BOUNDS FOR THE EIGENVALUES OF MATRICES

Limin Zou[1]

Youyi Jiang

School of Mathematics and Statistics
Chongqing Three Gorges University
Chongqing, 404100
P.R. China

Abstract. In this paper, we prove that all the eigenvalues of arbitrarily complex matrix lies in a closed disk with the radius involving the sum of the squares of the absolute values of the eigenvalues. So, any known upper bound on this sum of squares yields a "new" eigenvalues inclusion region. As applications, some existing results are obtained or improved.

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1. Introduction

Let M be an $n \times n$ complex matrix and λ_i $(i = 1, ..., n)$ be the eigenvalues of M. We denote by $||M||_F$, M^* and trM the Frobenius norm, conjugate transpose and trace of M, respectively. Let $i \in \mathbb{N}$ and $R_i(M) = \sum\limits_{j=1, j\neq i}^{n} |m_{ij}|$ be the sum of the absolute values of the non-diagonal entries in the ith row of $M = [m_{ij}]_{n\times n}$. For $z = a + bi \in \mathbb{C}$, the conjugate of z is denoted by $\bar{z} = a - bi$.

Estimation of eigenvalues has been, and still is, a hot topic of matrix analysis. There are many research papers published in a variety of journals each year, and different approaches have been taken for different purposes. Here, we briefly review three types of estimation methods for eigenvalues of matrices.

The well-known upper bound for $\sum\limits_{i=1}^{n} |\lambda_i|^2$ was presented by Schur in 1909 as follows:

[1]Corresponding author. E-mail address: limin-zou@163.com (L. Zou)

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \|M\|_F^2, \tag{1.1}$$

which is the first bound for $\sum_{i=1}^{n} |\lambda_i|^2$.

The famous Geršgorin disk theorem says that every eigenvalue of M lies in at least one of the Geršgorin discs centered at m_{ii} with radius $R_i(M)$; i.e.,

$$\lambda_i \in \bigcup_{i=1}^{n} \{z \in \mathbb{C} : |z - m_{ii}| \leq R_i(M)\}, \quad i = 1, ..., n.$$

The Geršgorin disks are a particular class of easily computed regions in the plane that are guaranteed to include the eigenvalues of a given matrix. Many authors, perhaps attracted by the geometrical elegance of the Geršgorin theory, have generalized the ideas and methods of this theory to obtain other types of inclusion [1, Chapter 6].

In 1994, Gu [2, Theorem 1] proved that every eigenvalue of M lies in the following disk centered at $\frac{trM}{n}$:

$$\left\{ z \in \mathbb{C} : \left| z - \frac{trM}{n} \right| \leq \sqrt{\frac{n-1}{n}\left(\|M\|_F^2 - \frac{|trM|^2}{n} \right)} \right\}. \tag{1.2}$$

Bhatia [3, p.24] pointed that results such as (1.2) are interesting because they give some information about the location of the eigenvalues of a matrix in terms of more easily computable function like the Frobenius norm and the trace.

In this paper, we prove that all the eigenvalues of M lies in the following disk centered at $\frac{trM}{n}$:

$$\left\{ z \in \mathbb{C} : \left| z - \frac{trM}{n} \right| \leq \sqrt{\frac{n-1}{n}\left(\sum_{i=1}^{n} |\lambda_i|^2 - \frac{|trM|^2}{n} \right)} \right\}. \tag{1.3}$$

The radius involves $\sum_{i=1}^{n} |\lambda_i|^2$, thus (1.3) gives the relationship between the first and third types of estimation methods. Any known upper bound for $\sum_{i=1}^{n} |\lambda_i|^2$ yields a "new" estimation of eigenvalues such as (1.2). As applications, some existing estimations are directly obtained by (1.3).

2. Main result

Theorem 2.1. *Let M be an $n \times n$ complex matrix. Then all the eigenvalues of M are located in the following disk:*

$$\left\{ z \in \mathbb{C} : \left| z - \frac{trM}{n} \right| \leq \sqrt{\frac{n-1}{n} \left(\sum_{i=1}^{n} |\lambda_i|^2 - \frac{|trM|^2}{n} \right)} \right\}.$$

Proof. We show that λ_1 is located in this disk, and similarly for others. By the Cauchy–Schwarz inequality, we have

$$\left| \sum_{i=2}^{n} \lambda_i \right|^2 \leq \sum_{i=2}^{n} 1^2 \cdot \sum_{i=2}^{n} |\lambda_i|^2 \leq (n-1) \sum_{i=2}^{n} |\lambda_i|^2.$$

Then

$$(2.1) \qquad \begin{aligned} (n-1)\,|\lambda_1|^2 + \left| \sum_{i=2}^{n} \lambda_i \right|^2 &\leq (n-1)\,|\lambda_1|^2 + (n-1) \sum_{i=2}^{n} |\lambda_i|^2 \\ &= (n-1) \sum_{i=1}^{n} |\lambda_i|^2. \end{aligned}$$

Since

$$\left| \sum_{i=2}^{n} \lambda_i \right|^2 = \left| \sum_{i=1}^{n} \lambda_i \right|^2 - \lambda_1 \sum_{i=1}^{n} \overline{\lambda_i} - \overline{\lambda_1} \sum_{i=1}^{n} \lambda_i + |\lambda_1|^2,$$

by (2.1), we have

$$n\,|\lambda_1|^2 - \lambda_1 \sum_{i=1}^{n} \overline{\lambda_i} - \overline{\lambda_1} \sum_{i=1}^{n} \lambda_i + \left| \sum_{i=1}^{n} \lambda_i \right|^2 \leq (n-1) \sum_{i=1}^{n} |\lambda_i|^2,$$

which implies

$$n\,|\lambda_1|^2 - \lambda_1 \sum_{i=1}^{n} \overline{\lambda_i} - \overline{\lambda_1} \sum_{i=1}^{n} \lambda_i + \frac{1}{n} \left| \sum_{i=1}^{n} \lambda_i \right|^2 \leq (n-1) \sum_{i=1}^{n} |\lambda_i|^2 - \frac{n-1}{n} \left| \sum_{i=1}^{n} \lambda_i \right|^2,$$

and then

$$|\lambda_1|^2 - \frac{\lambda_1}{n} \sum_{i=1}^{n} \overline{\lambda_i} - \frac{\overline{\lambda_1}}{n} \sum_{i=1}^{n} \lambda_i + \frac{1}{n^2} \left| \sum_{i=1}^{n} \lambda_i \right|^2 \leq \frac{n-1}{n} \sum_{i=1}^{n} |\lambda_i|^2 - \frac{n-1}{n^2} \left| \sum_{i=1}^{n} \lambda_i \right|^2.$$

Consequently,

$$\left(\lambda_1 - \frac{1}{n}\sum_{i=1}^{n}\lambda_i\right)\left(\overline{\lambda_1} - \frac{1}{n}\sum_{i=1}^{n}\overline{\lambda_i}\right) \le \frac{n-1}{n}\left(\sum_{i=1}^{n}|\lambda_i|^2 - \frac{1}{n}\left|\sum_{i=1}^{n}\lambda_i\right|^2\right),$$

which is equivalent to

$$\left|\lambda_1 - \frac{trM}{n}\right| \le \sqrt{\frac{n-1}{n}\left(\sum_{i=1}^{n}|\lambda_i|^2 - \frac{|trM|^2}{n}\right)}.$$

The proof is completed. ■

3. Applications

In section, we give some applications of our result.

Corollary 3.1. [2, Theorem 1]. *Let M be an $n \times n$ complex matrix. Then all the eigenvalues of M are located in the following disk:*

$$\left\{z \in \mathbb{C} : \left|z - \frac{trM}{n}\right| \le \sqrt{\frac{n-1}{n}\left(\|M\|_F^2 - \frac{|trM|^2}{n}\right)}\right\}.$$

This result is a corollary of Theorem 2.1 and the Schur's inequality (1.1).

Let M be an $n \times n$ complex matrix partitioned as

$$M = \begin{bmatrix} A_{k\times k} & B_{k\times(n-k)} \\ C_{(n-k)\times k} & D_{(n-k)\times(n-k)} \end{bmatrix}, \quad 1 \le k \le n-1,$$

where $A_{k\times k}$ is the principal submatrix of M. Tu [4, Theorem 1] obtained the following result:

$$\sum_{i=1}^{n}|\lambda_i|^2 \le \|M\|_F^2 - \max_{1\le k\le n-1}\left(\left\|B_{k\times(n-k)}\right\| - \left\|C_{(n-k)\times k}\right\|\right)^2. \tag{3.1}$$

The following result obtained by Zou and Jiang is a corollary of Theorem 2.1 and (3.1).

Corollary 3.2. [5, Theorem 2.1] *Let M be an $n \times n$ complex matrix. Then all the eigenvalues of M are located in the following disk:*

$$\left\{ z \in \mathbb{C} : \left| z - \frac{trM}{n} \right| \leq R \right\},$$

where

$$R = \sqrt{\frac{n-1}{n}\left(\|M\|_F^2 - \max_{1\leq k\leq n-1} \left(\left\|B_{k\times(n-k)}\right\|_F - \left\|C_{(n-k)\times k}\right\|_F \right)^2 - \frac{|trM|^2}{n} \right)}.$$

Kress et al. [6, Theorem 1] obtained an upper bound for $\sum\limits_{i=1}^{n} |\lambda_i|^2$ as follows:

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \sqrt{\|M\|_F^4 - \frac{1}{2}\|MM^* - M^*M\|_F^2}. \tag{3.2}$$

If M is non-normal, then (3.2) is obviously sharper than (1.1). The following result is a corollary of Theorem 2.1 and (3.2).

Corollary 3.3. *Let M be an $n \times n$ complex matrix. Then all the eigenvalues of M are located in the following disk:*

$$\left\{ z \in \mathbb{C} : \left| z - \frac{trM}{n} \right| \leq R \right\}, \tag{3.3}$$

where

$$R = \sqrt{\frac{n-1}{n}\left(\sqrt{\|M\|_F^4 - \frac{1}{2}\|MM^* - M^*M\|_F^2} - \frac{|trM|^2}{n} \right)}.$$

Note that the inequality (3.3) is a refinement of (1.2) for non-normal matrices.

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