A THEOREM ABOUT FINITE GROUPS
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\textbf{Abstract.} Let $G$ be a finite group. $G$ has the property that for any conjugacy class length, $G$ has exactly two conjugacy classes having such length. The paper classifies all possible structure of the finite group $G$ under a condition that $G'$ is nilpotent.

\textbf{Keywords:} finite groups, conjugacy class length, character.

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1. Introduction

In 1973, F. Mark ([1]) began to investigate the “S₃-conjecture”, where $S₃$ is the only finite group whose conjugacy classes all have different length. Many authors also studied this problem in special cases ([3], [4]). In 1994, J.P. Zhang ([5]) proved the conjecture is true if the group is a finite solvable group. In 2004, C.M. Boner and M.B. Ward ([2]) investigated the similar problem that if $G$ is a finite group with exactly two conjugacy classes of the same length, and $G'$ is nilpotent, then $G \cong Z_2, D_{10}, A_4$. This paper continues this kind of problems. We will prove that if $G$ is a finite group with the property $P^*$ ($P^*$ will be defined later), and $G'$ is nilpotent, then $G \cong Z_2, G \cong Z_2 \times S_3$ or $G \cong Z_3 \times Z_4$.

For the sake of convenience, we first give some notations. Let $x \in G$, we denote by $o(x)$ the order of $x$, $x^G$ denotes the conjugacy class containing $x$, $|x^G|$ denotes the length of the conjugacy class of $x$, $A \text{ char } G$ means that $A$ is a character subgroup of $G$. Let $\text{Irr}(G)$ be the set of all irreducible characters of $G$, $\text{Irr}^*(G)$ be the set of nonlinear irreducible characters of $G$. We call a finite group $G$ having property $P^*$ if $G$ has property that for any conjugacy class length, $G$ has exactly two conjugacy classes having such length. We assume that all groups in this paper are finite groups. All other symbols used in this paper are standard and the readers can consult [7] for more information.

We begin our work with the following lemmas.

2. Preliminaries

**Lemma 2.1.** ([5]) Suppose that $G$ is a finite solvable group. If the conjugacy classes of $G$ all have different length, then $G \cong S_3$.

In the following proof, we suppose that $G$ is a finite group with property $P^*$, and $G'$ is nilpotent.

**Lemma 2.2.** $Z(G) \cong Z_2$ and $G/G'$ is an Abelian 2-group.

**Proof.** Because $G$ contains exactly two conjugacy classes of length 1, and the conjugacy classes length of any elements of $Z(G)$ is 1, we have that $Z(G) \cong Z_2$. Let $z \in Z(G)$, $z \neq 1$, $x \in G$, and $o(x)$ is odd, then the length of conjugacy classes of $x$ and $zx$ are equal. $o(x)$ is odd implies $C_G(x) = C_G(x^2)$, which concludes that the conjugacy length of $x$ and $x^2$ are equal too. Since $G$ has property $P^*$, two elements among $x, x^2, zx$ must be conjugate. The order of $x$ and $zx$ are different implies that $x$ is conjugate to $x^2$. Therefore, there exists $g \in G$, such that $x^2 = x^g$. 
So we have that $x = x^{-1}x^g = [x, g] \in G'$, from which we know that $G/G'$ is an Abelian 2-group.

**Lemma 2.3.** Let $P_2 \in Syl_2(G)$. If $P_2$ is non-abelian, then $G' = Z_2H, P'_2 = Z_2$, where $Z_2 = Z(G), H$ is a 2'-Hall normal subgroup of $G$.

**Proof.** By Lemma 2.2, $G/G'$ is an Abelian 2-group. Since $G'$ is nilpotent, we can set $G' = L \times H$, where $L \leq P_2$. Then $P'_2 \leq L \leq P_2$. Since $L$ char $G' \sim G$, we have that $L \lhd G$ and $L \cap Z(P_2) \neq \emptyset$. Also, since $H$ char $G' \leq G, H \leq G$ and the elements of $L$ can commute with the elements of $H$. Thus $Z(G) = Z_2 = L \cap Z(P_2) \leq L$.

Now, we prove $L = Z_2$. Otherwise, since $L \leq G, L$ consists of conjugacy classes of $G, |L| = |Z_2| + |aG| + \ldots + |bG|$, where $a, b \in L$. Let $|aG|$ be the minimal length of the conjugacy classes of non-unit elements of $L$. Clearly, the $G$-conjugacy classes length of $L$ is power of 2. If $|aG| \geq 4$, we have a contradiction immediately since $|Z_2| = 2$. Therefore, $|aG| = 2$. But $G$ has another conjugacy class of length 2, let it be $xG$, that is $|xG| = 2$. If $x \in L$, then $|xG| + |aG| = 4$. Since $G$ has the property $P^*$, the other conjugacy classes length of $L$ is at least 4, with the same reason we have a contradiction. If $x \notin L$, it is easy to know that $|xG| = |(zx)G|$ and $|aG| = |(za)G|$, where $z \in Z(G)$. $G$ has property $P^*$ implies that $x$ is conjugate to $zx$, and $a$ is conjugate to $za$. Because $|aG| = 2$, we can conclude that for any element $g$ of $G, a^g = az$ or $a^g = a$. With the same reason we have that $x^g = zx$, or $x^g = x$. Furthermore, it is easy to prove for any $g \in G$, $(ax)^g = ax$, or $(ax)^g = zax$. So $|(ax)^G| = 2$ or $ax \in Z(G)$. Obviously, $ax \notin Z(G)$. We assert that $ax$ is not conjugate to $a$. Otherwise, there exists $g \in G$ such that $a = (ax)^g$, then $a = (ax)^g = ax$ or $a = (ax)^g = zax$, both of them are impossible. In the same way, we can prove that $ax$ is not conjugate to $x$. Therefore, $(ax)^G$ is the third conjugacy class with length 2 except $aG, xG$, it contradicts to the fact that $G$ has property $P^*$.

Therefore, $|L| = 2$, and $P'_2 = L = Z_2$.

**Lemma 2.4.** $P_2$ is an Abelian subgroup.

**Proof.** Let $|P_2| = 2^n$, if $P_2$ is not an Abelian subgroup, by Lemma 2.3, $G' = Z_2H, P'_2 = Z_2$. Let $N$ is a minimal normal subgroup of $G$, and $N \leq H$. Since $H \leq G'$, and $G'$ is nilpotent, $N$ is a $p$-group and $N \leq Z(H)$. Let $M = P_2N$. We consider a quotient group $\tilde{M} = M/Z_2 = \tilde{P}_2\tilde{N} \leq \tilde{G} = G/Z_2$, where $\tilde{P}_2 = P_2/Z_2, \tilde{N} = NZ_2/Z_2$. Clearly, $\tilde{N}$ is a minimal normal subgroup of $\tilde{M}$. If $\tilde{M} = M/Z_2$ is Abelian, then for any $x \in P_2, y \in N$, such that $y^{-1}y^x \in Z_2 \cap \tilde{N} = 1$, since $N \lhd G$. Therefore $xy = yx$ for any $x \in P_2, y \in N$. Since $N \leq Z(H)$, we have $N \leq Z(G) = Z_2$, a contradiction. Therefore, $M = M/Z_2$ is non-Abelian.
$\bar{P}_2$ is Abelian and $\bar{N}$ is a minimal normal subgroup of $\bar{M}$ imply that $\bar{M}' = \bar{N}$. Let $\chi \in \text{Irr}(\bar{M})$, $\chi(1) > 1$. Since $\bar{N}$ is an Abelian normal subgroup of $\bar{M}$, we have $\chi(1) | \bar{M} : \bar{N} = 2^{n-1}$ (Ref [6]). Also, since $\bar{P}_2$ is Abelian, for any $a \in \bar{P}_2$, $(|a^M|, \chi(1)) = 1$. Therefore, $\chi(a) = 0$, or $\rho(a) = \omega l(\text{Ref}[6])$, where $\rho$ is an irreducible representation of $\bar{M}$ which affords $\chi$, $I$ is a $\chi(1) \times \chi(1)$ identity matrix.

If $\rho(a) = \omega I$, since $\bar{M}' = \bar{N}$, for any $g \in \bar{N}$, $a^{-1}a^g = h \in \bar{N}$. Therefore, $\rho(a) = \rho(a)\rho(h)$, and $\rho(h) = I$ follows. So we have that $h \in \ker \chi$.

(i) if $h \neq 1$, then $1 < \bar{N} \cap \ker \chi < \bar{M}$. $\bar{N}$ is a minimal normal subgroup of $\bar{M}$, implies that $\bar{N} \leq \ker \chi$. It means that $\chi$ may be viewed as an irreducible character of $\bar{M}/\bar{N} \cong \bar{P}_2$. But $\bar{P}_2$ is Abelian, hence $\chi$ is one degree irreducible character, it contradicts to $\chi(1) > 1$.

(ii) if $h = 1$, then $a \in C_G(\bar{N})$. Let $C_G(\bar{N}) = \bar{K} \times H (\bar{K} \leq \bar{P}_2)$, then $\bar{K}$ char $C_G(\bar{N}) \triangleleft \bar{G}$, we have $\bar{K} \triangleleft \bar{G}$. So $KZ_2 \triangleleft G$. The elements of $H$ and $KZ_2$ can commute since $H$ is $2'$-Hall normal subgroup of $G$. Thus any $x \in KZ_2$, $|x^G|$ is power of 2. By class equation, $2^s = |KZ_2| = 2 + 2^{n_1} + ..., + 2^{n_i}$. If $n_1 \geq 2$, we have a contradiction immediately. If $n_1 = 1$, then $KZ_2$ contains an element $u$ with $|u^G| = 2$. So $G$ contains another element $v$ with $|v^G| = 2$ since $G$ has property $P^n$. It is easy to see that $H \leq C_G(v)$ and $v \in KZ_2$. Hence $2^t = |KZ_2| = 2 + 2 + 2^{n_3} + ..., + 2^{n_i}$. If $2 \leq n_3 \leq ..., \leq n_i$, a contradiction.

Therefore, for any $a \in \bar{P}_2$, $a \neq 1$, $\chi(a) = 0$.

Now, we restrict $\chi$ on $\bar{P}_2$, we have

$$((\chi, \chi)_{\bar{P}_2} = \frac{1}{|\bar{P}_2|} \sum_{a \in \bar{P}_2} \chi(a)a^{-1} = \frac{\chi(1)}{|\bar{P}_2|}.$$}

Therefore, $\chi^2(1) = |\bar{P}_2|((\chi, \chi)_{\bar{P}_2} = 2^{n-1}(\chi, \chi)_{\bar{P}_2}$. Let $\chi = n_1\theta_1 + n_2\theta_2 + ... + n_t\theta_t$, where $\theta_i \in \text{Irr}(\bar{P}_2)$ and $n_i \geq 1$. Since $\theta_i(1) = 1$, $\chi(1) = n_1 + n_2 + ... + n_t$, by orthogonality relation of irreducible character, $(\chi, \chi)_{\bar{P}_2} = n_1^2 + n_2^2 + ... + n_t^2 \geq n_1 + n_2 + ... + n_t = \chi(1)$, that is $\chi^2(1) = 2^{n-1}(\chi, \chi)_{\bar{P}_2} \geq 2^{n-1} \chi(1)$. Thus, we can conclude that $\chi(1) \geq 2^{n-1}$. Since $\chi(1)$ is power of 2, we have $(\chi, \chi)_{\bar{P}_2}$ is power of 2.

$\bar{M}' = \bar{N}$ means that the number of one degree irreducible characters of $\bar{M}'$ is $|\bar{M}/\bar{M}'| = |\bar{P}_2| = 2^{n-1}$. Let $|N| = p^m$, then

$$2^{n-1}p^m = |\bar{M}| = |\bar{M}/\bar{M}'| + \sum_{\chi \in \text{Irr}(\bar{M})} \chi^2(1) = 2^{n-1} + \sum_{\chi \in \text{Irr}(\bar{M})} (\chi, \chi)_{\bar{P}_2}.$$}

Therefore, $p^m = 1 + \sum_{\chi \in \text{Irr}(\bar{M})} (\chi, \chi)_{\bar{P}_2}$. Since $(\chi, \chi)_{\bar{P}_2} \geq \chi(1) \geq 2^{n-1}$ and $(\chi, \chi)_{\bar{P}_2}$ is power of 2, $p^m = 1 + 2^{n-1}\sum_{i} 2^{n_i}$. Hence $2^{n-1} | p^m - 1.$
We have known before that \( N < G \), and \( N \leq Z(H) \). So for any \( x \in N \), \( |x^G| \) is power of 2.

Clearly, \( |x^G| = |(zx)^G| \), they are not conjugate because \( o(x) \neq o(zx) \). Since \( x \in N \) and \( zx \notin N \) and \( G \) has property \( P^* \), the distinct conjugacy classes of elements of \( N \) have distinct length.

So \( p^m = |N| = 1 + |x_1^G| + |x_2^G| + \ldots + |x_s^G| = 1 + 2m_1 + 2m_2 + \ldots + 2m_s \), \( m_1 < m_2 < \ldots < m_s \). Because \( 2^{n-1}p^m - 1 \) and \( |P_2| = |\tilde{P}_2||Z_2| = 2^n \), we have \( s = 1 \), \( m_1 = n - 1 \). That is \( p^m - 1 = 2^{n-1} \). But \( N = 1 \cup x^G \), it implies that \( \tilde{P}_2 \) acts fixed-point-free on \( N \) by conjugacy. By [7] 7.24, \( \tilde{P}_2 \) is a cyclic group or generalized quaternion group. Since \( \tilde{P}_2 \) is Abelian, \( \tilde{P}_2 = P_2/Z(G) \) is a cyclic group, which implies that \( P_2 \) is Abelian. It contradicts to our hypothesis of \( P_2 \) non-Abelian.

In fact, we have proven that \( P_2 \) is an Abelian subgroup.

3. Main theorem

**Theorem 3.1.** Suppose that \( G \) is a finite group with property \( P^* \), and \( G' \) is nilpotent, then \( G \cong Z_2, G \cong Z_2 \times S_3 \), or \( G \cong Z_3 \times Z_4 \).

**Proof.** Since \( G \) has property \( P^* \), \( Z(G) = Z_2 = \langle z \rangle, o(z) = 2 \). Clearly, \( G \cong Z_2 \), if \( G \) is Abelian.

If \( G \) is not Abelian, let \( \sum_1 = \{ x^G | x \in G, zx \notin x^G \}, \sum_2 = \{ x^G | x \in G, zx \notin x^G \} \). Then \( G = \sum_1 \cup \sum_2 \). Clearly, \( x^G \in \sum_2 \) if and only if \( (zx)^G \in \sum_2 \). Let \( \sum_2 = \sum_{21} \cup \sum_{22} \), where \( \sum_{21} = \{ x_1^G, x_2^G, \ldots, x_l^G \}, \sum_{22} = \{ (zx_1)^G, (zx_2)^G, \ldots, (zx_l)^G \} \).

Because \( |x_i^G| = |(zx_i)^G|, i = 1, 2, \ldots, l \), and \( G \) has property \( P^* \), the length of conjugacy classes of \( \sum_{21} \) are distinct.

Firstly, we assert that \( \sum_1 = \emptyset \). Otherwise, let \( x^G \in \sum_1 \). By definition of \( \sum_1 \), \( zx \notin x^G \), so there exists \( g \in G \) such that \( zx = x^g \). We decompose \( x \) as production of a 2-element \( a \) and a 2'-element \( b \), \( x = ab = ba \). Let \( o(a) = 2^s, o(b) = t, (2, t) = 1 \). Since \( (ab)^t = ((ab)^t) = za^t = (a^t)^y \), if let \( y = a^t \), then \( zy \in y^G \). Since \( a \) is a 2-element and \( t \) is odd, \( y \) is a 2-element. Let \( P_2 \in Syl_2(G), y \in P_2 \). Since \( P_2 \) is Abelian, \( |y^G| \) is a odd number. On the other hand, since \( zy \in y^G \), \( Z(G) \) acts on conjugacy class \( y^G \). Obviously, the action is faithful. Hence \( 2 \mid |y^G| \), it is a contradiction. Thus \( \sum_1 = \emptyset \).

Now, considering \( \overline{G} = G/Z(G) \), we will prove that every conjugacy classes in \( G/Z(G) \) has distinct length.

Let \( \bar{x} \in G/Z(G) \), since \( \sum_1 = \emptyset \), \( zx \notin x^G \). It is easy to see that \( \overline{C_G(x)} \leq \overline{C_G(\bar{x})} \).

Let \( \bar{g} \in \overline{C_G(\bar{x})} \), then we have \( x^g = x, or x^g = xz \). By \( \sum_1 = \emptyset \), we can conclude that
$x^g = x$, that is $g \in C_G(x)$. Therefore, $C_G(\bar{x}) = \overline{C_G(x)}$. We have known that the conjugacy classes of $\sum_{21}$ has distinct length. So if let $\sum_{21}^* = \{(\bar{x})^G | x^G \in \sum_{21}\}$, then the conjugacy classes of $\sum_{21}^*$ has distinct length too. Also since $G/Z(G) = \sum_{21}^* = \{(\bar{x})^G | x^G \in \sum_2\}$, by Lemma 2.1, $G/Z(G) \cong S_3$. Therefore, $G$ is a group with order 12. But all finite groups of order 12 are the following:

$$Z_{12}, Z_2 \times Z_2 \times Z_3, A_4, G \cong Z_2 \times S_3, G \cong Z_3 \rtimes Z_4.$$ 

It is easy to check that, $G \cong Z_2 \times S_3$ or $G \cong Z_3 \rtimes Z_4$. The proof is completed.

References


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