrm{sdist}(\overline{\mathrm{E}}_2(\cdot)) = 2$.

| 2 | $a$ | $p$ | $b$ | $r$ | $t$ | $c$ | $d$ | $u$ | $v$ | $w$ | $e$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $a$ | $b$ | $r$ | $c$ | $u$ | $v$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ |
| $p$ | $t$ | $b$ | $u$ | $w$ | $d$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ |
| $b$ | $d$ | $u$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ |
| $r$ | $v$ | $c$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ |
| $t$ | $u$ | $w$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ |
| $c$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ |
| $d$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ |
| $u$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ |
| $v$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ |
| $w$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ |
| $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ |

**5.7 Remark.** The minimal SH-groupoid $G(\cdot)$ constructed in 2.8 and having the same semigroup distance has a seven-element underlying set. It follows from 5.5 that there are finite non-associative groupoids with the semigroup distance equal to 2 and having smaller underlying set. Does there exist a finite non-associative groupoid having at most five-element underlying set and having its semigroup distance equal to 2?

**5.8 Example.** The following table is just the table of the 16-element SH-groupoid $\overline{E}_3(\cdot)$ having $\mathrm{sdist}(\overline{\mathrm{E}}_3(\cdot)) = 3$.

| 3 | p | a | f | g | h | b | q | r | s | t | u | c | d | v | w | e |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p | f | h | b | r | q | t | v | w | d | e | e | e | e | e | e | e |
| a | g | b | q | t | u | c | e | e | e | e | e | e | e | e | e | e |
| f | b | s | t | w | d | e | e | e | e | e | e | e | e | e | e | e |
| g | q | u | c | e | e | e | e | e | e | e | e | e | e | e | e | e |
| h | r | t | t | e | e | e | e | e | e | e | e | e | e | e | e | e |
| b | t | d | e | e | e | e | e | e | e | e | e | e | e | e | e | e |
| q | c | e | e | e | e | e | e | e | e | e | e | e | e | e | e | e |
| r | v | e | e | e | e | e | e | e | e | e | e | e | e | e | e | e |
| s | w | e | e | e | e | e | e | e | e | e | e | e | e | e | e | e |
| t | e | e | e | e | e | e | e | e | e | e | e | e | e | e | e | e |
| u | e | e | e | e | e | e | e | e | e | e | e | e | e | e | e | e |
| c | e | e | e | e | e | e | e | e | e | e | e | e | e | e | e | e |
| d | e | e | e | e | e | e | e | e | e | e | e | e | e | e | e | e |
| v | e | e | e | e | e | e | e | e | e | e | e | e | e | e | e | e |
| w | e | e | e | e | e | e | e | e | e | e | e | e | e | e | e | e |
| e | e | e | e | e | e | e | e | e | e | e | e | e | e | e | e | e |

## 6. Comments and open problems

**6.1.** It was proved in 2.5, 2.6 and 2.8 that there are finite non-associative groupoids having their semigroup distance equal to 1 resp. to 2. It follows from 5.5 that the construction using a fundamental restriction of stratified SH-groupoids $E_n(\cdot)$ is not minimal with respect to the number of elements contained in the underlying set $\overline{E}_n$.

**6.2.** Examples 5.7 and 5.9 show that the number of elements of the underlying set $\overline{E}_n$ quickly grows up with the increasing positive integer $n$.

**6.3.** The construction of finite non-associative groupoids $\overline{E}_n(\cdot)$ is based on the use of primitive extensions of the minimal SH-groupoids $V(\cdot)$ (see 2.1) having the semigroup distance equal to 1. It is possible to use also primitive extensions of the minimal SH-groupoid $W(\cdot)$ (see 2.1) and having semigroup distance equal to 2. It seems to be clear that a lesser number of elements in the underlying set can bring a bigger semigroup distance in this case. But, in that case, the proofs should be more complicated and the result should be the same.

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Accepted: 01.07.2014