YOUNG TYPE INEQUALITIES FOR MATRICES

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Abstract. In this note, we present a refinement of an inequality due to Hirzallah and Kittaneh [Linear Algebra Appl., 308 (2000), 77-84]. Meanwhile, we also obtain an improvement of a result shown by Kittaneh and Manasrah [Linear Multilinear Algebra, 59 (2011), 1031-1037].

Keywords: Young type inequalities; Hilbert-Schmidt norm; positive semidefinite matrices; Kantorovich constant.

MSC (2010) Subject Classification: 15A60.

1. Introduction

Let $M_n$ be the space of $n \times n$ complex matrices. For $A = (a_{ij}) \in M_n$, the Hilbert-Schmidt norm of $A$ is defined by

$$\|A\|_2 = \left( \sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$  

It is known that the Hilbert-Schmidt norm is unitarily invariant.

The classical Young inequality for scalar says that if $a$, $b \geq 0$ and $0 \leq v \leq 1$, then

$$a^v b^{1-v} \leq va + (1 - v)b$$

with equality if and only if $a = b$. By using Young’s inequality, we can obtain some results of Heinz mean. For more information on Heinz inequality for matrices the reader is referred to [1]–[3].

The Kontorovich constant is defined as

$$K(t, 2) = \frac{(t + 1)^2}{4t}$$

for $t > 0$. Zuo, Shi, Fujii [4] proved that if $a$, $b \geq 0$ and $0 \leq v \leq 1$, then

$$K(h, 2)^r a^v b^{1-v} \leq va + (1 - v)b,$$  

(1.1)
where $h = \frac{a}{b}$, $r = \min \{v, 1-v\}$. This is a refinement of the classical Young inequality.

Let $A, X, B \in M_n$ such that $A$ and $B$ are positive semidefinite. Kosaki [5] and Bhatia-Parthasarathy [6] proved that if $0 \leq v \leq 1$, then

\begin{equation}
\|A^vXB^{1-v}\|_2^2 \leq \|vAX + (1-v)XB\|_2^2.
\end{equation}

This is a matrix version of Young inequality. Hirzallah and Kittaneh [7] proved that if $0 \leq v \leq 1$, then

\begin{equation}
\|A^vXB^{1-v}\|_2^2 + v_0^2 \|AX - XB\|_2^2 \leq \|vAX + (1-v)XB\|_2^2,
\end{equation}

where $v_0 = \min \{v, 1-v\}$. Inequality (1.3) is an improvement of inequality (1.2).

Kittaneh-Manasrah [8] and He-Zou [9] proved that if $0 \leq v \leq 1$, then

\begin{equation}
\|vAX + (1-v)XB\|_2^2 \leq \|A^vXB^{1-v}\|_2^2 + s_0^2 \|AX - XB\|_2^2,
\end{equation}

where $s_0 = \max \{v, 1-v\}$. Inequality (1.4) is a reverse inequality of (1.3).

In this note, we present refinements of inequalities (1.3) and (1.4).

2. Main results

In this section, we first give a refinement of inequality (1.3). To achieve it, we need the following lemma.

**Lemma 2.1.** If $a, b \geq 0$ and $0 \leq v \leq 1$, then

\begin{equation}
K(h, 2)^r (a^v b^{1-v})^2 + v_0^2 (a-b)^2 \leq (va + (1-v)b)^2,
\end{equation}

where $h = \frac{a}{b}$, $v_0 = \min \{v, 1-v\}$, $r = \min \{2v_0, 1-2v_0\}$.

**Proof.** If $v = \frac{1}{2}$, inequality (2.1) becomes equality. If $v < \frac{1}{2}$, then by (1.1), we have

\[(va + (1-v)b)^2 - v_0^2 (a-b)^2 = (va + (1-v)b)^2 - v^2 (a-b)^2 = 2vab + (1-2v)b^2 \geq K(h, 2)^r (a^v b^{1-v})^2.\]

If $v > \frac{1}{2}$, then by (1.1), we have

\[(va + (1-v)b)^2 - v_0^2 (a-b)^2 = (va + (1-v)b)^2 - (1-v)^2 (a-b)^2 = (2v-1) a^2 + 2 (1-v) ab \geq K(h, 2)^r (a^v b^{1-v})^2.\]

This completes the proof. \hfill \blacksquare
Theorem 2.2. Let $A, X, B \in M_n$ such that $A$ and $B$ are positive semidefinite. Suppose that the spectral decomposition of $A, B$ are $A = U\Lambda_1 U^*$, $B = V\Lambda_2 V^*$ respectively, where $\Lambda_1 = \text{diag}(\lambda_1, \ldots, \lambda_n), \Lambda_2 = \text{diag}(\mu_1, \ldots, \mu_n), \lambda_i, \mu_i \geq 0, i = 1, \ldots, n$. Let

$$K = \min \left\{ K \left( \frac{\lambda_i}{\mu_j}, 2 \right), i, j = 1, \ldots, n \right\}.$$ 

Then

$$K^r \parallel A^v XB^{1-v} \parallel_2^2 + v_0^2 \parallel AX - XB \parallel_2^2 \leq \parallel vAX + (1 - v) XB \parallel_2^2,$$  \hspace{1cm} (2.2)

where $v_0 = \min \{ v, 1 - v \}, r = \min \{2v_0, 1 - 2v_0 \}$.

Proof. Let $Y = U^* XV = [y_{ij}]$. We have as in [10],

$$\parallel vAX + (1 - v) XB \parallel_2^2 = \sum_{i,j=1}^n (v\lambda_i + (1 - v) \mu_j)^2 |y_{ij}|^2,$$

$$\parallel A^v XB^{1-v} \parallel_2^2 = \sum_{i,j=1}^n (\lambda_i^v \mu_j^{1-v})^2 |y_{ij}|^2,$$

$$\parallel AX - XB \parallel_2^2 = \sum_{i,j=1}^n (\lambda_i - \mu_j)^2 |y_{ij}|^2.$$  

Inequality (2.2) deduces from inequality (2.1) and above equalities. This completes the proof. \hfill $\blacksquare$

Next, we show an improvement of inequality (1.4). To do this, we need the following lemma.

Lemma 2.3. If $a, b \geq 0$ and $0 \leq v \leq 1$, then

$$\left( va + (1 - v) b \right)^2 \leq K(h, 2)^{-r} \left( a^v b^{1-v} \right)^2 + s_0^2 (a - b)^2,$$  \hspace{1cm} (2.3)

where $h = \frac{a}{b}, s_0 = \max \{ v, 1 - v \}, r = \min \{2s_0 - 1, 2 - 2s_0 \}$.

Proof. If $v = \frac{1}{2}$, inequality (2.3) becomes equality. If $v < \frac{1}{2}$, then by (1.1), we have

$$s_0^2 (a - b)^2 - (va + (1 - v) b)^2 = (1 - v)^2 (a - b)^2 - (va + (1 - v) b)^2$$

$$= (1 - 2v) a^2 - 2 (1 - v) ab$$

$$= (1 - 2v) a^2 + 2va b - 2ab$$

$$\geq K(h, 2)^{-r} \left( a^v b^{1-v} \right)^2 - 2ab$$

$$\geq -K(h, 2)^{-r} \left( a^v b^{1-v} \right)^2.$$  

If $v > \frac{1}{2}$, then by (1.1), we have

$$s_0^2 (a - b)^2 - (va + (1 - v) b)^2 = v^2 (a - b)^2 - (va + (1 - v) b)^2$$

$$= (2v - 1) b^2 - 2va b$$

$$= (2v - 1) b^2 + 2 (1 - v) ab - 2ab$$

$$\geq K(h, 2)^{-r} \left( a^v b^{1-v} \right)^2 - 2ab$$

$$\geq -K(h, 2)^{-r} \left( a^v b^{1-v} \right)^2.$$  

\hfill $\blacksquare$
This completes the proof.

**Theorem 2.4.** Let \( A, X, B \in M_n \) such that \( A \) and \( B \) are positive semidefinite. Suppose that the spectral decomposition of \( A, B \) are \( A = U\Lambda_1 U^* \), \( B = V\Lambda_2 V^* \) respectively, where \( \Lambda_1 = \text{diag}(\lambda_1, ..., \lambda_n) \), \( \Lambda_2 = \text{diag}(\mu_1, ..., \mu_n) \), \( \lambda_i, \mu_i \geq 0 \), \( i = 1, ..., n \). Let

\[
K = \min \left\{ K \left( \frac{\lambda_i}{\mu_j} \right), i, j = 1, ..., n \right\}.
\]

Then

\[
\| vAX + (1-v)XB \|_2^2 \leq K^{-r} \| A^r XB^{1-r} \|_2^2 + s_0^2 \| AX - XB \|_2^2,
\]

where \( s_0 = \max \{v, 1-v\} \), \( r = \min \{ 2v_0, 1 - 2v_0 \} \).

**Proof.** The result follows from inequality (2.3) and by using an argument similar to that used for the proof of Theorem 2.2. This completes the proof.

Since \( K(t, 2) = \frac{(t+1)^2}{4t} \geq 1 \) for \( t > 0 \), it follows that inequalities (2.2) and (2.4) are refinements of inequalities (1.3) and (1.4) respectively.

**References**


Accepted: 06.05.2014