SOME CHARACTERIZATIONS OF INTRA-REGULAR ABEL GRASSMANN'S GROUPOIDS

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Abstract. In this paper, we give some characterizations of a new non-associative structure, namely intra-regular AG-groupoids by the properties of its $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals.

Keywords and phrases: AG-groupoid, left invertive law, medial law, paramedial law and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal.

1. Introduction

The fuzzy set theory was developed by Zadeh in 1965 [15]. This theory plays an important role in the real life problems involving uncertainties. Fuzzy set theory can be applied to several basic notations of algebras. Rosenfeld in 1971, introduced the concept of fuzzy set theory in groups [13]. Mordeson et, al. [7] have discussed the applications of fuzzy set theory in fuzzy coding, fuzzy automata and finite state machines. In today's world, many theories have been developed to deal with such uncertainties like fuzzy set theory, theory of vague sets, theory of soft ideals, theory of intuitionistic fuzzy sets and theory of rough sets. The theory of soft sets (see [4, 5]) has many applications in different fields such as the smoothness of functions, game theory, operations research, Riemann integration etc.

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Fuzzy set theory on semigroups has already been developed. In [8] Murali defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set was defined in [10]. Bhakat and Das [1], [2] gave the concept of (α, β)-fuzzy subgroups by using the "belongs to" relation $\in$ and "quasi-coincident with" relation q between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroups, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. In [12] regular semigroups were characterized by the properties of their $(\in, \in \vee q)$-fuzzy ideals. In [11] semigroups were characterized by the properties of their $(\in, \in \vee q)$-fuzzy ideals.

An AG-groupoid is a mid structure between a groupoid and a commutative semigroup. Mostly it works like a commutative semigroup. For instance, $a^2b^2 = b^2a^2$, for all a, b holds in a commutative semigroup, while this equation also holds for an AG-groupoid with left identity e, moreover $ab = (ba)e$ for any subset $\{a, b\}$ of an AG-groupoid. Now our aim is to discover some logical investigations for regular and intra-regular AG-groupoids using the new generalized concept of fuzzy sets.

In this paper, we discuss the $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals in a new non-associative algebraic structure, that is, in AG-groupoids and develop some new results. We characterize intra-regular AG-groupoids by the properties of their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals.

2. Preliminaries

A groupoid $(S, .)$ is called an AG-groupoid (LA-semigroup in some articles [9]), if its elements satisfy left invertive law: $(ab)c = (cb)a$. In an AG-groupoid medial law [3], $(ab)(cd) = (ac)(bd)$, holds for all a, b, c, $d \in S$. It is also known that in an AG-groupoid with left identity, the paramedial law: $(ab)(cd) = (db)(ca)$, holds for all a, b, c, $d \in S$. If an AG-groupoid contains left identity, the following law holds,

$$a(bc) = b(ac), \text{for all } a, b, c \in S. \tag{1}$$

Let S be an AG-groupoid. By AG-subgroupoid of S we means a non-empty subset A of S such that $A^2 \subseteq A$, and by a left (right) ideal of S we mean a non-empty subset L of S such that $SL \subseteq L$ $(LS \subseteq L)$. By two-sided ideals or simply ideal, we mean a non-empty subset of S which is both a left and a right ideal of S. An AG-subgroupoid B of S is called a bi-ideal of S if $(BS)B \subseteq B$. A non-empty subset B of S is called a generalized bi-ideal of S if $(BS)B \subseteq B$.

A fuzzy subset f of a given set S is described as an arbitrary function $f : S \to [0, 1]$, where $[0, 1]$ is the usual closed interval of real numbers. For any two fuzzy subsets f and g of S, $f \leq g$ means that, $f(x) \leq g(x)$ for all x in S. Let f and g be any fuzzy subsets of an AG-groupoid S, then the product $f \circ g$ is defined by

$$(f \circ g)(a) = \begin{cases} \bigvee_{a=bc} \{f(b) \wedge g(c)\} & \text{if there exist } b,\ c \in S, \text{ such that } a = bc, \\ 0 & \text{otherwise.} \end{cases}$$

The following definitions are available in [7].

A fuzzy subset f of an AG-groupoid S is called a fuzzy AG-subgroupoid of S if $f(xy) \geq f(x) \wedge f(y)$ for all $x, y \in S$.

A fuzzy subset f of an AG-groupoid S is called a fuzzy left (right) ideal of S if $f(xy) \geq f(y)$ $(f(xy) \geq f(x))$ for all $x, y \in S$.

A fuzzy subset f of an AG-groupoid S is called a fuzzy ideal of S if it is both fuzzy left and fuzzy right ideal of S.

A fuzzy subset f of an AG-groupoid S is called a fuzzy generalized bi-ideal of S if $f((xy)z) \geq f(x) \wedge f(z)$, for all x, y and $z \in S$.

A fuzzy generalized bi-ideal f of S is called a fuzzy bi-ideal of S if it is fuzzy AG-subgroupoid of S. Let $F(S)$ denote the collection of all fuzzy subsets of an AG-groupoid S with left identity, then $(F(S), \circ)$ becomes an AG-groupoid with left identity S, that is $(F(S), \circ)$ satisfies left invertive law, medial law, paramedial law and property (1). Note that S can be considered as a fuzzy subset of S itself and we write $S = C_S$, that is, $S(x) = 1$ for all $x \in S$. Moreover, $S \circ S = S$.

Definition 1 [15] Let X be a non-empty set. A fuzzy subset f of X is define as a mapping from X into $[0, 1]$, where $[0, 1]$ is the usual interval of real numbers. We denote by $\mathcal{F}(X)$ the set of all fuzzy subsets of X.

A fuzzy subset f of X of the form

$$f(y) = \begin{cases} r(\neq 0) & \text{if } y = x, \\ 0 & \text{otherwise,} \end{cases}$$

is said to be a fuzzy point with support x and value r and is denoted by x_r, where $r \in (0, 1]$.

3. $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals in AG-groupoids

The following definitions are available in [14].

In what follows, let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. For any $B \subseteq A$, we define $X^\delta_{\gamma B}$ be the fuzzy subset of X by $X^\delta_{\gamma B}(x) \geq \delta$ for all $x \in B$ and $X^\delta_{\gamma B}(x) \leq \gamma$ otherwise. Clearly, $X^\delta_{\gamma B}$ is the characteristic function of B if $\gamma = 0$ and $\delta = 1$.

For a fuzzy point x_r and a fuzzy subset f of X, we say that

(1) $x_r \in_\gamma f$ if $f(x) \geq r > \gamma$.
(2) $x_r q_\delta f$ if $f(x) + r > 2\delta$.
(3) $x_r \in_\gamma \vee q_\delta f$ if $x_r \in_\gamma f$ or $x_r q_\delta f$.

Now, we introduce a new relation on $\mathcal{F}(X)$, denoted as "$\subseteq \vee q_{(\gamma,\delta)}$", as follows.

For any $f, g \in \mathcal{F}(X)$, by $f \subseteq \vee q_{(\gamma,\delta)} g$ we mean that $x_r \in_\gamma f$ implies $x_r \in_\gamma \vee q_\delta g$ for all $x \in X$ and $r \in (\gamma, 1]$. Moreover, f and g are said to be (γ, δ)-equal, denoted by $f =_{(\gamma,\delta)} g$, if $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} f$.

The proofs of the following lemmas are the same as in [14].

Lemma 1 *Let f and g are fuzzy subsets of $\mathcal{F}(X)$. Then $f \subseteq \vee q_{(\gamma,\delta)} g$ if and only if* $\max\{f(x), \gamma\} \geq \min\{g(x), \delta\}$ *for all $x \in X$.*

Lemma 2 *Let $f, g, h \in \mathcal{F}(X)$. If $f \subseteq \vee q_{(\gamma,\delta)} g$ and $g \subseteq \vee q_{(\gamma,\delta)} h$, then $f \subseteq \vee q_{(\gamma,\delta)} h$.*

It is shown in [14] that "$=_{(\gamma,\delta)}$" is an equivalence relation on $\mathcal{F}(X)$. It is also notified that $f =_{(\gamma,\delta)} g$ if and only if $\max\{\min\{f(x), \delta\}, \gamma\} = \max\{\min\{g(x), \delta\}, \gamma\}$ for all $x \in X$.

Lemma 3 *Let A, B be any non empty subsets of an AG-groupoid S with left identity. Then we have*

(1) *$A \subseteq B$ if and only if $X^\delta_{\gamma A} \subseteq \vee q_{(\gamma,\delta)} X^\delta_{\gamma B}$, where $r \in (\gamma, 1]$ and $\gamma, \delta \in [0, 1]$.*
(2) $X^\delta_{\gamma A} \cap X^\delta_{\gamma B} =_{(\gamma,\delta)} X^\delta_{\gamma(A\cap B)}$.
(3) $X^\delta_{\gamma A} \circ X^\delta_{\gamma B} =_{(\gamma,\delta)} X^\delta_{\gamma(AB)}$.

Proof. It is the same in [14]. ■

Definition 2 A fuzzy subset f of an AG-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy AG-subgroupoid of S if for all $x, y \in S$ and $t, s \in (\gamma, 1]$, it satisfies $x_t \in_\gamma f$, $y_s \in_\gamma f$ implies that $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.

Theorem 1 *Let f be a fuzzy subset of an AG groupoid S. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy AG subgroupoid of S if and only if* $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$, *where $\gamma, \delta \in [0, 1]$.*

Proof. Let f be a fuzzy subset of an AG-groupoid S which is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subgroupoid of S. Assume that there exist $x, y \in S$ and $t \in (\gamma, 1]$, such that

$$\max\{f(xy), \gamma\} < t \leq \min\{f(x), f(y), \delta\}.$$

Then $\max\{f(xy), \gamma\} < t$. This implies that $f(xy) < t$, which further implies that $(xy)_{\min t} \overline{\in_\gamma \vee q_\delta} f$ and $\min\{f(x), f(y), \delta\} \geq t$. Therefore $\min\{f(x), f(y)\} \geq t$ $\Rightarrow f(x) \geq t > \gamma$, $f(y) \geq t > \gamma$, implies that $x_t \in_\gamma f$, $y_s \in_\gamma f$. But $(xy)_{\min\{t,s\}} \overline{\in_\gamma \vee q_\delta} f$ a contradiction to the definition. Hence

$$\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\} \text{ for all } x, y \in S.$$

Conversely, assume that there exist $x, y \in S$ and $t, s \in (\gamma, 1]$ such that $x_t \in_\gamma f$, $y_s \in_\gamma f$ but $(xy)_{\min\{t,s\}} \overline{\in_\gamma \vee q_\delta} f$, then $f(x) \geq t > \gamma$, $f(y) \geq s > \gamma$, $f(xy) < \min\{f(x), f(y), \delta\}$ and $f(xy) + \min\{t, s\} \leq 2\delta$. It follows that $f(xy) < \delta$ and so $\max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\}$, this is a contradiction. Hence $x_t \in_\gamma f$, $y_s \in_\gamma f$ implies that $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$ for all x, y in S. ■

Definition 3 A fuzzy subset f of an AG-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (respt-right) ideal of S if for all $x, y \in S$ and $t, s\ (\gamma, 1]$ it satisfies $y_t \in_\gamma f$ implies that $(xy)_t \in_\gamma \vee q_\delta f$ (respt $x_t \in_\gamma f$ implies $((xy)_t \in_\gamma \vee q_\delta f)$.

Example 1 Consider an AG-groupoid $S = \{1, 2, 3\}$ in the following multiplication table.

$\circ$	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

Define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.41 & \text{if } x = 1, \\ 0.44 & \text{if } x = 2, \\ 0.42 & \text{if } x = 3. \end{cases}$$

Then, we have

- f is an $(\in_{0.1}, \in_{0.1} \vee q_{0.11})$-fuzzy right ideal,
- f is not an $(\in, \in \vee q_{0.11})$-fuzzy right ideal.

Example 2 Let $S = \{1, 2, 3\}$ and the binary operation $\circ$ be defined on S as follows:

$\circ$	1	2	3
1	2	2	2
2	2	2	2
3	1	2	2

Then, clearly, $(S, \circ)$ is an AG-groupoid. Define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.3 & \text{if } x = 1, \\ 0.7 & \text{if } x = 2, \\ 0.8 & \text{if } x = 3. \end{cases}$$

Then, we have

- f is an $(\in_{0.4}, \in_{0.4} \vee q_{0.45})$-fuzzy AG-subgroupoid of S,
- f is not an $(\in_{0.4}, \in_{0.4} \vee q_{0.45})$-fuzzy right ideal of S.

Example 3 Let $S = \{1, 2, 3\}$ and the binary operation " $\cdot$ " be defined on S as follows:

$\cdot$	1	2	3
1	1	1	1
2	1	1	1
3	1	2	1

Then clearly $(S, \cdot)$ is an AG-groupoid. Define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.8 & \text{if } x = 1, \\ 0.6 & \text{if } x = 2, \\ 0.5 & \text{if } x = 3. \end{cases}$$

Then it is easy to see that f is an $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$-fuzzy ideal of S.

Theorem 2 *A fuzzy subset f of an AG-groupoid S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (respt right) ideal of S if and only if*

$$\max\{f(xy), \gamma\} \geq \min\{f(y), \delta\} \text{ (respt } \max\{f(xy), \gamma\} \geq \min\{f(x), \delta\}).$$

Proof. Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. If exist $x, y \in S$ and $t \in (\gamma, 1]$ such that

$$\max\{f(xy), \gamma\} < t \leq \min\{f(y), \delta\}.$$

Then $\max\{f(xy), \gamma\} < t \leq \gamma$ this implies that $(xy)_t \bar{\in}_\gamma f$ which further implies that $(xy)_t \overline{\in_\gamma \vee q_\delta} f$. As $\min\{f(y), \delta\} \geq t > \gamma$ which implies that $f(y) \geq t > \gamma$, this implies that $y_t \in_\gamma f$. But $(xy)_t \overline{\in_\gamma \vee q_\delta} f$ a contradiction to the definition. Thus

$$\max\{f(xy), \gamma\} \geq \min\{f(y), \delta\}.$$

Conversely, assume that there exist $x, y \in S$ and $t, s \in (\gamma, 1]$ such that $y_s \in_\gamma f$ but $(xy)_t \overline{\in_\gamma \vee q_\delta} f$, then $f(y) \geq t > \gamma$, $f(xy) < \min\{f(y), \delta\}$ and $f(xy) + t \leq 2\delta$. It follows that $f(xy) < \delta$ and so $\max\{f(xy), \gamma\} < \min\{f(y), \delta\}$ which is a contradiction. Hence $y_t \in_\gamma f$ this implies that $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$ (respt $x_t \in_\gamma f$ implies $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$) for all x, y in S. ∎

Definition 4 A fuzzy subset f of an AG-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S if for all x, y and $z \in S$ and $t, s \in (\gamma, 1]$, the following conditions hold.

(1) if $x_t \in_\gamma f$ and $y_s \in_\gamma f$ implies that $(xy)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.

(2) if $x_t \in_\gamma f$ and $z_s \in_\gamma f$ implies that $((xy)z)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$.

Example 4 Define a fuzzy subset f on S in example 2 as follows:

$$f(x) = \begin{cases} 0.44 & \text{if } x = 1, \\ 0.6 & \text{if } x = 2, \\ 0.7 & \text{if } x = 3. \end{cases}$$

Then, we have

- f is an $(\in_{0.4}, \in_{0.4} \vee q_{0.45})$-fuzzy bi-ideal of S,
- f is not an $(\in_{0.4}, \in_{0.4} \vee q_{0.45})$-fuzzy right ideal of S.

Theorem 3 *A fuzzy subset f of an AG-groupoid S is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S if and only if*

(I) $\max\{f(xy),\gamma\} \geq \min\{f(x), f(y), \delta\}$.

(II) $\max\{f((xy)z),\gamma\} \geq \min\{f(x), f(z), \delta\}$.

Proof. (1) $\Leftrightarrow$ (I) is the same as Theorem 1.

(2) $\Rightarrow$ (II) Assume that $x, y \in S$ and $t, s \in (\gamma, 1]$ such that

$$\max\{f((xy)z),\gamma\} < t \leq \min\{f(x), f(z), \delta\}.$$

Then $\max\{f((xy)z),\gamma\} < t$ which implies that $f((xy)z) < t$ this implies that $((xy)z)_t \bar{\in}_\gamma f$ which further implies that $((xy)z)_t \overline{\in_\gamma \vee q_\delta} f$. Also $\min\{f(x), f(z), \delta\} \geq t > \gamma$, this implies that $f(x) \geq t > \gamma$, $f(z) \geq t > \gamma$ implies that $x_t \in_\gamma f$, $z_t \in_\gamma f$. But $((xy)z)_t \overline{\in_\gamma \vee q_\delta} f$, a contradiction. Hence

$$\max\{f((xy)z),\gamma\} \geq \min\{f(x), f(z), \delta\}.$$

(II) $\Rightarrow$ (2) Assume that x, y in S and $t, s \in (\gamma, 1]$, such that $x_t \in_\gamma f$, $z_s \in_\gamma f$ but $((xy)z)_{\min\{t,s\}} \overline{\in_\gamma \vee q_\delta} f$, then $f(x) \geq t > \gamma$, $f(z) \geq s > \gamma$, $f((xy)z) < \min\{f(x), f(y), \delta\}$ and $f((xy)z) + \min\{t, s\} \leq 2\delta$. It follows that $f((xy)z) < \delta$ and so $\max\{f((xy)z),\gamma\} < \min\{f(x), f(y), \delta\}$, a contradiction. Hence $x_t \in_\gamma f$, $z_s \in_\gamma f$ implies that $((xy)z)_{\min\{t,s\}} \in_\gamma \vee q_\delta f$ for all x, y in S. ∎

Lemma 4 *Let f be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy AG-subgroupoid and g be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of an AG-groupoid S. Then $(f \circ g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S.*

Proof. Let f and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy AG-subgroupoid an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of an AG-groupoid S with left identity,respectively. So for any y in S, there exist a and b in S such that $y = ab$. Therefore, $xy = x(ab) = a(xb)$.

Then

$$\begin{aligned}
\min\{f \circ g(y), \delta\} &= \min\left\{ \bigvee_{y=ab} \{f(a) \wedge g(b)\}, \delta \right\} \\
&= \bigvee_{y=ab} \{\min\{\min\{f(a), \delta\}, \min\{g(b), \delta\}\}\} \\
&\leq \bigvee_{xy=a(xb)} \{\min\{\max\{f(a), \gamma\}, \max\{g(xb), \gamma\}\}\} \\
&= \bigvee_{xy=a(xb)} \{\max\{\min\{f(a), g(xb)\}, \gamma\}\} \\
&\leq \bigvee_{xy=ac} \{\max\{\min\{f(a), g(c)\}, \gamma\}\} \\
&= \max\{(f \circ g)(xy), \gamma\}.
\end{aligned}$$

∎

Corollary 1 *Let f and g be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals of an AG-groupoid S with left identity. Then $(f \circ g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S.*

Theorem 4 *Let f and g be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals of an AG-groupoid S with left identity. Then $(f \circ g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S.*

Proof. Let f and g be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideals of an AG-groupoid S. Since S is an AG-groupoid, there exist x, y in S such that $x = ab$ and $y = y_1 y_2$, for some a, b, y_1, y_2 in S. Then by medial law we get

$$xy = (ab)y = (ab)(y_1 y_2) = (ay_1)(by_2).$$

Then we have

$$\begin{aligned}
\min\{f \circ g(x), \delta\} &= \min\left\{\bigvee_{x=ab}\{f(a) \wedge g(b)\}, \delta\right\} \\
&= \bigvee_{x=ab}\{\min\{\min\{f(a), \delta\}, \min\{g(b), \delta\}\}\} \\
&\leq \bigvee_{xy=(ay_1)(by_2)}\{\min\{\max\{f(ay_1), \gamma\}, \max\{g(by_2), \gamma\}\}\} \\
&= \bigvee_{xy=(ay_1)(by_2)}\{\max\{\min\{f(ay_1), g(by_2)\}, \gamma\}\} \\
&\leq \bigvee_{xy=cd}\{\max\{\min\{f(c), g(d)\}, \gamma\}\} \\
&= \max\{(f \circ g)(xy), \gamma\}.
\end{aligned}$$

■

Theorem 5 *Let f and g be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals of an AG-groupoid S. Then $(f \circ g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S.*

Proof. Assume that f and g are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals of an AG-groupoid S. Consider on the contrary that there exist $a, b \in S$ and $t, s \in (\gamma, 1]$ such that

$$\max\{(f \circ g)(ab), \gamma\} < t \leq \min\{(f \circ g)(a), (f \circ g)(b), \delta\}.$$

Then $\min\{(f \circ g)(a), (f \circ g)(b), \delta\} \geq t$ implies that $(f \circ g)(a) \geq t > \gamma, (f \circ g)(b) \geq t > \gamma$, this implies that $a_t \in_\gamma (f \circ g), b_t \in_\gamma (f \circ g)$ and $\max\{(f \circ g)(ab), \gamma\} < t$ implies that $(f \circ g)(ab) < t$ implies that $(f \circ g)(ab) + s < 2\delta$ further implies that $(ab)_t \overline{\in_\gamma \vee q_\delta} (f \circ g)$. But $(ab)_t \overline{\in_\gamma \vee q_\delta} (f \circ g)$, a contradiction. Hence

$$\max\{(f \circ g)(ab), \gamma\} \geq \min\{(f \circ g)(a), (f \circ g)(b), \delta\}.$$

Assume that $\max\{(f \circ g)((ax)b), \gamma\} < t \leq \min\{(f \circ g)(a), (f \circ g)(b), \delta\}$, then $\min\{(f \circ g)(a), (f \circ g)(b), \delta\} \geq t$ implies that $(f \circ g)(a) \geq t$, $(f \circ g)(b) \geq t$, this implies that $a_t \in_\gamma (f \circ g), b_t \in_\gamma (f \circ g)$ and $\max\{(f \circ g)((ax)b), \gamma\} < t$

implies that $(f \circ g)((ax)b) < t$ further implies that $((ax)b)_t \bar{\in}_\gamma (f \circ g)$ implies that $((ax)b)_t \overline{\in_\gamma \vee q_\delta} (f \circ g)$. But $((ax)b)_t \overline{\in_\gamma \vee q_\delta} (f \circ g)$, a contradiction. Hence

$$\max\{(f \circ g)((ax)b), \gamma\} \geq \min\{(f \circ g)(a), (f \circ g)(b), \delta\}.$$

Hence $(f \circ g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S. ∎

Lemma 5 *A subset B of an AG-groupoid S is a bi-ideal if and only if $X^\delta_{\gamma B}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S.*

Proof. (i) Let B be a bi-ideal and assume that $x, y \in B$ then for any a in S we have $(xa)y \in B$, thus $X^\delta_{\gamma B}((xa)y) \geq \delta$. Now since $x, y \in B$ so $X^\delta_{\gamma B}(x) \geq \delta$, $X^\delta_{\gamma B}(y) \geq \delta$ which clearly implies that $\min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y)\} \geq \delta$. Thus

$$\max\{X^\delta_{\gamma B}((xa)y), \gamma\} = X^\delta_{\gamma B}((xa)y).$$
$$\min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}, \delta\} = \delta.$$

Hence $\max\{X^\delta_{\gamma B}((xa)y), \gamma\} \geq \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\}$.

(ii) Let $x \in B, y \notin B$, then $(xa)y \notin B$, for all a in S. This implies that $X^\delta_{\gamma B}((xa)y) \leq \gamma$, $X^\delta_{\gamma B}(x) \geq \delta$ and $X^\delta_{\gamma B}(y) < \gamma$. Therefore

$$\max\{X^\delta_{\gamma B}((xa)y), \gamma\} = \gamma \text{ and } \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\} = X^\delta_{\gamma B}(y).$$

Hence $\max\{X^\delta_{\gamma B}((xa)y), \gamma\} \geq \min\{X^\delta_{\gamma B}(x), X^\delta_{\geq \gamma B}(y), \delta\}$.

(iii) Let $x \notin B, y \in B$ implies that $(xa)y \notin B$, for all a in S. This implies that $\max\{X^\delta_{\gamma B}((xa)y), \gamma\} \geq \delta, X^\delta_{\gamma B}(x) < \gamma, X^\delta_{\gamma B}(y) \geq \delta$ and $\min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\} = X^\delta_{\gamma B}(x)$. Therefore

$$\max\{X^\delta_{\gamma B}((xa)y), \gamma\} \geq \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\}.$$

(iv) Let $x, y \notin B$ which implies that $(xa)y \notin B$, for all a in S. This implies that $\min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y)\} \leq \gamma$, $X^\delta_{\gamma B}((xa)y) \leq \gamma$. Thus

$$\max\{X^\delta_{\gamma B}((xa)y), \gamma\} = \gamma \text{ and}$$
$$\min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\} \leq \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y)\} \leq \gamma.$$

Hence $\max\{X^\delta_{\gamma B}((xa)y), \gamma\} \geq \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\}$.

If $(xa)y \in B$, then $\min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y)\} \leq \gamma$, $X^\delta_{\gamma B}((xa)y) \geq \delta$. Thus

$$\max\{X^\delta_{\gamma B}((xa)y), \gamma\} = X^\delta_{\gamma B}((xa)y) \text{ and}$$
$$\min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\} \leq \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y)\} \leq \gamma.$$

Hence $\max\{X^\delta_{\gamma B}((xa)y), \gamma\} \geq \min\{X^\delta_{\gamma B}(x), X^\delta_{\gamma B}(y), \delta\}$.

Conversely, let $(b_1 s)b_2 \in (BS)B$, where $b_1, b_2 \in B$ and $s \in S$.

Now, by hypothesis $\max\{X^\delta_{\gamma B}((b_1 s)b_2), \gamma\} \geq \min\{X^\delta_{\gamma B}(b_1), X^\delta_{\gamma B}(b_2), \delta\}$.

Since $b_1, b_2 \in B$, therefore $X^{\delta}_{\gamma B}(b_1) \geq \delta$ and $X^{\delta}_{\gamma B}(b_2) \geq \delta$ which implies that $\min\{X^{\delta}_{\gamma B}(b_1), X^{\delta}_{\gamma B}(b_2), \delta\} = \delta$. Thus

$$\max\{X^{\delta}_{\gamma B}((b_1 s)b_2), \gamma\} \geq \delta.$$

This clearly implies that $\max X^{\delta}_{\gamma B}((b_1 s)b_2) \geq \delta$. Therefore $(b_1 s)b_2 \in B$. Hence B is a bi-ideal of S. ∎

Similarly, we can prove the following lemmas.

Lemma 6 *A subset A of an AG-groupoid S is closed if and only if $X^{\delta}_{\gamma A}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy AG-subgroupoid of S.*

Lemma 7 *For any fuzzy subset f of an AG-groupoid S with left identity, $f \subseteq \vee q_{(\gamma,\delta)} S \circ f$.*

Proof. Let f be a fuzzy subset of an AG-groupoid S with left identity. Therefore, for any a in S there exist p and q in S such that $a = pq$. Thus

$$\begin{aligned} max\{(S \circ f)(a), \gamma\} &= \max\left\{\bigvee_{a=pq}\{\{S(p) \wedge f(q)\}, \gamma\}\right\} \\ &\geq \max\{\min\{S(e) \wedge f(a)\}, \gamma\} \\ &= \max\{\min\{1 \wedge f(a)\}, \gamma\} \\ &= \max\{f(a)\}, \gamma\} \\ &\geq \min\{f(a), \delta\}. \end{aligned}$$

Hence by Lemma 1, $f \subseteq \vee q_{(\gamma,\delta)} S \circ f$. ∎

4. Intra-regular AG-groupoids

An element a of an AG-groupoid S is called **intra-regular** if there exist $x, y \in S$ such that $a = (xa^2)y$ and S is called **intra-regular**, if every element of S is intra-regular. In this section, we discuss the characterizations of intra-regular AG-groupoids.

Example 5 *Let $S = \{1, 2, 3, 4, 5, 6\}$, the following table shows that S is an intra-regular AG-groupoid.*

$\circ$	1	2	3	4	5	6
1	2	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	6	3	4	5
4	1	2	5	6	3	4
5	1	2	4	5	6	3
6	1	2	3	4	5	6

It is easy to see that $(S, \circ)$ is an AG-groupoid and is non-commutative and non-associative structure because $(3 \circ 4) \neq (4 \circ 3)$ and $(3 \circ 6) \circ 4 \neq 3 \circ (6 \circ 4)$. Also $1 = (3 \circ 1^2) \circ 1$, $2 = (2 \circ 2^2) \circ 2$, $3 = (4 \circ 3^2) \circ 5$, $4 = (4 \circ 4^2) \circ 6$, $5 = (6 \circ 5^2) \circ 5$, $6 = (6 \circ 6^2) \circ 6$. *Therefore, $(S, \circ)$ is an intra-regular AG-groupoid.*

Theorem 1 *Let S be an AG-groupoid with left identity, then the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) *For every subset L and bi-ideal B of S, $L \cap B \subseteq LB$.*

(iii) *For every $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy subset f and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal g, $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g$.*

Proof. (i) $\Rightarrow$ (iii) Let f and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of an intra-regular AG-groupoid S with left identity, respectively. Now since S is intra-regular, therefore, for any a in S there exist x, y in S such that $a = (xa^2)y$. Since $S = S^2$, so for any y there exists y_1, y_2 such that $y = y_1 y_2$. Now, using (1) and the left invertive law, the medial law and the paramedial law, we get

$$\begin{aligned} a &= (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a \\ &= (y\,(xa))\left(\left(xa^2\right)y\right) = \left(y\left(xa^2\right)\right)((xa)\,y) = \left((y_1y_2)\left(xa^2\right)\right)((xa)\,(y_1y_2)) \\ &= \left(\left(a^2x\right)(y_2y_1)\right)((y_2y_1)\,(ax)) = \left(\left(a^2x\right)(y_2y_1)\right)(a\,((y_2y_1)\,x)) \\ &= a[\{\left(a^2x\right)(y_2y_1)\}\{(y_2y_1)\,x\}] = a[\{((y_2y_1)\,x)\,a^2\}\,((y_2y_1)\,x)] \\ &= a[\{a^2\,(x\,(y_2y_1))\}\{(y_2y_1)\,x\}] = a[\{\{(y_2y_1)\,x\}\,(x\,(y_2y_1))\}a^2] \\ &= a[ua^2], \text{ where } u = \{(y_2y_1)\,x\}\,(x\,(y_2y_1)). \end{aligned}$$

Now

$$\begin{aligned} ua^2 &= u\left[(xa^2)y\right]^2 = u\left[(xa^2)^2y^2\right] = (xa^2)^2\left[uy^2\right] = [x^2(a^2)^2]\left[uy^2\right] \\ &= [(a^2)^2x^2]\left[uy^2\right] = [(uy^2)x^2](a^2a^2) = (a^2a^2)[x^2(uy^2)] \\ &= ([x^2(uy^2)]a^2)a^2 = (a^2[x^2(uy^2)])a^2 \\ &= (a^2q)a^2, \text{ where } q = x^2(uy^2). \end{aligned}$$

Thus $a = a((a^2q)a^2)$, where $q = x^2(uy^2)$ and $u = \{(y_2y_1)\,x\}\,(x\,(y_2y_1))$.

For any a in S, there exist p and q in S such that $a = pq$, then

$$\begin{aligned} \max\{(f \circ g)(a), \gamma\} &= \max\left\{\bigvee_{a=pq}\{\{f(p) \wedge g(q)\}, \gamma\}\right\} \\ &\geq \max\{\min\{f(a)), g((a^2q)a^2)\}, \gamma\} \\ &= \min\{\max\{f(a)), g((a^2q)a^2)\}, \gamma\} \\ &= \min\{\max\{f(a)), \gamma\}, \max\{g((a^2q)a^2), \gamma\}\} \\ &\geq \min\{\min\{f(a), \delta\}, \min\{g(a), \delta\}\} \\ &= \min\{(f \cap g)(a), \delta\}. \end{aligned}$$

Thus, by Lemma 1, $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g$.

(iii) ⇒ (ii) Let L and B be any subset and bi-ideal of S, respectively. Then, by Lemma 3 and (iii), we get

$$X^{\delta}_{\gamma(L\cap B)} =_{(\gamma,\delta)} X^{\delta}_{\gamma L} \cap X^{\delta}_{\gamma B} \subseteq \vee q_{(\gamma,\delta)} X^{\delta}_{\gamma L} \circ X^{\delta}_{\gamma B} =_{(\gamma,\delta)} X^{\delta}_{\gamma LB}.$$

Hence, by Lemma 3, we get $L \cap B \subseteq LB$.

(ii) ⇒ (i) Since Sa is an bi-ideal and any subset of S containing a. Therefore, by (ii) medial law, paramedial law and (1), we get

$$\begin{aligned} a \in Sa \cap Sa &\subseteq (Sa)(Sa) = (SS)(aa) = (a^2S)S \\ &= ((aa)(SS))S = ((SS)(aa))S = (Sa^2)S. \end{aligned}$$

Hence S is intra-regular. ■

Remark 1 If S is an intra-regular AG-groupoid with left identity, then for every $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy subset f of S, $f \subseteq \vee q_{(\gamma,\delta)} f \circ S$.

Theorem 2 *Let S be an AG-groupoid with left identity, then the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) *For every bi-ideal B and any subset A of S, $B \cap A \subseteq BA$.*

(iii) *For every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal f and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset g of S, $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g$.*

(iv) *For every $(\in_\gamma, \in_\gamma \vee q_\delta)$ generalized fuzzy bi-ideal f and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset g of S, $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g$.*

Proof. (i) ⇒ (iv) Let f and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset of an intra-regular AG-groupoid S with left identity, respectively. Now since S is intra-regular, therefore for any a in S, there exist x, y in S such that $a = (xa^2)y$. Now using (1), the left invertive law, the medial law and the paramedial law, we get

$$\begin{aligned} (xa^2)y &= (a(xa))y = (y(xa))a = [(y_1y_2)(xa)]a = [(ax)(y_2y_1)]a \\ &= [((y_2y_1)x)a]a = [((y_2y_1)x)((xa^2)y)]a \\ &= [(tx)((xa^2)y)]a, \text{ where } t = y_2y_1 \\ (tx)((xa^2)y) &= (xa^2)((tx)y) = (a(xa))((tx)y) \\ &= [((tx)y)(xa))]a = [(u(xa)]a, \text{ where } u = (tx)y \\ u(xa) &= u[x\{(xa^2)y)\}] = u[(xa^2)(xy)] = (xa^2)(u(xy)) \\ &= ((xy)u)(a^2x) = a^2[\{(xy)u\}x] = [x\{(xy)u\}]a^2 = a[\{x((xy)u\})a] \\ &= a(pa), \text{ where } p = (x((xy)u))a. \end{aligned}$$

Thus $a = [(a(pa))a]a = [(aq)a]a$, where $q = pa$.

For any a in S, there exist p and q in S such that $a = pq$. Then

$$\begin{aligned} max\{f \circ g(a), \gamma\} &= \max\left\{ \bigvee_{a=pq} \{\{f(p) \wedge g(q)\}, \gamma\}\} \right\} \\ &= \max\left\{ \bigvee_{a=pq} \{\min\{f(p), g(q)\}, \gamma\}\} \right\} \\ &\geq \min\{\max\{f((aq)a), g(a), \gamma\} \\ &= \min\{\max\{f((aq)a), \gamma\}, \max\{g(a), \gamma\}\} \\ &\geq \min\{\min\{f(a), \delta\}, \min\{g(a), \delta\}\} \\ &= \min\{(f \cap g)(a), \delta\}. \end{aligned}$$

Thus, by Lemma 1, $f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g$.

(iv) $\Rightarrow$ (iii) It is obvious.

(iii) $\Rightarrow$ (ii) Let A be any subset and B be an bi- ideal of S. Then, by Lemma 3 and (iii), we get

$$X^{\delta}_{\gamma(B\cap A)} =_{(\gamma,\delta)} X^{\delta}_{\gamma B} \cap X^{\delta}_{\gamma A} \subseteq \vee q_{(\gamma,\delta)} X^{\delta}_{\gamma B} \circ X^{\delta}_{\gamma A} =_{(\gamma,\delta)} X^{\delta}_{\gamma BA}$$

Hence, by Lemma 3, we get $B \cap A \subseteq BA$.

(ii) $\Rightarrow$ (i) Since Sa is a bi-ideal containing a, so using (ii), we get

$$a \in Sa \cap Sa = Sa^2 = (Sa^2)S.$$

Hence S is intra-regular. ∎

Corollary 2 *For an AG-groupoid S with left identity, the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) *For all bi-ideal B_1 and B_2, $B_1 \cap B_2 \subseteq B_1 B_2$.*

(iii) *$f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g$, for all $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy bi-ideals f and g.*

(iv) *$f \cap g \subseteq \vee q_{(\gamma,\delta)} f \circ g$, for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideals f and g.*

Definition 5 A fuzzy subset f of an AG-groupoid S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideal if $x^2_t \in_\gamma f$ implies that $x_t \in_\gamma \vee q_\delta$ for all $x \in S$ and $t \in (\gamma, 1]$.

Example 6 Define a fuzzy subset f on S in Example 2 as given:

$$f(x) = \begin{cases} 0.3 & \text{if } x = 1, \\ 0.2 & \text{if } x = 2, \\ 0.4 & \text{if } x = 3. \end{cases}$$

Then, it is easy to see that f is an $(\in_{0.2}, \in_{0.2} \vee q_{0.3})$-fuzzy semiprime ideal of S.

Example 7 Define a fuzzy subset f on S in Example 5 as given:

$$f(x) = \begin{cases} 0.7 & \text{if } x = 1, \\ 0.6 & \text{if } x = 2, \\ 0.68 & \text{if } x = 3, \\ 0.63 & \text{if } x = 4, \\ 0.52 & \text{if } x = 5, \\ 0.5 & \text{if } x = 6. \end{cases}$$

Then, it is easy to see that f is an $(\in_{0.4}, \in_{0.4} \vee q_{0.5})$-fuzzy semiprime ideal of S.

Theorem 3 *A fuzzy subset f of an AG-groupoid S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideal if and only if* $\max\{f(x), \gamma\} \geq \min\{f(x^2), \delta\}$.

Proof. Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime ideal of S. If exist $x, y \in S$ and $t \in (\gamma, 1]$ such that $\max\{f(x), \gamma\} < t \leq \min\{f(x^2), \delta\}$. Then $\max\{f(x), \gamma\} < t$ implies that $x_t \bar{\in}_\gamma f$ implies that $x_t \overline{\in_\gamma \vee q_\delta} f$. As $\min\{f(x^2), \delta\} \geq t > \gamma$ this implies that $f(x^2) \geq t > \gamma$ implies that $x_t^2 \in_\gamma f$. But $x_t \overline{\in_\gamma \vee q_\delta} f$, a contradiction to the definition of semiprime ideals. Thus, we have $\max\{f(x), \gamma\} \geq \min\{f(x^2), \delta\}$.

Conversely, assume that there exist x, y in S and $t \in (\gamma, 1]$ such that $x_t^2 \in_\gamma f$ but $x_t \overline{\in_\gamma \vee q_\delta} f$, then $f(x^2) \geq t > \gamma$, $f(x) < \min\{f(x^2), \delta\}$ and $f(x) + t \leq 2\delta$. It follows that $f(x) < \delta$ and so $\max\{f(x), \gamma\} < \min\{f(x^2), \delta\}$ which is a contradiction to the definition of semiprime ideals. Hence, $x_t^2 \in_\gamma f$ implies that $(x^2)_t \in_\gamma \vee q_\delta f$, for all x, y in S. ∎

Theorem 4 *For a non empty subset I of an AG-groupoid S with left identity, the following conditions are equivalent.*

(i) *I is semiprime.*

(ii) *$X_{\gamma I}^\delta$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime.*

Proof. (i) $\Rightarrow$ (ii) Let I be a semiprime ideal of an AG-groupoid S. Let a be any element of S such that $a \in I$, then I is an ideal, so $a^2 \in I$. Hence $X_{\gamma I}^\delta(a), X_{\gamma I}^\delta(a^2) \geq \delta$ which implies that $\max\{X_{\gamma I}^\delta(a), \gamma\} \geq \min\{X_{\gamma I}^\delta(a^2), \delta\}$.

Now, let $a \notin I$, since I is semiprime, thus $a^2 \notin I$. This implies that $X_{\gamma I}^\delta(a) \leq \gamma$ and $X_{\gamma I}^\delta(a^2) \leq \gamma$. Therefore, $\max\{X_{\gamma I}^\delta(a), \gamma\} \geq \min\{X_{\gamma I}^\delta(a^2), \delta\}$. Hence, we have $\max\{X_{\gamma I}^\delta(a), \gamma\} \geq \min\{X_{\gamma I}^\delta(a^2), \delta\}$ for all $a \in S$.

(ii) $\Rightarrow$ (i) Let $X_{\gamma I}^\delta$ is fuzzy semiprime. If $a^2 \in I$, for some a in S, this implies that $X_{\gamma I}^\delta(a^2) \geq \delta$. Now, since $X_{\gamma I}^\delta$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy semiprime. Thus, $\max\{X_{\gamma I}^\delta(a), \gamma\} \geq \min\{X_{\gamma I}^\delta(a^2), \delta\}$. Therefore $\max\{X_{\gamma I}^\delta(a), \gamma\} \geq \delta$. But $\delta > \gamma$, so $X_{\gamma I}^\delta(a) \geq \delta$. Thus, $a \in I$. Hence, I is semiprime. ∎

Lemma 8 *Let f be a fuzzy subset of an AG-groupoid S. Then f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S if and only if* $\max\{f(a), \gamma\} \geq \min\{((f \circ S) \circ f)(a), \delta\}$.

Proof. Assume that f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of an AG-groupoid S. If $a \in S$, then there exist c, d, p and q in S such that $a = pq$ and $p = cd$. Since f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S, we have $\max\{f((cd)q), \gamma\} \geq \min\{f(c), f(q), \delta\}$. Therefore,

$$\begin{aligned}
\min\{((f \circ S) \circ f)(a), \delta\} &= \min\left\{\bigvee_{a=pq} \{(f \circ S)(p) \wedge f(q)\}, \delta\right\} \\
&= \min\left\{\bigvee_{a=pq} \{\bigvee_{p=cd} \{f(c) \wedge S(d)\} \wedge f(q)\}, \delta\right\} \\
&= \min\left\{\bigvee_{a=(cd)q} \{\{f(c) \wedge 1 \wedge f(q)\}, \delta\}\right\} \\
&= \min\left\{\bigvee_{a=(cd)q} \{f(c) \wedge f(q)\}, \delta\right\} \\
&= \bigvee_{a=(cd)q} \{\min\{f(p), f(q), \delta\}\} \\
&\leq \bigvee_{a=(cd)q} \{\max\{f((cd)q), \gamma\}\} \\
&= \max\{f(a), \gamma\}.
\end{aligned}$$

Hence, $\max\{f(a), \gamma\} \geq \min\{((f \circ S) \circ f)(a), \delta\}$.

Conversely, assume that $\max\{f(a), \gamma\} \geq \min\{((f \circ S) \circ f)(a), \delta\}$. Let a in S, there exist c, d and q in S such that $a = (cd)q$. Then we have

$$\begin{aligned}
\max\{f((cd)q), \gamma\} &= \max\{f(a), \gamma\} \\
&\geq \min\{((f \circ S) \circ f)(a), \delta\} \\
&= \min\left\{\bigvee_{a=bc} \{(f \circ S)(b) \wedge f(c), \delta\}\right\} \\
&\geq \min\{\{(f \circ S)(cd) \wedge f(q)\}, \delta\} \\
&= \min\left\{\bigvee_{cd=st} \{f(s) \wedge S(t)\} \wedge f(q)\}, \delta\right\} \\
&\geq \min\{\min(f(c), f(q), \delta\} \\
&= \min\{(f(c), f(q), \delta\}.
\end{aligned}$$

Hence $\max\{f((cd)q), \gamma\} \geq \min\{f(c), f(q), \delta\}$. ∎

Lemma 9 *Every* $(\in_\gamma \in_\gamma \vee q_\delta)$*-fuzzy right ideal of an AG-groupoid* S *is an* $(\in_\gamma \in_\gamma \vee q_\delta)$*-fuzzy bi-ideal of* S*.*

Proof. Let f be an $(\in_\gamma \in_\gamma \vee q_\delta)$-fuzzy right ideal of an AG-groupoid S. For any a in S, there exist p, q, s and t in S, such that

$$\begin{aligned}
\min\{((f \circ S) \circ f)(a), \delta\} &= \min\{\bigvee_{a=pq} \{(f \circ S)(p) \wedge f(q)\}, \delta\} \\
&= \min\{\bigvee_{a=pq} \{\bigvee_{p=st} \{f(s) \wedge S(t)\} \wedge f(q)\}, \delta\} \\
&= \min\{\bigvee_{a=(st)q} \{f(s) \wedge f(q)\}, \delta\} \\
&= \min\{\bigvee_{a=(st)q} [\min\{f(s), f(q)\}], \delta\} \\
&= \bigvee_{a=(st)q} \min[\min\{f(s), \delta\}, \min\{f(q), \delta\}] \\
&\leq \bigvee_{a=(st)q=(qt)s} \min\{\max\{f(st)q, \gamma\}, \max\{f(qt)s, \gamma\}\} \\
&= \min\{\max\{f(a), \gamma\}, \max\{f(a), \gamma\}\} \\
&= \max\{f(a), \gamma\}.
\end{aligned}$$

Hence $\max\{f(a), \gamma\} \geq \min\{((f \circ S) \circ f)(a), \delta\}$. Hence, by Lemma 8, f is an $(\in_\gamma \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S. ■

Theorem 5 *Let S be an intra-regular AG-groupoid with left identity, then the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) *For every ideal of S is semiprime.*

(iii) *For every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S is fuzzy semiprime.*

(iv) *For every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S is fuzzy semiprime.*

(v) *For every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S is fuzzy semiprime.*

Proof. (i) $\Rightarrow$ (v) Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of an intra-regular AG-groupoid S with left identity. Now, since S is intra-regular so for each $a \in S$ there exist x, y in S such that $a = (xa^2)y$. Now, using (1) and the left invertive law, we get

$$\begin{aligned}
a &= (xa^2)y = [x(aa)]y = [a(xa)]y = [y(xa)]a \\
&= [y\{x((xa^2)y)\}]a = [y\{(xa^2)(xy)\}]a \\
&= [(xa^2)\{y(xy)\}]a = [\{(xy)y\}(a^2x)]a = [a^2((y^2x)x)]a \\
&= [a((y^2x)x)]a^2 = [\{(xa^2)y\}((y^2x)x)]a^2 \\
&= [\{((y^2x)x)y\}(xa^2)]a^2 = [(a^2x)\{y((y^2x)x)\}]a^2 \\
&= [\{\{y((y^2x)x)\}x\}a^2]a^2 = [a^2\{x\{y((y^2x)x)\}\}]a^2 \\
&= (a^2q)a^2, \text{ where } q = \{((y^2x)x)\}x\}y\}.
\end{aligned}$$

Thus

$$\begin{aligned}\max\{f(a),\gamma\} &= \max\{f((a^2q)a^2),\gamma\}\\ &\geq \min\{f(a^2),f(a^2),\delta\}\\ &= \min\{f(a^2),\delta\}.\end{aligned}$$

(v) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (ii) It follows from Lemma 3.

(ii) $\Rightarrow$ (i) Assume that every ideal is semiprime and since Sa^2 is an ideal containing a^2. Thus, we have

$$a \in Sa^2 = (SS)a^2 = (a^2S)S = (Sa^2)S.$$

Hence S is an intra-regular AG-groupoid. ∎

5. Conclusions

In this paper, we characterize intra-regular AG-groupoids with left identity using the properties of their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-deals. In our future research work we will focus on considering other types of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals intra-regular AG-groupoids. We remark that this research work will give a new direction for applications of fuzzy set theory particularly in algebraic logic, non-classical logic, fuzzy coding, fuzzy finite state mechanics and fuzzy languages.

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References

[1] Bhakat, S.K., Das, P., *On the definition of a fuzzy subgroup*, Fuzzy Sets and Systems, 51 (1992), 235-241.

[2] Bhakat, S.K., Das, P., $(\in, \in \vee q)$*-fuzzy subgroups*, Fuzzy Sets and Systems, 80 (1996), 359-368.

[3] Kazim, M.A., Naseeruddin, M., *On almost semigroups*, The Alig. Bull. Math., 2 (1972), 1-7.

[4] Feng, F., Li, C.X., Davvaz, B., Ali, M.I., *Soft sets combined with fuzzy sets and rough sets: a tenta approach*, Soft Computing, 14 (2010), 899–911.

[5] Feng, F., Liu, X.Y., Leoreanu-Fotea, V., Jun, Y.B., *Soft sets and soft rough sets*, Inform. Sci., 181 (2011), 1125–1137.

[6] Goodman, I.R., *Fuzzy sets as equivalence classes of random sets.* In: Recent Developments in Fuzzy Sets and Possibility Theory (R. Yager, Ed.), Pergamon, New York, 1982.

[7] Mordeson, J.N., Malik, D.S., Kuroki, N., *Fuzzy semigroups*, Springer-Verlag, Berlin, Germany, 2003.

[8] Murali, V., *Fuzzy points of equivalent fuzzy subsets*, Inform. Sci., 158 (2004), 277-288.

[9] Mushtaq, Q., Yusuf, S.M., *On LA-semigroupss*, The Alig. Bull. Math., 8 (1978), 65-70.

[10] Pu, P.M., Liu, Y.M., *Fuzzy topology I, neighborhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl., 76 (1980), 571-599.

[11] Shabir, M., Jun, Y.B., Nawaz, Y., *Semigroups characterized by* $(\in, \in \vee q_k)$*-fuzzy ideals*, Comput. Math. Appl., 60 (2010), 1473-1493.

[12] Shabir, M., Jun, Y.B., Nawaz, Y., *Characterizations of regular semigroups by* (α, β)*-fuzzy ideals*, Comput. Math. Appl., 59 (2010), 161-175.

[13] Rosenfeld, A., *Fuzzy groups*, J. Math. Anal. Appl., 35 (1971), 512-517.

[14] Yin, Y., Zhan, J., *Characterization of ordered semigroups in terms of fuzzy soft ideals*, Bull. Malays. Math. Sci. Soc., (2) 35 (4) (2012), 997–1015.

[15] Zadeh, L.A., *Fuzzy sets*, Inform. Control, 8 (1965), 338-353.

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