IRREDUCIBLE IDEALS IN RINGS

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Abstract. It is well known that the ideals of any ring form an algebraic lattice under the set inclusion ordering, in which the finitely generated ideals are precisely the compact elements. Strongly irreducible ideals of a ring were studied by W.J. Heinzer, L.J. Ratliff and D.E. Rush [3] and $\alpha-$irreducible and $\alpha-$strongly irreducible ideals of a ring were characterized by X. Lu and H. Qin [8]. In this paper, we extend these results for elements of a general algebraic lattice and obtain the results on ideals of rings and on submodules of an $R-$ module as consequences of our general results. Also, we characterize algebraic lattices satisfying the ascending chain condition.

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1. Introduction and preliminaries

The concepts of irreducible elements and prime elements are crucial in the study of the structure theory of general algebraic systems, in particular, in that of distributive lattices. For example, the prime ideals play a vital role in the pioneering work of M.H. Stone [5] on the representation theory of distributive lattices. The prime ideals of a lattice are precisely the prime elements in the lattice of its ideals. In a general lattice, prime elements are also called strongly irreducible elements.

First, we recall certain elementary concepts and notations from Theory of Lattices [1]. A partially ordered set $(L, \leq)$ is called a lattice (complete lattice) if every two element subset (respectively, every arbitrary subset) of L has both infimum and supremum in L; For any subset A of L, we write $\inf A$ or $\bigwedge A$ or $\bigwedge\limits_{a \in A} a$ for the infimum (greatest lower bound) of A and $\sup A$ or $\bigvee A$ or $\bigvee\limits_{a \in A} a$ for the supremum (least upper bound) of A. If A is a finite set $\{a_1, a_2, \cdots, a_n\}$, then we write $\bigwedge\limits_{i=1}^{n} a_i$ or $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ for the inf A and $\bigvee\limits_{i=1}^{n} a_i$ or $a_1 \vee a_2 \vee \cdots \vee a_n$ for the sup A. If $(L, \leq)$ is a lattice, then $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$ give us two binary operations $\wedge$ and $\vee$ on L which are both associative, commutative and idempotent and satisfy the absorption laws $(a \wedge (a \vee b) = a = a \vee (a \wedge b))$. Conversely if $\wedge$ and $\vee$ are binary operations on a non empty set L which satisfy all the above properties and if the partial order $\leq$ on L is defined by $a = a \wedge b \Leftrightarrow a \leq b \Leftrightarrow a \vee b = b$, then $(L, \leq)$ is a lattice in which $a \wedge b$ and $a \vee b$ are respectively the infimum and supremum of any $\{a, b\}$.

A lattice $(L, \leq)$ is called bounded if it has smallest element 0 (that is, $0 \leq a$ for all $a \in L$) and the largest element 1 (that is, $a \leq 1$ for all $a \in L$). A complete lattice is necessarily bounded. Logically, the infimum and supremum of the empty set, if they exist, are the largest element 1 and smallest element 0 respectively. An element a of a complete lattice L is called compact if, for any $X \subseteq L, a \leq \sup X \Longrightarrow a \leq \sup F$ for some finite subset F of X. A complete lattice L is called an algebraic lattice if every element of L is the supremum of a set of compact elements in L. An element m in a partially ordered set $(L, \leq)$ is called maximal if $m \leq a \in L$ implies $m = a$.

2. Algebraic Lattices with ACC

Let us recall that a partially ordered set $(P, \leq)$ is said to satisfy the ascending chain condition (ACC) if any increasing sequence in P terminates at a finite stage; that is, if $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$ is a sequence in P such that $a_n \leq a_{n+1}$ for all n, then there exists n_0 such that $a_{n_0} = a_{n_0+k}$ for all $k > 0$. It is well known that $(P, \leq)$ satisfies ACC if and only if every non empty subset of P has a maximal element. In the following, we characterize algebraic lattices satisfying the ascending chain condition.

Theorem 2.1. *Let $(L, \leq)$ be an algebraic lattice. Then $(L, \leq)$ satisfies the ACC if and only if every element of L is compact.*

Proof. Suppose that $(L, \leq)$ satisfies ACC. Let $a \in L$. We can assume that $a \neq 0$, since the smallest element 0 is always compact. Let X be a subset of L such that $a \leq \sup X$. Since $a \neq 0$, X is non empty. Consider the set

$$A = \{\sup F \mid F \text{is a finite subset of } X\}.$$

A is a non empty subset of L. Since L satisfies ACC, A has a maximal element. Let m be a maximal element in A. Then, $m = \sup F$, for some finite subset F

of X. For any $x \in X$, we have $m = \sup F \leq \sup(F \cup \{x\}) \in A$ and, by the maximal element of $m, m = \sup(F \cup \{x\})$, so that $x \leq m$ for all $x \in X$. Therefore, $m = \sup X$ and $a \leq m = \sup F$ and F is a finite subset of X. Thus, a is compact.

Conversely, suppose that every element of L is compact. Let $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$ be an increasing sequence in L and $a = \sup_{n \in \mathbb{Z}^+} \{a_n\}$. Then a is compact and hence $a = a_{n_1} \vee a_{n_2} \vee \cdots \vee a_{n_r}$ for some $n_1, n_2, \cdots, n_r \in \mathbb{Z}^+$. If $n = max\{n_1, n_2, \cdots, n_r\}$, then $a = a_{n_1} \vee a_{n_2} \vee \cdots \vee a_{n_r} = a_n$ and hence $a_k \leq a_n$ for all k. Therefore $a_n = a_{n+k}$ for all $k \in \mathbb{Z}^+$. Thus $(L, \leq)$ satisfies the ascending chain condition. ∎

Recall that an ideal I of a commutative ring R with unity is called finitely generated if there exist $a_1, a_2, \cdots, a_n \in R$ such that

$$I = \{r_1 a_1 + r_2 a_2 + \cdots + r_n a_n \mid r_i \in R\}.$$

Also, R is said to be a Noetherian ring if the lattice of ideals of R satisfies ascending chain condition. It is well known that the ideals of any ring form an algebraic lattice, under the set inclusion order, in which the compact elements are precisely the compact elements. We record the following well known result as a consequence of the above theorem.

Corollary 2.2. *The following are equivalent to each other for any ring R.*

(1) *R is a Noetherian ring.*

(2) *Every ideal of R is compact in the lattice of ideals of R.*

(3) *Every ideal of R is finitely generated.*

(4) *Any nonempty class of ideals has a maximal member.*

On the same lines as discussed above, the submodule of an $R-$module (where R is any ring) form an algebraic lattice under the set inclusion ordering in which the finitely generated $R-$submodules are precisely the compact elements. An $R-$module is said to be Noetherian if the lattice of $R-$submodules satisfies the ascending chain condition.

Corollary 2.3. *The following are equivalent to each other for any module M over a ring R.*

(1) *M is a Noetherian $R-$ module.*

(2) *Every submodule of M is compact in the lattice of submodules of M.*

(3) *Every submodule of M is finitely generated.*

(4) *Any non empty class of submodules of M has a maximal member.*

It is known that a complete lattice is algebraic if and only if it is isomorphic to the lattice of subuniverses of an (universal) algebra. In view of this, we have the following

Corollary 2.4. *The following are equivalent to each other for any universal algebra A.*

(1) *The lattice of subuniverses of A satisfies ACC.*

(2) *Every subuniverse of A is compact in the lattice of subuniverses of A.*

(3) *Any subuniverse of A is finitely generated.*

(4) *Any class of subuniverses of A has a maximal member.*

3. $\alpha-$Irreducibility

In this section, we recall the notions of irreducible element and strongly irreducible element in a general complete lattice and discuss certain important properties of these in order to facilitate characterizations of these among ideals of a ring.

Definition 3.1. Let $(L, \wedge, \vee)$ be a bounded lattice and $1 \neq p \in L$.

(1) p is said to be irreducible if, for any a and $b \in L$,
$p = a \wedge b \Longrightarrow p = a$ or $p = b$.

(2) p is said to be strongly irreducible or prime if, for any a and $b \in L$,
$a \wedge b \leq p \Longrightarrow a \leq p$ or $b \leq p$.

Clearly, every prime element in any bounded lattice is irreducible and the converse is not true. However, if the lattice is distributive, then every irreducible element is prime. In the following, we write $\mid A \mid$ for the cardinality of a set A.

Definition 3.2. Let $(L, \leq)$ be a complete lattice, $1 \neq p \in L$ and α a cardinal number.

(1) p is said to be $\alpha-$irreducible if, for any $A \subseteq L$ with $\mid A \mid \leq \alpha$,
$p = \inf A \Longrightarrow p \in A$.

(2) p is said to be $\alpha-$strongly irreducible or $\alpha-$prime if, for any $A \subseteq L$ with $\mid A \mid \leq \alpha$,
$\inf A \leq p \Longrightarrow a \leq p$ for some $a \in A$.

An element p is irreducible (prime) if and only if p is $\alpha-$irreducible (respectively $\alpha-$prime) for all positive integers α. Also, clearly every $\alpha-$ prime element is $\alpha-$ irreducible and the converse is not true; for, consider the set N of non negative integers together with the partial order defined by '$a \leq b \Longleftrightarrow b$ divides a'. Then $(N, \leq)$ is a complete lattice in which, for any subset A, infimum and supremum of A are precisely the LCM and GCD of A respectively. Here 2 is $\alpha-$ irreducible for all cardinal numbers α. But 2 is not $\alpha-$prime, since the infimum of the set of all odd primes is 0 and $0 < 2$.

However, if the lattice L satisfies the infinite join distributivity (that is, $x \vee (\inf A) = \inf\{x \vee a \mid a \in A\}$ for all $x \in L$ and $A \subseteq L$), then every $\alpha-$irreducible element in L is $\alpha-$prime. Note that the complete lattice $(N, \leq)$ discussed above does not satisfy the infinite join distributivity. The following two theorems are proved in 2.2 and 2.7 of [6].

Theorem 3.3. *An element $p \neq 1$ in an algebraic lattice L is $\alpha-$irreducible for every cardinal number α if and only if there exists a compact non zero element $c \in L$ such that p is a maximal element in the set $\{x \in L \mid c \nleq x\}$.*

Theorem 3.4. *An element $p \neq 1$ in an algebraic lattice L is $\alpha-$prime for every cardinal number α if and only if there exists a compact non zero element $c \in L$ such that p is the unique maximal element in the set $\{x \in L \mid c \nleq x\}$.*

Also, it is proved in 2.5 of [6] that any element of an algebraic lattice L is the infimum of a set of elements in L which are $\alpha-$irreducible for all cardinal numbers α.

4. Irreducible ideals of a ring

It is well known that ideals of a ring R form a complete lattice under the set inclusion order, in which the infimum of any class $\{I\}_{\alpha\in\Delta}$ of ideals of R is simply the set intersection, while the supremum is the ideal generated $\bigcup\limits_{\alpha\in\Delta} I_\alpha$. Let us denote the set of all ideals of a ring R by $\mathscr{I}(R)$.

Theorem 4.1. *Let I be a proper ideal of a ring R. Then I is $\alpha-$irreducible in the lattice $\mathscr{I}(R)$ for every cardinal number α if and only if there exists a finite non empty subset F of R such that I is a maximal member in the set of all ideals of R not containing F.*

Proof. First, observe that the lattice $\mathscr{I}(R)$ of ideals of R is an algebraic lattice in which the compact elements are precisely the finitely generated ideals of R. Also, for any ideal J of R and for any subset F or R, we have $F \subseteq J$ if and only if $< F > \subseteq J$, where $< F >$ is the ideal generated by F in R. Now, the theorem follows from Theorem 3.3. ∎

Note that an ideal I of a ring R with unity is a proper ideal if and only if I does not contain the unity element. This, together with the above theorem, implies the following.

Corollary 4.2. *Let R be a ring with unity. Then, any maximal ideal of R is $\alpha-$irrducible in $\mathscr{I}(R)$ for all cardinal numbers α.*

A proof analogous to that of Theorem 4.1 together with Theorem 3.4 yield the following.

Theorem 4.3. *A proper ideal I of a ring R is $\alpha-$prime in $\mathscr{I}(R)$ for all cardinal numbers α if and only if there exists a non empty finite subset F of R such that I is the unique maximal member in the set of all ideals of R not containing F.*

Recall that a ring with unity is called a local ring if it has a unique maximal ideal. The following is an immediate consequence of the above theorem.

Corollary 4.4. *In a local ring, the maximal ideal is $\alpha-$ prime for all cardinal numbers α.*

Example 4.5. Consider the ring $\mathbb{Z}$ of integers. It is well known that any ideal of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for some non negative integer n. $\mathbb{Z}$ is a Noetherian ring and every ideal of $\mathbb{Z}$ is compact in the lattice $\mathscr{I}(\mathbb{Z})$ of ideals of $\mathbb{Z}$. Since

$$\{0\} = \cap\{p\mathbb{Z} \mid p \text{ is a prime number}\},$$

$\{0\}$ is not $\alpha-$irreducible for any infinite cardinal α, However, $\{0\}$ is irreducible, since

$$\{0\} = n\mathbb{Z} \cap m\mathbb{Z} \implies n = 0 \text{ or } m = 0.$$

It can be proved that $\mathscr{I}(\mathbb{Z})$ is a distributive lattice and that a non zero ideal $n\mathbb{Z}$ is irreducible (and hence prime) if and only if $n = p^r$ for some prime number p and a positive integer r. Also, any ideal of $\mathbb{Z}$ is not $\alpha-$prime, for an infinite cardinal α; for, let $n\mathbb{Z}$ be a non zero proper ideal of $\mathbb{Z}$. Then $n > 1$. Consider the set

$$X = \{p\mathbb{Z} \mid p \text{ is a prime number not dividing } n\}.$$

Then $\inf X = \{0\} \subseteq n\mathbb{Z}$ and $p\mathbb{Z} \nsubseteq n\mathbb{Z}$ for all $p\mathbb{Z} \in X$. Thus $n\mathbb{Z}$ is not $\alpha-$prime. On the other hand, $n\mathbb{Z}$ is $\alpha-$irreducible for all cardinals α if and only if $n = p^r$ for some prime number p and a positive integer r.

Theorems 4.1 and 4.3 can be extended to submodules of an $R-$module M, since a submodule of M is compact in the lattice of submodules of M if and only if it is finitely generated over R.

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