A NEW APPROACH TO A CERTAIN GENERALIZED INTEGRAL TRANSFORM FOR CERTAIN SPACE OF BOEHMIANS

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Abstract. In this article, we introduce a generalization of Fourier and Hartely transforms. The transform we have obtained has been investigated on certain space of distributions. Two spaces of Boehmians are also established. The extended transform is then obtained and is well-defined, linear, one-to-one and onto mapping. More properties are also illustrated.

Keywords: $\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}$ Transform; Fourier Transform Pair; Distribution Space; Generalized Function; Boehmian; Hartley Transform Pair.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers and g be an integrable function defined on $\mathbb{R}$. The transform we consider in this article is given by the integral equation

$$(1)\qquad \iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(g)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\tau)\left(\alpha\cos(\gamma\xi\tau) + \beta\sin(\gamma\xi\tau)\right) d\tau.$$

The inversion formula is recovered from our transform (1) as

$$(2)\qquad g(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(g)(\xi)\left(\frac{\gamma}{\alpha}\cos(\gamma\xi\tau) + \frac{\gamma}{\beta}\sin(\gamma\xi\tau)\right) d\xi.$$

Let $\gamma = \alpha = \beta = 1$, then the shortness of (1) and (2) reduces to the Hartley transform pair [2], [8], [12]. On the other hand, a substitution of $\alpha = 1$, $\beta = i$, $\gamma = 1$, in (1) and (2) , describes a Fourier transform pair [9].

Denote by $E^{'}(\mathbb{R})$ the space of distributions of compact supports. Then, the extention $\widetilde{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}$ of $\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}$ to a distribution $g \in E^{'}(\mathbb{R})$ can be given as

$$\widetilde{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}(g)(\xi) = \frac{1}{\sqrt{2\pi}} \langle g(\tau), \alpha \cos(\gamma\xi\tau) + \beta \sin(\gamma\xi\tau) \rangle . \tag{3}$$

This definition is indeed well-defined by the smoothness of $\alpha \cos(\gamma\xi\tau)+\beta \sin(\gamma\xi\tau)$. Therefore, $\widetilde{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}$ justifies its following properties:

(i) $\widetilde{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}$ is linear;

(ii) $\widetilde{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}$ is continuous;

(iii) $\widetilde{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}$ is analytic.

The justification of those properties follows in fact from basic properties of distributions.

In what follows, we spread the discussion to further space of Boehmians over two sections.

In Section 2, we construct the image and preimage spaces of Boehmians. In Section 3, we discuss the transform in the context of Boehmian spaces and obtain some properties.

2. The constructed spaces of Boehmian

We assume the reader is acquainted with the concept of Boehmian spaces. For further constructions, reader can check the citations [1], [3], [4], [6], [7], [10], [11], [13] of this article.

Let us first define some auxiliary mappings that are useful to our next investigation of Boehmian spaces.

Denote by D the Schwartz space of test functions of bounded support. Then, for every $g \in E^{'}$ and $v \in D$ we introduce the operation $\bullet$ defined by

$$(g \bullet v)\,\ell(\varsigma) = g(v * \ell)(\varsigma), \tag{4}$$

where $\ell \in E$, and $*$ is the usual convolution product of two functions of first kind [9], [5].

To establish the first space of Boehmians we are requested to establish the following theorems.

Theorem 1. *Let $g \in E^{'}$ and $v \in D$; then we have $g \bullet v \in E^{'}$.*

Proof. Let $\ell \in E$ and $\mathbb{K}$ be a compact subset of $\mathbb{R}$; then, by (4), we have

$$(g \bullet v)\,\ell(\varsigma) = g(v * \ell)(\varsigma). \tag{5}$$

So, to prove the theorem, it is sufficient to show that $v*\ell \in E$. By the property [9],

$$D^k f * g = D^k f * g$$

we derive that

$$\frac{\mathtt{d}^k}{\mathtt{d}\varsigma^k}(v*\ell)(\varsigma) = v * \frac{\mathtt{d}^k}{\mathtt{d}\varsigma^k}\ell(\varsigma).$$

Hence, allowing $\mathbb{K}$ traverse the compact subsets of $\mathbb{R}$ implies

$$\begin{aligned}\sup_{\varsigma\in\mathbb{K}}\left|\frac{\mathtt{d}^k}{\mathtt{d}\varsigma^k}(v*\ell)(\varsigma)\right| &= \sup_{\varsigma\in\mathbb{K}}\left|v*\tfrac{\mathtt{d}^k}{\mathtt{d}\varsigma^k}\ell(\varsigma)\right| \\ &\leq \eta^*\int_{\mathbb{K}}\left|\frac{\mathtt{d}^k}{\mathtt{d}\varsigma^k}\ell(\varsigma-x)\right|dx,\end{aligned}$$

η^* is certain positive constant.

This gives

$$\sup_{\varsigma\in\mathbb{K}}\left|\frac{\mathtt{d}^k}{\mathtt{d}\varsigma^k}(v*\ell)(\varsigma)\right| < \infty.$$

Hence, $v*\ell \in E$. Thus, $g \bullet \ell \in E'$.

This completes the proof of the theorem. ∎

Theorem 2. *Let $g_1, g_2 \in E'$ and $\ell \in D, \eta \in \mathbb{C}$; then we have*

$$\eta(g_1+g_2)\bullet\ell = (\eta g_1 + \eta g_2)\bullet\ell.$$

The proof of this theorem is straightforward. Hence, we prefer to omit the details. ∎

Theorem 3. *Let $g_n \to g$ in E' and $v \in D$; then $g_n \bullet v \to g \bullet v$ as $n \to \infty$.*

Proof. For $\ell \in E$, we can write

$$(g_n \bullet v - g\bullet v)\ell = ((g_n - g)\bullet v)\ell = (g_n - g)(v*\ell). \tag{6}$$

Right hand side of equation (6) is well defined by Theorem 1.

Considering the limit as $n \to \infty$, the right hand side of (6) tends to 0 as $n \to \infty$.

Therefore,

$$(g_n \bullet v \to g \bullet v)\ell \to 0$$

as $n \to \infty$.

Hence $g_n \bullet v \to g \bullet v$ as $n \to \infty$.

The proof of the theorem is completed. ∎

As final in this construction, we merely need to establish the following theorem.

Theorem 4. *Let $g \in E'$ and $(\delta_n) \in \Delta$; then we have $g \bullet \delta_n \to g$ as $n \to \infty$.*

Proof. Let $\ell \in E$ and $(\delta_n) \in \Delta$; then, since (δ_n) is delta sequence, we have $\delta_n * \ell$ as $n \to \infty$. Hence,

$$(g \bullet \delta_n)\, \ell = g\,(\delta_n * \ell) \to g \text{ as } n \to \infty. \tag{7}$$

Thus our theorem is completely proved. ∎

The space $\boldsymbol{B}\left(E', D, \Delta, \bullet\right)$ is therefore considered as a Boehmian space.

The sum and multiplication by a scalar of two Boehmians can be defined in a natural way

$$\left[\frac{f_n}{\epsilon_n}\right] + \left[\frac{g_n}{\tau_n}\right] = \left[\frac{f_n \bullet \tau_n + g_n \bullet \epsilon_n}{\epsilon_n \bullet \tau_n}\right]$$

and

$$\eta \left[\frac{f_n}{\epsilon_n}\right] = \left[\frac{\eta f_n}{\epsilon_n}\right], \eta \in \mathbb{C}.$$

The operation $\bullet$ and the differentiation are defined by

$$\left[\frac{f_n}{\epsilon_n}\right] \bullet \left[\frac{g_n}{\tau_n}\right] = \left[\frac{f_n \bullet g_n}{\epsilon_n \bullet \tau_n}\right] \text{ and } D^\alpha \left[\frac{f_n}{\epsilon_n}\right] = \left[\frac{D^\alpha f_n}{\epsilon_n}\right].$$

If $\left[\frac{f_n}{\epsilon_n}\right] \in \boldsymbol{B}\left(E', D, \Delta, \bullet\right)$ and $\phi \in D$, then we have

$$\left[\frac{f_n}{\epsilon_n}\right] \bullet \phi = \left[\frac{f_n \bullet \phi}{\epsilon_n}\right].$$

A sequence of Boehmians (β_n) in $\boldsymbol{B}\left(E', D, \Delta, \bullet\right)$ is said to be δ-convergent to a Boehmian β in $\boldsymbol{B}\left(E', D, \Delta, \bullet\right)$, denoted by $\beta_n \xrightarrow{\delta} \beta$, if there exists a delta sequence (ϵ_n) such that $(\beta_n \bullet \epsilon_n), (\beta \bullet \epsilon_n) \in E', \forall k, n = 1, 2, 3, ...,$and

$$(\beta_n \bullet \epsilon_k) \to (\beta \bullet \epsilon_k) \text{ as } n \to \infty, \text{in } E', \; k = 1, 2, 3, ...$$

The following lemma is equivalent for the statement of δ-convergence:

$\beta_n \xrightarrow{\delta} \beta$ (as $n \to \infty$) in $\boldsymbol{B}\left(E', D, \Delta, \bullet\right)$ if and only if there is $f_{n,k}, f_k \in E'$ and $(\epsilon_k) \in \Delta$ such that $\beta_n = \left[\frac{f_{n,k}}{\epsilon_k}\right], \beta = \left[\frac{f_k}{\epsilon_k}\right]$ and for each $k = 1, 2, 3, ...,$

$$f_{n,k} \to f_k \text{ as } n \to \infty \text{ in } E'.$$

A sequence (β_n) of Boehmians in $\boldsymbol{B}\left(E', D, \Delta, \bullet\right)$ is said to be *a* Δ-convergent to a Boehmian β in $\boldsymbol{B}\left(E', D, \Delta, \bullet\right)$, denoted by $\beta_n \xrightarrow{\Delta} \beta$, if there exists a $(\epsilon_n) \in \Delta$ such that

$$(\beta_n - \beta) \bullet \epsilon_n \in E',$$

$\forall n = 1, 2, 3, ...$ and $(\beta_n - \beta) \bullet \epsilon_n \to 0$ as $n \to \infty$ in E'.

Let us now consider another space of Boehmians.

Let W be the space of $\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}$ transforms of distributions in E'; then we define a product $\circledast$ as

$$(w \circledast v)(\xi) = \int_{\mathbb{R}} w(\xi + x)\, v(x)\, dx, \tag{8}$$

$\xi \in \mathbb{R}$. Then, (8) can be simply written as

$$(w \circledast v)(\xi) = v * \widetilde{w}(\xi),$$

where $\widetilde{w}(\xi) = w(-\xi)$.

Theorem 5. *Let $g \in E', v \in D$; then*

$$\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(g \bullet v) = \iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(g) \circledast v.$$

Proof. Over compact subsets $\mathbb{K}$ of $\mathbb{R}$ we by (3) have that

$$\begin{aligned} \iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(g \bullet v)(\xi) &= \langle (g \bullet v)(\tau), \alpha \cos \gamma\xi\tau + \beta \sin \gamma\xi\tau \rangle \\ \text{i.e.} \quad &= \langle g(\tau), \langle v(x), \alpha \cos \gamma\xi(\tau + x) + \beta \sin \gamma\xi(\tau + x)\rangle\rangle \\ \text{i.e.} \quad &= \int_{\mathbb{R}} \langle g(\tau), \alpha \cos \gamma\xi(\tau + x) + \beta \sin \gamma\xi(\tau + x)\rangle\, v(x)\, dx. \end{aligned}$$

Therefore, we have obtained

$$\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(g \bullet v)(\xi) = \left(\iota_{\alpha,\mathbb{R}}^{\beta,\gamma} g \circledast v\right)(\xi).$$

This completes the proof of the theorem. ∎

Theorem 6. *Let $w \in W, v \in D$; then $w \circledast v \in W$.*

Proof. The assumption that $w \in W$ implies $w = \iota_{\alpha,\mathbb{R}}^{\beta,\gamma} g$ for some $g \in E'$.

Therefore, we get

$$w \circledast v = \iota_{\alpha,\mathbb{R}}^{\beta,\gamma} g \circledast v. \tag{9}$$

By Theorem 5, equation (9) gives

$$w \circledast v = \iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(g \bullet v). \tag{10}$$

Theorem 1, therefore, implies $w \circledast v \in E'$.

This completes the proof of the theorem. ∎

The proofs of the following two theorems have similar techniques.

Theorem 7. *$w_n, w \in W, (\delta_n) \in \Delta$ and $v \in D$. Then we get*

(1) *Let* $w_n \to w$ *in* W *and* $v \in D$; *then* $w_n \circledast v \to w \circledast v$.

(2) *Let* $w \in W$ *and* $(\delta_n) \in \Delta$; *then* $w \circledast \delta_n \to w$ *as* $n \to \infty$.

Theorem 8. *Let* $w_1, w_2 \in W, v \in D$; *then for* $\eta_1, \eta_2 \in \mathbb{C}$, *we get that*

$$(\eta_1^* w_1 + \eta_2^* w_2) \circledast v = \eta_1^* (w_1 \circledast v) \times \eta_2^* (w_2 \circledast v)$$

where $v \in D$.

The Boehmian space $\boldsymbol{B}(W, D, \Delta, \circledast)$ is therefore constructed.

Addition, multiplication by scalars and convergence on $\boldsymbol{B}(W, D, \Delta, \circledast)$ are similar to that of $\boldsymbol{B}\left(E', D, \Delta, \bullet\right)$. Hence details are avoided.

3. The generalized $\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}$ transform of Boehmians

Let $\left[\frac{g_n}{\delta_n}\right] \in \boldsymbol{B}\left(E', D, \Delta, \bullet\right)$; then, we define the generalized transform $\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}$ of $\left[\frac{g_n}{\delta_n}\right]$ as

$$\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}\left[\frac{g_n}{\delta_n}\right] = \left[\frac{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(g_n)}{\delta_n}\right] \tag{11}$$

in the space $\boldsymbol{B}(W, D, \Delta, \circledast)$.

Theorem 9. *The mapping* $\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}$ *is well-defined.*

Proof. Let $\left[\frac{f_n}{\epsilon_n}\right] = \left[\frac{g_n}{\tau_n}\right] \in \boldsymbol{B}\left(E', D, \Delta, \bullet\right)$; then we have

$$\frac{f_n}{\epsilon_n} \text{ is equivalent to } \frac{g_n}{\tau_n} \text{ in } \boldsymbol{B}\left(E', D, \Delta, \bullet\right).$$

Therefore, $f_n \bullet \tau_m = g_m \bullet \epsilon_n, \forall m, n \in N$. The action of $\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}$ jointly with Theorem 5 imply

$$\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(f_n) \circledast \tau_m = \iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(g_m) \circledast \epsilon_n,$$

$\forall m, n \in N$. Hence

$$\frac{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(f_n)}{\epsilon_n} \text{ is equivalent to } \frac{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(g_n)}{\tau_n} \text{ in } \boldsymbol{B}(W, D, \Delta, \circledast).$$

Hence, we have obtained

$$\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}\left[\frac{f_n}{\epsilon_n}\right] = \widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}\left[\frac{g_n}{\tau_n}\right].$$

This completes the proof of the theorem. ■

Theorem 10. *The mapping* $\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}$ *is linear.*

Proof. Let $\left[\frac{f_n}{\epsilon_n}\right], \left[\frac{g_n}{\tau_n}\right] \in \boldsymbol{B}\left(E', D, \Delta, \bullet\right), \kappa, \eta \in \mathbb{R}$; then

$$\begin{aligned}
\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}\left(\kappa\left[\frac{f_n}{\epsilon_n}\right]+\eta\left[\frac{g_n}{\tau_n}\right]\right) &= \widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}\left(\left[\frac{\kappa f_n}{\epsilon_n}\right]+\left[\frac{\eta g_n}{\tau_n}\right]\right) \\
\text{i.e.} \qquad &= \widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}\left(\left[\frac{(\kappa f_n)\bullet\tau_n+(\eta g_n)\bullet\epsilon_n}{\epsilon_n\bullet\tau_n}\right]\right) \\
\text{i.e.} \qquad &= \left[\frac{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}\left((\kappa f_n)\bullet\tau_n+(\eta g_n)\bullet\epsilon_n\right)}{\epsilon_n\bullet\tau_n}\right].
\end{aligned}$$

Linearity of $\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}$ and (11) imply

$$\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}\left(\kappa\left[\frac{f_n}{\epsilon_n}\right]+\eta\left[\frac{g_n}{\tau_n}\right]\right)=\left[\frac{\kappa\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}\left(f_n\bullet\tau_n\right)+\eta\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}\left(g_n\bullet\epsilon_n\right)}{\epsilon_n\circledast\tau_n}\right].$$

Theorem 5 gives

$$\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}\left(\kappa\left[\frac{f_n}{\epsilon_n}\right]+\eta\left[\frac{g_n}{\tau_n}\right]\right)=\left[\frac{\kappa\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}\left(f_n\right)\circledast\tau_n+\eta\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}\left(g_n\right)\circledast\epsilon_n}{\epsilon_n\circledast\tau_n}\right].$$

Thus, we have

$$\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}\left(\kappa\left[\frac{f_n}{\epsilon_n}\right]+\eta\left[\frac{g_n}{\tau_n}\right]\right)=\left[\frac{\kappa\eta_{\alpha,\mathbb{R}}^{\beta,\gamma}\left(f_n\right)}{\epsilon_n}\right]+\left[\frac{\eta\eta_{\alpha,\mathbb{R}}^{\beta,\gamma}\left(g_n\right)}{\tau_n}\right]$$

Hence, we got that

$$\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}\left(\kappa\left[\frac{f_n}{\epsilon_n}\right]+\eta\left[\frac{g_n}{\tau_n}\right]\right)=\kappa\left[\frac{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}\left(f_n\right)}{\epsilon_n}\right]+\eta\left[\frac{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}\left(g_n\right)}{\tau_n}\right]$$

The theorem is completely proved. ■

Theorem 11. *The mapping* $\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}$ *is an isomorphism from* $\boldsymbol{B}\left(E', D, \Delta, \bullet\right)$ *into* $\boldsymbol{B}\left(W, D, \Delta, \circledast\right)$.

Proof. We first prove that $\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}$ is injective mapping from $\boldsymbol{B}\left(E', D, \Delta, \bullet\right)$ into $\boldsymbol{B}\left(W, D, \Delta, \circledast\right)$.

Assume that

$$\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}\left[\frac{f_n}{\epsilon_n}\right]=\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}\left[\frac{g_n}{\tau_n}\right]$$

in $\boldsymbol{B}(W, D, \Delta, \circledast)$.

Then, by (11), we have

$$\left[\frac{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(f_n)}{\epsilon_n}\right] = \left[\frac{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(g_n)}{\tau_n}\right].$$

Hence, we derive that

$$\frac{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(f_n)}{\epsilon_n} \text{ and } \frac{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(g_n)}{\tau_n}$$

are equivalent quotients in $\boldsymbol{B}(W, D, \Delta, \circledast)$.

Thus, the concept of equivalent classes of $\boldsymbol{B}(W, D, \Delta, \circledast)$ implies

$$\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(f_n) \circledast \tau_m = \iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(g_m) \circledast \epsilon_n.$$

Theorem 5 then gives

$$\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(f_n \circledast \tau_n) = \iota_{\alpha,\mathbb{R}}^{\beta,\gamma}(g_n \circledast \epsilon_n).$$

Injectivity of $\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}$ and the concept of equivalent classes of $\boldsymbol{B}\left(E', D, \Delta, \bullet\right)$ implies

$$\frac{f_n}{\epsilon_n} \text{ are equivalent } \frac{g_n}{\tau_n} \text{ in } \boldsymbol{B}\left(E', D, \Delta, \bullet\right).$$

Therefore $\left[\frac{f_n}{\epsilon_n}\right]$ and $\left[\frac{g_n}{\tau_n}\right]$ are equivalent.

This proves the first part.

Surjectivity of $\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}$ is obvious.

The theorem is completely proved. ∎

The proofs of the following theorems are obvious.

Theorem 12. *If* $\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}\left[\frac{g_n}{\delta_n}\right] = 0$, *then*

$$\left[\frac{g_n}{\delta_n}\right] = 0$$

in the sense of $\boldsymbol{B}\left(E', D, \Delta, \bullet\right)$.

Theorem 13. *If* (β_n) *is sequence in* $\boldsymbol{B}\left(E', D, \Delta, \bullet\right)$ *such that* $\beta_n \overset{\Delta}{\to} \beta$ *as* $n \to \infty$, *then*

$$\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}(\beta_n) \xrightarrow{\Delta} \widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}(\beta)$$

as $n \to \infty$ *in* $\boldsymbol{B}(W, D, \Delta, \circledast)$ *on compact subsets.*

Theorem 14. *Let* $\left[\frac{g_n}{\delta_n}\right] \in \boldsymbol{B}\left(E', D, \Delta, \bullet\right)$; *then*

$$\widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}\left(\left[\frac{g_n}{\delta_n}\right] \circledast \epsilon_n\right) = \widehat{\iota_{\alpha,\mathbb{R}}^{\beta,\gamma}}\left(\epsilon_n \circledast \left[\frac{g_n}{\delta_n}\right]\right),$$

$(\epsilon_n) \in \Delta$.

Readers can check the proofs from the citations given by the same author.

References

[1] Al-Omari, S.K.Q., Kilicman, A., *Some Remarks on the Extended Hartley-Hilbert and Fourier-Hilbert Transforms of Boehmians*, Abstract and Applied Analysis, vol. 2013 (2013), 1-6.

[2] Al-Omari, S.K.Q., *Notes for Hartley Transforms of Generalized Functions*, Italian J. Pure Appl. Math., 28 (2011), 21-30.

[3] Al-Omari, S.K.Q., Kilicman, A., *On Diffraction Fresnel Transforms for Boehmians*, Abstract and Applied Analysis, Volume 2011, 1-15.

[4] Boehme, T.K., *The Support of Mikusinski Operators*, Tran. Amer. Math. Soc.176 (1973), 319-334.

[5] Banerji, P.K., Al-Omari, S.K.Q., Debnath, L., *Tempered Distributional Fourier Sine (Cosine) Transform*, Integ. Trans. Spl. Funct., 17, 11 (2006), 759-768.

[6] Al-Omari, S.K.Q., Kilicman, A., *Note on Boehmians for Class of Optical Fresnel Wavelet Transforms*, Journal of Function Spaces and Applications, vol. 2012 (2012), 1-12.

[7] Mikusinski, P., *Convergence of Boehmians,* Japanese Journal of Mathematics, 9 (1) (1983), 159–179.

[8] Millane, R.P., *Analytic Properties of the Hartley Transform and their Applications*, Proc. IEEE, 82 (3) (1994), 413-428.

[9] Pathak, R.S., *Integral Transforms of Generalized Functions and their Applications*, Gordon and Breach Science Publishers, Australia, Canada, India, Japan, 1997.

[10] Al-Omari, S.K.Q., Loonker D., Banerji P.K., Kalla, S.L., *Fourier Sine (Cosine) Transform for Ultradistributions and their Extensions to Tempered and UltraBoehmian Spaces*, Integ. Trans. Spl. Funct., 19 (6) (2008), 453–462.

[11] Al-Omari, S.K.Q., Kilicman, A., *On Generalized Hartley-Hilbert and Fourier-Hilbert Transforms*, Advances in Difference Equations, vol. 2012 (2012), 232-244.

[12] Bracewell, R.N., *The Hartley Transform,* New York, Oxford Univ., 1986.

[13] Al-Omari, S.K.Q., Kilicman, A., *An Estimate of Sumudu Transform for Boehmians*, Advances in Difference Equations, vol. 2013, (2013), 1-11.

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