HYERS-ULAM STABILITY OF LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH CONSTANT COEFFICIENT

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Abstract. Y. Li and Y. Shen [12] have proved the Hyers-Ulam stability of differential equation $y''(x)+\alpha y'(x)+\beta y(x)=0$, in the condition that its characteristic equation has two different positive roots. In this paper, we prove that the differential equation $y''(x)+\alpha y'(x)+\beta y(x)=0$ has the Hyers-Ulam stability, no matter whether its characteristic roots are real or complex. Therefore the results obtained in this paper improve and extend the ones of [12].

Keywords: Hyers–Ulam stability; differential equations; characteristic equation.

2010 Mathematics Subject Classification: 34K20, 39A10.

1. Introduction

In 1940, Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin, in which he discussed a number of important unsolved problems (see [1]). Among those was the question concerning the stability of homomorphisms: Let G_1be a group and G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given any $\delta>0$, does there exist an $\varepsilon>0$ such that if a function $h: G_1\to G_2$ satisfies the inequality

$$d(h(xy),h(x)h(y))<\varepsilon \text{ for all } x,y\in G_1,$$

then there exists a homomorphism $H:\ G_1\to G_2$ with $d(h(x),H(x))<\delta$ for all $x\in G_1$?

The problem for the case of approximately additive mappings was solved by Hyers [2] when G_1 and G_2 are Banach spaces and the result of Hyers was generalized by Rassias (see [3]). Since then, the stability problems of functional equations have been extensively investigated by several mathematicians (see [3]-[5]).

C. Alsina and R. Ger were the first authors who investigated the Hyers-Ulam stability of differential equations. In 1998, they proved in [6] the following: Assume that a differential equation $f:\ I\to$ R is a solution of the differential inequality

$$|y'(t)-y(t)|\leq\varepsilon,$$

where I is an open subinterval of R. Then there exists a solution $f_0:\ I\to$ R of the differential equation $y'(t)=y(t)$ such that $|f(t)-f_0(t)|\leq\varepsilon$, for all $t\in\ I$.

The result of Hyers-Ulam stability for first-order linear differential equations has been generalized by many researchers (see [7]-[11]). Jung [11] proved the generalized Hyers-Ulam stability of differential equations of the form

$$ty'(t)+\alpha y(t)+\beta t^r x_0=0$$

and also applied this result to the investigation of the Hyers-Ulam stability of the differential equation

$$t^2y''(t)+\alpha ty'(t)+\beta y(t)=0.$$

Recently, Y. Li and Y. Shen [12] discussed the Hyers-Ulam stability of linear differential equations of second order $y''(x)+\alpha y'(x)+\beta y(x)=0$, in the condition that the characteristic equation $\lambda^2+\alpha\lambda+\beta=0$ has two positive roots.

The aim of this paper is to investigate the Hyers-Ulam stability of the following linear differential equations of second order.

$$y''(x)+\alpha y'(x)+\beta y(x)=0 \tag{1.1}$$

and

$$y''(x)+\alpha y'(x)+\beta y(x)=f(x), \tag{1.2}$$

where $y:\ [a,\ b]\to$ C is a twice continuously differentiable function, $f:\ [a,\ b]\to$ C is continuous function and $\alpha,\beta\in R$.

First of all, we give the definition of the Hyers-Ulam stability.

Definition 1.1. We say that equation (1.1) has the Hyers-Ulam stability if there exists a constant $K\geq 0$ with the following property, for every $\varepsilon>0$, if

$$||y''(x)+\alpha y'(x)+\beta y(x)||\leq\varepsilon,$$

then there exists some $u:\ [a,\ b]\to$ C which is a twice continuously differentiable function satisfying $u''(x)+\alpha u'(x)+\beta u(x)=0$ such that

$$||y(x)-u(x)||\leq K\varepsilon.$$

We call such K a Hyers-Ulam stability constant for equation (1.1).

2. Main results

In the following theorem, we prove the Hyers-Ulam stability of the differential equation (1.1).

Theorem 2.1. *Equation* (1.1) *has the Hyers-Ulam stability, where* $y : [a, b] \to \mathrm{C}$ *is a twice continuously differentiable function.*

Proof. Let $\varepsilon > 0$ and $y : [a, b] \to \mathrm{C}$ be a twice continuously differentiable function such that $||y'' + \alpha y' + \beta y|| \le \varepsilon$.

We will show that there exists a constant K independent of ε and y such that $||y - u|| \le K\varepsilon$ for some twice continuously differentiable function $u : [a, b] \to \mathrm{C}$ satisfying

$$u'' + \alpha u' + \beta u = 0.$$

Let λ_1 and λ_2 be the roots of the characteristic equation

$$\lambda^2 + \alpha\lambda + \beta = 0.$$

Define $g(x) = y'(x) - \lambda_1 y(x)$, then

$$\begin{aligned} ||g'(x) - \lambda_2 g(x)|| &= ||y''(x) - \lambda_1 y'(x) - \lambda_2 y'(x) + \lambda_1\lambda_2 y(x)|| \\ &= ||y''(x) + \alpha y'(x) + \beta y(x)|| \le \varepsilon. \end{aligned}$$

Let $Z(x) = e^{-\lambda_2(x-a)}g(x)$, for each $x \in [a, b]$, then

$$\begin{aligned} ||Z(x) - Z(t)|| &= ||e^{-\lambda_2(x-a)}g(x) - e^{-\lambda_2(t-a)}g(t)|| \\ &= \Big\| \int_t^x \frac{d}{dv}(e^{-\lambda_2(v-a)}g(v))dv \Big\| \\ &= \Big\| \int_t^x e^{-\lambda_2(v-a)}[g'(v) - \lambda_2 g(v)]dv \Big\| \\ &\le \varepsilon \Big| \int_t^x e^{||\lambda_2||(v-a)}dv \Big| \le e^{||\lambda_2||(b-a)}(b-a)\varepsilon, \end{aligned} \tag{2.1}$$

for any $x, t \in [a, b]$.

For any $x \in [a, b]$, it follows from (2.1) that

$$||g(x) - e^{\lambda_2(x-b)}g(b)|| = ||e^{\lambda_2(x-a)}(Z(x) - Z(b))|| \le e^{2||\lambda_2||(b-a)}(b-a)\varepsilon.$$

Let $g_1(x) = e^{\lambda_2(x-b)}g(b)$, then $g_1(x)$ satisfies

$$g_1'(x) - \lambda_2 g_1(x) = 0 \tag{2.2}$$

and

$$||g(x) - g_1(x)|| \le e^{2||\lambda_2||(b-a)}(b-a)\varepsilon.$$

Since $g(x) = y'(x) - \lambda_1 y(x)$, we have that

$$||y'(x) - \lambda_1 y(x) - g_1(x)|| \leq e^{2||\lambda_2||(b-a)}(b-a)\varepsilon.$$

Let $W(x) = e^{-\lambda_1(x-a)}y(x) - \int_a^x e^{-\lambda_1(v-a)}g_1(v)dv$, for each $x \in [a,b]$. By an argument similar to the above, we can show that there exists

$$u(x) = e^{\lambda_1(x-b)}y(b) - e^{\lambda_1(x-a)} \int_x^b e^{-\lambda_1(v-a)}g_1(v)dv$$

such that

$$||y-u|| \leq e^{2(||\lambda_2||+||\lambda_1||)(b-a)}(b-a)^2\varepsilon$$

and the twice continuously differentiable function $u : [a,b] \to \mathrm{C}$ satisfying

$$u'(x) - \lambda_1 u(x) = g_1(x). \tag{2.3}$$

Finally, it follows from (2.2) and (2.3) that $u'' + \alpha u' + \beta u = 0$. Thus, the proof is completed. ■

Theorem 2.2. *Assume that $f : [a,b] \to \mathrm{C}$ is continuous function such that $f(x)$ is integrable on $[a,c]$ for each $c \in [a,b]$. If a twice continuously differentiable function $y : [a,b] \to \mathrm{C}$ satisfies the differential inequality*

$$||y''(x) + \alpha y'(x) + \beta y(x) - f(x)|| \leq \varepsilon, \tag{2.4}$$

for all $x \in [a,\ b]$, then (2.4) has the Hyers-Ulam stability.

Proof. Similar to the proof of Theorem 2.1. Let λ_1 and λ_2 be the roots of characteristic equation $\lambda^2 + \alpha\lambda + \beta = 0$.

Define $g(x) = y'(x) - \lambda_1 y(x)$, we have

$$\begin{aligned} ||g'(x) - \lambda_2 g(x) - f(x)|| &= ||y''(x) - \lambda_1 y'(x) - \lambda_2 y'(x) + \lambda_1\lambda_2 y(x) - f(x)|| \\ &= ||y''(x) + \alpha y'(x) + \beta y(x) - f(x)|| \leq \varepsilon. \end{aligned}$$

Let $g_1(x) = e^{\lambda_2(x-b)}g(b) - e^{\lambda_2(x-a)} \int_x^b e^{-\lambda_2(v-a)}f(v)dv$, then $g_1(x)$ satisfies

$$g_1'(x) - \lambda_2 g_1(x) - f(x) = 0$$

and

$$||g(x) - g_1(x)|| \leq e^{2||\lambda_2||(b-a)}(b-a)\varepsilon.$$

Since $g(x) = y'(x) - \lambda_1 y(x)$, we have

$$||y'(x) - \lambda_1 y(x) - g_1(x)|| \leq e^{2||\lambda_2||(b-a)}(b-a)\varepsilon.$$

By an argument similar to Theorem 2.1, we can show that there exists

$$u(x) = e^{\lambda_1(x-b)}y(b) - e^{\lambda_1(x-a)} \int_x^b e^{-\lambda_1(v-a)} g_1(v)dv$$

such that

$$||y - u|| \leq e^{2(||\lambda_2||+||\lambda_1||)(b-a)}(b-a)^2\varepsilon$$

and the twice continuously differentiable function $u : [a, b] \to \mathrm{C}$ satisfies

$$u'' + \alpha u' + \beta u - f(x) = 0.$$

Thus, the proof is completed. ∎

Remark 2.3. The roots of characteristic equation $\lambda^2 + \alpha\lambda + \beta = 0$ of equation (1.1) can be real or complex.

Acknowledgements. This research was supported by Scientific Research Fund of Yunnan Provincial Education Department (No. 2013C157).

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Accepted: 19.02.2014