YOUNG TYPE INEQUALITIES FOR MATRICES

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Abstract. In this note, we present a refinement of an inequality due to Hirzallah and Kittaneh [*Linear Algebra Appl.*, 308 (2000), 77-84]. Meanwhile, we also obtain an improvement of a result shown by Kittaneh and Manasrah [*Linear Multilinear Algebra*, 59 (2011), 1031-1037].

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1. Introduction

Let M_n be the space of $n \times n$ complex matrices. For $A = (a_{ij}) \in M_n$, the Hilbert-Schmidt norm of A is defined by

$$\|A\|_2 = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}.$$

It is known that the Hilbert-Schmidt norm is unitarily invariant.

The classical Young inequality for scalar says that if $a,\ b \geq 0$ and $0 \leq v \leq 1$, then

$$a^v b^{1-v} \leq va + (1-v)\, b$$

with equality if and only if $a = b$. By using Young's inequality, we can obtain some results of Heinz mean. For more information on Heinz inequality for matrices the reader is referred to [1]–[3].

The Kontorovich constant is defined as

$$K(t,2) = \frac{(t+1)^2}{4t}$$

for $t > 0$. Zuo, Shi, Fujii [4] proved that if $a,\ b \geq 0$ and $0 \leq v \leq 1$, then

$$K(h,2)^r\, a^v b^{1-v} \leq va + (1-v)\, b, \tag{1.1}$$

where $h = \frac{a}{b}$, $r = \min\{v,\ 1-v\}$. This is a refinement of the classical Young inequality.

Let $A, X, B \in M_n$ such that A and B are positive semidefinite. Kosaki [5] and Bhatia-Parthasarathy [6] proved that if $0 \leq v \leq 1$, then

$$\left\|A^v X B^{1-v}\right\|_2^2 \leq \|vAX + (1-v)\, XB\|_2^2 . \tag{1.2}$$

This is a matrix version of Young inequality. Hirzallah and Kittaneh [7] proved that if $0 \leq v \leq 1$, then

$$\left\|A^v X B^{1-v}\right\|_2^2 + v_0^2 \|AX - XB\|_2^2 \leq \|vAX + (1-v)\, XB\|_2^2 , \tag{1.3}$$

where $v_0 = \min\{v,\ 1-v\}$. Inequality (1.3) is an improvement of inequality (1.2). Kittaneh-Manasrah [8] and He-Zou [9] proved that if $0 \leq v \leq 1$, then

$$\|vAX + (1-v)\, XB\|_2^2 \leq \left\|A^v X B^{1-v}\right\|_2^2 + s_0^2 \|AX - XB\|_2^2 , \tag{1.4}$$

where $s_0 = \max\{v,\ 1-v\}$. Inequality (1.4) is a reverse inequality of (1.3).

In this note, we present refinements of inequalities (1.3) and (1.4).

2. Main results

In this section, we first give a refinement of inequality (1.3). To achieve it, we need the following lemma.

Lemma 2.1. *If $a,\ b \geq 0$ and $0 \leq v \leq 1$, then*

$$K(h,2)^r \left(a^v b^{1-v}\right)^2 + v_0^2 (a-b)^2 \leq (va + (1-v)\, b)^2 , \tag{2.1}$$

where $h = \frac{a}{b}$, $v_0 = \min\{v,\ 1-v\}$, $r = \min\{2v_0,\ 1-2v_0\}$.

Proof. If $v = \frac{1}{2}$, inequality (2.1) becomes equality. If $v < \frac{1}{2}$, then by (1.1), we have

$$\begin{aligned} (va + (1-v)\, b)^2 - v_0^2 (a-b)^2 &= (va + (1-v)\, b)^2 - v^2 (a-b)^2 \\ &= 2vab + (1-2v)\, b^2 \\ &\geq K(h,2)^r \left(a^v b^{1-v}\right)^2 . \end{aligned}$$

If $v > \frac{1}{2}$, then by (1.1), we have

$$\begin{aligned} (va + (1-v)\, b)^2 - v_0^2 (a-b)^2 &= (va + (1-v)\, b)^2 - (1-v)^2 (a-b)^2 \\ &= (2v-1)\, a^2 + 2(1-v)\, ab \\ &\geq K(h,2)^r \left(a^v b^{1-v}\right)^2 . \end{aligned}$$

This completes the proof. ∎

Theorem 2.2. *Let $A, X, B \in M_n$ such that A and B are positive semidefinite. Suppose that the spectral decomposition of A, B are $A = U\Lambda_1 U^*$, $B = V\Lambda_2 V^*$ respectively, where $\Lambda_1 = diag\left(\lambda_1, ..., \lambda_n\right), \Lambda_2 = diag\left(\mu_1, ..., \mu_n\right), \lambda_i,\ \mu_i \geq 0$, $i = 1, ..., n$. Let*

$$K = \min\left\{K\left(\frac{\lambda_i}{\mu_j}, 2\right), i, j = 1, ..., n\right\}.$$

Then

$$K^r \left\|A^v X B^{1-v}\right\|_2^2 + v_0^2 \left\|AX - XB\right\|_2^2 \leq \left\|vAX + (1-v) XB\right\|_2^2, \tag{2.2}$$

where $v_0 = \min\{v,\ 1-v\}$, $r = \min\{2v_0,\ 1-2v_0\}$.

Proof. Let $Y = U^* X V = [y_{ij}]$. We have as in [10],

$$\begin{aligned} \left\|vAX + (1-v) XB\right\|_2^2 &= \sum_{i,j=1}^{n} \left(v\lambda_i + (1-v)\mu_j\right)^2 |y_{ij}|^2, \\ \left\|A^v X B^{1-v}\right\|_2^2 &= \sum_{i,j=1}^{n} \left(\lambda_i^v \mu_j^{1-v}\right)^2 |y_{ij}|^2, \\ \left\|AX - XB\right\|_2^2 &= \sum_{i,j=1}^{n} \left(\lambda_i - \mu_j\right)^2 |y_{ij}|^2. \end{aligned}$$

Inequality (2.2) deduces from inequality (2.1) and above equalities. This completes the proof. ∎

Next, we show an improvement of inequality (1.4). To do this, we need the following lemma.

Lemma 2.3. *If $a,\ b \geq 0$ and $0 \leq v \leq 1$, then*

$$\left(va + (1-v) b\right)^2 \leq K(h, 2)^{-r} \left(a^v b^{1-v}\right)^2 + s_0^2 (a-b)^2, \tag{2.3}$$

where $h = \dfrac{a}{b}$, $s_0 = \max\{v,\ 1-v\}$, $r = \min\{2s_0 - 1,\ 2 - 2s_0\}$.

Proof. If $v = \frac{1}{2}$, inequality (2.3) becomes equality. If $v < \frac{1}{2}$, then by (1.1), we have

$$\begin{aligned} s_0^2 (a-b)^2 - \left(va + (1-v) b\right)^2 &= (1-v)^2 (a-b)^2 - \left(va + (1-v) b\right)^2 \\ &= (1-2v) a^2 - 2(1-v) ab \\ &= (1-2v) a^2 + 2vab - 2ab \\ &\geq K(h, 2)^r \left(a^{1-v} b^v\right)^2 - 2ab \\ &\geq -K(h, 2)^{-r} \left(a^v b^{1-v}\right)^2. \end{aligned}$$

If $v > \frac{1}{2}$, then by (1.1), we have

$$\begin{aligned} s_0^2 (a-b)^2 - \left(va + (1-v) b\right)^2 &= v^2 (a-b)^2 - \left(va + (1-v) b\right)^2 \\ &= (2v-1) b^2 - 2vab \\ &= (2v-1) b^2 + 2(1-v) ab - 2ab \\ &\geq K(h, 2)^r \left(a^{1-v} b^v\right)^2 - 2ab \\ &\geq -K(h, 2)^{-r} \left(a^v b^{1-v}\right)^2. \end{aligned}$$

This completes the proof. ∎

Theorem 2.4. *Let $A, X, B \in M_n$ such that A and B are positive semidefinite. Suppose that the spectral decomposition of A, B are $A = U\Lambda_1 U^*$, $B = V\Lambda_2 V^*$ respectively, where $\Lambda_1 = diag\,(\lambda_1, ..., \lambda_n)\,, \Lambda_2 = diag\,(\mu_1, ..., \mu_n)\,, \lambda_i,\ \mu_i \geq 0$, $i = 1, ..., n$. Let*

$$K = \min\left\{K\left(\frac{\lambda_i}{\mu_j}, 2\right), i, j = 1, ..., n\right\}.$$

Then

$$\|vAX + (1-v)\,XB\|_2^2 \leq K^{-r}\left\|A^v X B^{1-v}\right\|_2^2 + s_0^2\,\|AX - XB\|_2^2\,, \tag{2.4}$$

where $s_0 = \max\{v,\ 1-v\}$, $r = \min\{2v_0,\ 1-2v_0\}$.

Proof. The result follows from inequality (2.3) and by using an argument similar to that used for the proof of Theorem 2.2. This completes the proof. ∎

Since $K(t, 2) = \dfrac{(t+1)^2}{4t} \geq 1$ for $t > 0$, it follows that inequalities (2.2) and (2.4) are refinements of inequalities (1.3) and (1.4) respectively.

References

[1] Hu, X., *Some inequalities for unitarily invariant norms*, J. Math. Inequal., 6 (2012), 615-623.

[2] Fu, X., He, C., *On some inequalities for unitarily invariant norms*, J. Math. Inequal., 7 (2013), 727-737.

[3] Zou, L., *Inequalities related to Heinz and Heron means*, J. Math. Inequal., 7 (2013), 389-397.

[4] Zuo, H., Shi, G., Fujii, M., *Refined Young inequality with Kantorovich constant*, J. Math. Inequal., 5 (2011), 551-556.

[5] Kosaki, H., *Arithmetic-geometric mean and related inequalities for operators*, J. Funct. Anal., 156 (1998), 429-451.

[6] Bhatia, R., Parthasarathy, K.R., *Positive definite functions and operator inequalities*, Bull. London Math. Soc., 32 (2000), 214-228.

[7] Hirzallah, O., Kittaneh, F., *Matrix Young inequalities for the Hilbert-Schmidt norm*, Linear Algebra Appl., 308 (2000), 77-84.

[8] Kittaneh, F., Manasrah, Y., *Reverse Young and Heinz inequalities for matrices*, Linear Multilinear Algebra, 59 (2011), 1031-1037.

[9] He, C., Zou, L., *Some inequalities involving unitarily invariant norms*, Math. Inequal. Appl., 15 (2012), 767-776.

[10] Kittaneh, F., Manasrah, Y., *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl., 361 (2010), 262-269.

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