RIGHT ALTERNATIVE RINGS WITH $x(yz)-y(xz)$ IN THE CENTER

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Abstract. In [1] it was proved that if R is a prime right alternative ring of char. $\neq 2, 3$ with $(R, R, U) \subseteq U$ or $S(x^2, x, y) = 0$, then either $U = C$ or R is strongly $(-1, 1)$. In this paper first we prove that if R is a prime right alternative ring with $x(yz) - y(xz) \in U$, then $(R, R, U) \subseteq U$. Using this we prove that either $U = C$ or R is strongly $(-1, 1)$.

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1. Introduction

Throughout this paper R represents a right alternative ring of char. $\neq 2$. We denote $S(x, y, z) = (x, y, z)+(y, z, x)+(z, x, y)$. A right alternative ring satisfying the identity $[[R, R], R] = 0$ is called a strongly $(-1, 1)$ ring. The set C defined by $C = \{c \in N(R)/[c, R] = 0\}$ is called the center of R and $N_a(R) = \{v \in R/(x, x, v) = 0, \forall x \in R\}$ is called the alternative nucleus of R. A ring R is said to be commutative center if $U = \{u \in U|[u, R] = 0\}$.

In any right alternative ring we have the following identities:

$$(y,x,z)+(y,z,x)=0, \tag{1.1}$$

$$[xy,z]=x[y,z]+[x,z]y+2(x,y,z)+(z,x,y), \tag{1.2}$$

$$2S(x,y,z)=[[x,y],z]+[[y,z],x]+[[z,x],y], \tag{1.3}$$

$$(z,x^2,y)=(z,x,xy+yx), \tag{1.4}$$

$$(z,x,xy)=(z,x,y)x, \tag{1.5}$$

$$(wx,y,z)+(w,x,[y,z])=w(x,y,z)+(w,y,z)x, \tag{1.6}$$

$$([w,x],y,z)-[w,(x,y,z)]+[x,(w,y,z)]=(x,w,[y,z])-(w,x,[y,z]), \tag{1.7}$$

$$\left.\begin{aligned}((a,y,z),b,c)=((a,b,c),y,z)-(a,b,(c,y,z))-(a,(b,y,z),c)\\ +(a,b,c)[y,z]-(a,b,c[y,z])+(a,b,[y,z])c.\end{aligned}\right\} \tag{1.8}$$

For $u\in U$ the following identities and relations hold in R.[1]

$$2[x,(z,y,u)]=[x,(u,y,z)]=(x,[y,z],u), \tag{1.9}$$

$$([x,y],[z,w],u)=0, \tag{1.10}$$

$$(R,[R,R],U)\subseteq U, \tag{1.11}$$

$$(a,a,(x,y,u))=0, \tag{1.12}$$

$$((x,y,u),z,w)=2(w,z,(x,y,u)), \tag{1.13}$$

$$3(c,a,(a,y,u))=((a,a,c),y,u), \tag{1.14}$$

$$3[x,(x,z,(a,b,u))]=[(x,x,z),(a,b,u)]. \tag{1.15}$$

In a right alternative ring of char. $\neq 2$, we have $(R,R,U)\subseteq U$ if and only if

$$(R,[R,R],U)=0, \tag{1.16}$$

by (1.9). Also in any right alternative ring of char. $\neq 2$

$$(N(R)\cap U)=C. \tag{1.17}$$

First, we prove the following lemmas.

Lemma 1. *If R is a right alternative ring with $x(yz)-y(xz)\in U$, then $(R,R,U)\subseteq U$.*

Proof. By hypothesis, $x(yz)-y(xz)\in U$.

$$(xy)z-(x,y,z)+(y,x,z)-(yx)z\in U.$$

$$-(x,y,z)+(y,x,z)+[x,y]z\in U. \tag{1.18}$$

By using semi-jacobian identity, we get

$$-[xz, y] + x[z, y] + (x, z, y) \in U. \tag{1.19}$$

Now, by taking $y = u \in U$ and using the definition of U, we get $(x, z, u) \in U$. Therefore, $(R, R, U) \subseteq U$. ∎

Lemma 2. *If R is a right alternative ring with $x(yz) - y(xz) \in U$, then the ideal generated by $(R, R, U) \in R$ is $\langle (R, R, U) \rangle = (R, R, U) + R(R, R, U)$.*

Proof. Since $x(yz) - y(xz) \in U$, from Lemma 1, we have $(R, R, U)) \subseteq U$.

Now, we obviously have $(R, R, U)R = R(R, R, U)$ and

$$R(R(R, R, U)) \subseteq (R, R, (R, R, U)) + R^2(R, R, U) \subseteq (R, R, U) + R(R, R, U).$$

This and (1.1) gives

$$\begin{aligned}(R(R, R, U))R &\subseteq (R, (R, R, U), R) + R((R, R, U)R) \\ &\subseteq (R, R, (R, R, U)) + R(R(R, R, U)) \\ &\subseteq (R, R, U) + R(R, R, U).\end{aligned}$$

This proves Lemma 2. ∎

Lemma 3. *If R is a right alternative ring of char. $\neq 2$ with $x(yz) - y(xz) \in U$, then $K = x \in R/(x, R, U) = x(R, R, U) = 0$ is an ideal of R such that*

$$K\langle (R, R, U) \rangle = 0 \text{ and } [[R, R], R] \subseteq K.$$

Proof. We let $x \in K$. Using $U \subseteq Na(R)$ and (1.1), we first note that

$$0 = (x, R, U) = (R, x, U) = (x, U, R).$$

Since $(R, R, U) \subseteq U$, we have

$$\begin{aligned}(xR)(R, R, U) &= x(R(R, R, U)) = x((R, R, U)R) \\ &= (x(R, R, U))R = 0\end{aligned}$$

and

$$(Rx)(R, R, U) = R(x(R, R, U)) = 0.$$

Now, $(R, R, U) \subseteq U$ means $0 = (R, [x, R], U) = ([x, R], R, U)$ by using (1.16) and $U \subseteq N_a(R)$. Then this and (1.6) shows

$$(xR, R, U) = (Rx, R, U) \subseteq R(x, R, U) + (R, R, U)x + (R, x, [R, U]) = x(R, R, U) = 0.$$

Thus it follows that K is an ideal of R. Using this, Lemma 2, and $(R, R, U) \subseteq U$, we then see

$$K\langle (R, R, U) \rangle = K(R, R, U) + R(R, R, U) = (KR)(R, R, U) \subseteq K(R, R, U) = 0.$$

That is

$$K\langle (R,R,U)\rangle = 0.$$

Finally, we show that $[[R,R],R] \subseteq K$.

First, by $U \subseteq N_a(R)$ and (1.16), we have

$$([R,R],R],R,U) = -(R,[[R,R],R],U) = 0.$$

Then, by using (1.2), $(R,R,U) \subseteq U$, (1.16) and (1.6), we see

$$\begin{aligned}[[R,R],R](R,R,U) &\subseteq [[R,R](R,R,U),R] + [R,R][(R,R,U),R] \\ &\quad +2([R,R],(R,R,U),R) + (R,[R,R],(R,R,U)) \\ &= [[R,R](R,R,U),R] \\ &\subseteq [([R,R]R,R,U),R] + [([R,R],R,U)R,R] \\ &\quad +[([R,R],R,[R,U]),R] \\ &= [([R,R],R,U)R,R] = 0,\end{aligned}$$

by $U \subseteq N_a(R)$ and (1.16).

This completes the proof of the lemma. ∎

2. Main result

Theorem 1 *If R is a prime right alternative ring of char. $\neq 2$ with $x(yz) - y(xz) \in U$, then either $U = C$ or R is strongly $(-1,1)$.*

Proof. From Lemma 1 we have $(R,R,U) \subseteq U$.

Since $(R,R,U) \subseteq U$, from Lemma 3, we have

$$K\langle (R,R,U)\rangle = 0 \text{ and } [[R,R],R] \subseteq K.$$

Since R is prime, either $K = 0$ or $\langle (R,R,U)\rangle = 0$.

If $K = 0$, then $[[R,R],R] = 0$.

If $\langle (R,R,U)\rangle = 0$, then $U \subseteq U \cap N(R) = C$ by (1.17).

Thus either $U = C$ or R is strongly $(-1,1)$.

This proves the theorem. ∎

References

[1] Kleinfeld, E., Smith, H.F., *On centers and Nuclei in prime right alternative rings*, Comm. Algebra, 22 (3) (1994), 829-855.

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