CHARACTERIZATIONS OF REGULAR ABEL-GRASSMANN'S GROUPOIDS BY THE PROPERTIES OF THEIR $(\in, \in \vee q_k)$-FUZZY IDEALS

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Abstract. In this paper, we characterize some properties of regular Abel-Grassmann's groupoid in terms of its $(\in, \in \vee q_k)$-fuzzy ideals.

Keywords and phrases: AG-groupoid, left invertive law, medial law, paramedial law and $(\in, \in \vee q_k)$-fuzzy ideals.

1. Introduction

Fuzzy set theory and its applications are growing day by day in various branches of Science like mathematics, computer science, engineering, physics, management sciences, medical science, operational research, artificial intelligence, robotics, expert system and various other fields of sciences. Fuzzy mappings are used in fuzzy image processing, fuzzy decision making, fuzzy linear programming and fuzzy data bases. It is used in mechanical engineering, industrial engineering, computer engineering and civil engineering. Also the uses of fuzzification can be found in fuzzy systems, genetic algorithms mechanics and economics.

Mordeson et al. [20] has discovered the grand exploration of fuzzy semigroups, where theory of fuzzy semigroups is explored along with the applications of fuzzy semigroups in fuzzy coding, fuzzy finite state mechanics and fuzzy languages and the use of fuzzification in automata and formal language has widely been explored.

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Moreover the complete l-semigroups have wide range of applications in the theories of automata, formal languages and programming. It is worth mentioning that some recent investigations of l-semigroups are closely connected with algebraic logic and non-classical logic. Fuzzy sets are also closely related to other soft computing models such as rough sets [23], random sets [7] and soft sets [5], [6], [19]. Zadeh further discussed the relationships between fuzzy set theory and probability theory [31].

An AG-groupoid is a mid structure between a groupoid and a commutative semigroup. If an AG-groupoid contains left identity then this left identity is unique. However an AG-groupoid with right identity becomes a commutative semigroup with identity. Moreover every commutative AG-groupoid becomes a commutative semigroup. Mostly an AG-groupoid works like a commutative semigroup. For instance $a^2b^2 = b^2a^2$, for all a, b holds in a commutative semigroup, while this equation also holds for an AG-groupoid with left identity e, moreover $ab = (ba)e$ for any subset $\{a, b\}$ of an AG-groupoid. Now our aim is to discover some logical investigations for regular AG-groupoids using the new generalized concept of fuzzy sets. It is therefore concluded that this research work will give a new direction for applications of fuzzy set theory particularly in algebraic logic, non-classical logics, fuzzy coding, fuzzy finite state mechanics and fuzzy languages.

In [21], Murali gave the idea of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set is defined in [27]. Bhakat and Das [1], [2], gave the idea of (α, β)-fuzzy subgroups by using the "belongs to" relation $\in$ and "quasi-coincident with" relation q between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroups, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. Davvaz defined $(\in, \in \vee q)$-fuzzy subnearrings and ideals of a near ring in [4]. Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup in [10]. In [29], regular semigroups are characterized by the properties of their $(\in, \in \vee q)$-fuzzy ideals. In [28], semigroups are characterized by the properties of their $(\in, \in \vee q_k)$-fuzzy ideals.

In this paper we have introduced $(\in, \in \vee q_k)$-fuzzy ideals in a new non-associative algebraic structure, that is, in an AG-groupoid and developed some new results. We have defined a regular AG-groupoid and characterized it by the properties of its $(\in, \in \vee q_k)$-fuzzy ideals.

2. AG-groupoids

A groupoid $(S, .)$ is called AG-groupoid, if its elements satisfy left invertive law: $(ab)c = (cb)a$. In an AG-groupoid medial law [12], $(ab)(cd) = (ac)(bd)$, holds for all a, b, c, $d \in S$. It is also known that in an AG-groupoid with left identity, the paramedial law: $(ab)(cd) = (db)(ca)$, holds for all a, b, c, $d \in S$. If an AG-groupoid contains left identity, the following law holds,

$$a(bc) = b(ac), \text{for all } a, b, c \in S. \tag{1}$$

Let S be an AG-groupoid. By AG-subgroupoid of S we means a non-empty subset A of S such that $A^2 \subseteq A$, by a left (right) ideal of S we mean a non-empty subset L of S such that $SL \subseteq L$ ($RS \subseteq R$) and by a quasi-ideal of S we mean a non-empty subset Q of S such that $QS \cap SQ \subseteq Q$. By two-sided ideal or simply ideal, we mean a non-empty subset of S which is both a left and a right ideal of S. An AG-subgroupoid B of S is called bi-ideal of S if $(BS)B \subseteq B$. A subset B of S is called generalized bi-ideal of S if $(BS)B \subseteq B$.

A fuzzy subset f of a given set S is described as an arbitrary function $f : G \longrightarrow [0, 1]$, where $[0, 1]$ is the usual closed interval of real numbers. For any two fuzzy subsets f and g of S, $f \leq g$ means that, $f(x) \leq g(x)$ for all x in S. The symbols $f \cap g$ and $f \cup g$ will means that the following fuzzy subsets of S

$$(f \cap g)(x) = \min\{f(x),\ g(x)\} = f(x) \wedge g(x)$$
$$(f \cup g)(x) = \max\{f(x),\ g(x)\} = f(x) \vee g(x)$$

for all x in S.

Let f and g be any fuzzy subsets of an AG-groupoid S, then the product $f \circ g$ is defined by

$$(f \circ g)(a) = \begin{cases} \bigvee\limits_{a=bc} \{f(b) \wedge g(c)\}, & \text{if there exist } b,\, c \in S, \text{ such that } a = bc \\ 0, & \text{otherwise.} \end{cases}$$

A fuzzy subset f of an AG-groupoid S is called a fuzzy AG-subgroupoid of S if $f(xy) \geq f(x) \wedge f(y)$ for all $x,\, y \in S$.

A fuzzy subset f of an AG-groupoid S is called fuzzy left (right) ideal of S if $f(xy) \geq f(y)$ ($f(xy) \geq f(x)$) for all $x,\, y \in S$.

A fuzzy subset f of an AG-groupoid S is called fuzzy two-sided ideal of S if it is both fuzzy left and fuzzy right ideal of S.

A fuzzy AG-subgroupoid f of an AG-groupoid S is called fuzzy bi-ideal of S if $f((xy)z) \geq f(x) \wedge f(z)$, for all x, y and $z \in S$.

A fuzzy subset f of an AG-groupoid S is called fuzzy generalized bi-ideal of S if $f((xy)z) \geq f(x) \wedge f(z)$, for all x, y and $z \in S$.

A fuzzy subset f of an AG-groupoid S is called fuzzy quasi-ideal of S if $(f \circ S)(x) \wedge (S \circ f)(x) \leq f(x)$, for all $x \in S$.

Let $F(S)$ denote the collection of all fuzzy subsets of an AG-groupoid S with left identity, then $(F(S), \circ)$ becomes an AG-groupoid with left identity S, that is $(F(S), \circ)$ satisfies left invertive law, medial law, paramedial law and property (1). Note that S can be considered as a fuzzy subset of S itself and we write $S = C_S$, that is, $S(x) = 1$ for all $x \in S$. Moreover $S \circ S = S$.

The characteristic function C_A for a subset A of an AG-groupoid S is defined by

$$C_A(x) = \begin{cases} 1, \text{ if } x \in A, \\ 0, \text{ if } x \notin A. \end{cases}$$

The proof of the following three lemmas are the same as in [20].

Definition 1 A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q)$-fuzzy AG-subgroupoid of S if

$$x_t \in f, y_r \in f \Rightarrow (xy)_{t\wedge r} \in \vee q f$$

for all $x, y \in S$ and $t, r \in (0,1]$.

Theorem 1 *Let f be a fuzzy subset of S. Then f is an $(\in, \in \vee q)$-fuzzy AG-subgroupoid of S if and only if $f(xy) \geq \min\{f(x), f(y), \frac{1-k}{2}\}$ for all $x, y \in S$.*

Proof. It is similar to the proof of Theorem 12 in [28]. ∎

Definition 2 A fuzzy subset f of an AG-groupoid S is called an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S if it satisfies the following condition:

$$y_t \in f \Rightarrow (xy)_t \in \vee q f \ \ (\text{resp. } y_t \in f \Rightarrow (yx)_t \in \vee q f)$$

for all $x, y \in S$ and $t \in (0,1]$.

A fuzzy subset f of S is called an $(\in, \in \vee q)$-fuzzy ideal of S if it is both an $(\in, \in \vee q)$-fuzzy left ideal and an $(\in, \in \vee q)$-fuzzy right ideal of S.

Theorem 2 *A fuzzy subset f of S is an $(\in, \in \vee q)$-fuzzy left (resp. right) ideal of S, if and only if*

$$f(xy) \geq \min\left\{f(y), \frac{1-k}{2}\right\} \ \left(\textit{resp. } f(xy) \geq \min\left\{f(x), \frac{1-k}{2}\right\}\right).$$

Proof. It is similar to the proof of Lemma 5 in [28]. ∎

Definition 3 Let S be an AG-groupoid, and f be a fuzzy subset of S. Then f is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S, if for all x, y,$z \in S$ and t, $r \in (0,1]$, we have

$$x_t \in f, z_r \in f \Rightarrow ((xy)z)_{t\wedge r} \in \vee q f.$$

An $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S is called an $(\in, \in \vee q)$-fuzzy bi-ideal of S if it is also an $(\in, \in \vee q)$-fuzzy AG-subgroupoid of S.

Definition 4 A fuzzy subset f of an AG-groupoid $\mathcal{S}$ is called an $(\in, \in \vee q)$-fuzzy quasi-ideal of S if it satisfies $f(x) \geq \min(f \circ C_S(x), C_S \circ f(x), \frac{1-k}{2})$, where C_S is the fuzzy subset of S mapping every element of S on 1.

Theorem 3 *Let f be a fuzzy subset of an AG-groupoid S. Then f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S if and only if it satisfies:*

(i) $f(xy) \geq \min\{f(x),\ f(y), \frac{1-k}{2}\}$, *for all $x, y \in S$.*

(ii) $f((xy)z) \geq \min\{f(x), f(z), \frac{1-k}{2}\}$, *for all $x, y, z \in S$.*

Definition 5 An element a of an AG-groupoid S is called *regular* if there exist x in S such that $a = (ax)a$ and S is called *regular*, if every element of S is regular.

Definition 6 An element a of an AG-groupoid S is called *intra-regular* if there exist $x, y \in S$ such that $a = (xa^2)y$ and S is called *intra-regular*, if every element of S is intra-regular.

Lemma 1 *For an AG-groupoid S, the following holds.*

(i) *A non empty subset I of AG-groupoid S is an ideal if and only if $(C_I)_k$ is $(\in, \in \vee q_k)$-fuzzy ideal.*

(ii) *A non empty subset L of AG-groupoid S is left ideal if and only if $(C_L)_k$ is $(\in, \in \vee q_k)$-fuzzy left ideal.*

(iii) *A non empty subset R of AG-groupoid S is right ideal if and only if $(C_R)_k$ is $(\in, \in \vee q_k)$-fuzzy right ideal.*

(iv) *A non empty subset B of AG-groupoid S is bi-ideal if and only if $(C_B)_k$ is $(\in, \in \vee q_k)$-fuzzy bi-ideal.*

(v) *A non empty subset Q of AG-groupoid S is quasi-ideal if and only if $(C_Q)_k$ is $(\in, \in \vee q_k)$-fuzzy quasi-ideal.*

Lemma 2 *Let A, B be non empty subsets of an AG-groupoid S. Then the following holds.*

(i) $(C_{A\cap B})_k = (C_A \wedge_k C_B)$.

(ii) $(C_{A\cup B})_k = (C_A \vee_k C_B)$.

(iii) $(C_{AB})_k = (C_A \circ_k C_B)$.

Lemma 3 *Let S be an AG-groupoid. If $a = a(ax)$, for some x in S. Then $a = a^2y$, for some y in S.*

Proof. Using the medial law, we get

$$a = a(ax) = [a(ax)](ax) = (aa)((ax)x) = a^2y, \text{ where } y = (ax)x. \qquad \blacksquare$$

Lemma 4 *Let S be an AG-groupoid with left identity. If $a = a^2x$, for some x in S. Then $a = (ay)a$, for some y in S.*

Proof. Using the medial law, the left invertive law, (1), the paramedial law and the medial law, we get

$$\begin{aligned} a &= a^2x = (aa)x = ((a^2x)(a^2x))x = ((a^2a^2)(xx))x = (xx^2)(a^2a^2) \\ &= a^2((xx^2)a^2)) = ((xx^2)a^2)a)a = ((aa^2)(xx^2))a = ((x^2x)(a^2a))a \\ &= [a^2\{(x^2x)a\}]a = [\{a(x^2x)\}(aa)]a = [a(\{a(x^2x)\}a)]a \\ &= (ay)a, \text{ where } y = \{a(x^2x)\}a. \end{aligned}$$

$\blacksquare$

Lemma 5 *In AG-groupoid S, with left identity, the following holds.*

(i) $(aS)\,a^2 = (aS)\,a.$

(ii) $(aS)\,((aS)\,a) = (aS)\,a.$

(iii) $S\,((aS)\,a) = (aS)\,a.$

(iv) $(Sa)\,(aS) = a\,(aS)\,.$

(v) $(aS)\,(Sa) = (aS)\,a.$

(vi) $[a(aS)]S = (aS)a.$

(vii) $[(Sa)S](Sa) = (aS)(Sa).$

(viii) $(Sa)S = (aS).$

(ix) $S(Sa) = Sa.$

(x) $Sa^2 = a^2S.$

Proof. It is easy. ∎

Lemma 6 *Every intra-regular AG-groupoid with left identity is regular but the converse is not true.*

Proof. It is easy. ∎

For the converse of Lemma 6, see the following example.

Example 1 Let us consider an AG-groupoid $S = \{1, 2, 3\}$ in the following multiplication table.

$\circ$	1	2	3
1	1	1	1
2	1	1	3
3	1	2	1

It is easy to check that $\{1, 2\}$ is the quasi-ideal of S. Clearly S is regular because $1 = 1 \circ 1$, $2 = (2 \circ 3) \circ 2$ and $3 = (3 \circ 2) \circ 3$. But is not intra-regular AG-groupoid.

Let us define a fuzzy subset f on S as follows:

$$f(x) = \begin{cases} 0.9 & \text{for } x = 1 \\ 0.8 & \text{for } x = 2 \\ 0.6 & \text{for } x = 3 \end{cases}$$

Then, clearly, f is an $(\in, \in \vee q_k)$-fuzzy ideal of S.

Theorem 4 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For bi-ideal B, ideal I and left ideal L of S, $B \cap I \cap L \subseteq (BI)\,L$.*

(iii) *$B\,[a] \cap I\,[a] \cap L\,[a] \subseteq (B\,[a]\,I\,[a])\,L\,[a]$, for some a in S.*

Proof. (i) $\Rightarrow$ (ii) Assume that B, I and L are bi-ideal, ideal and left ideal of a regular AG-groupoid S respectively. Let $a \in B \cap I \cap L$. This implies that $a \in I$, $a \in B$ and $a \in L$. Since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)\,a = (((ax)\,a)\,x)\,a = ((xa)\,(ax))\,a = (a\,((xa)\,x))\,a = (B\,((SI)\,S))\,L = (BI)\,L$. Thus, $B \cap I \cap L \subseteq (BI)\,L$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) $B\,[a] = a \cup a^2 \cup (aS)\,a$, $I\,[a] = a \cup Sa \cup aS$ and $L\,[a] = a \cup Sa$ are principle bi-ideal, principle ideal and principle left ideal of S generated by a respectively. Thus, by (iii), Lemma 5, (1), left invertive law and paramedial law we have,

$$\begin{aligned}
&(a \cup a^2 \cup (aS)\,a) \cap (a \cup Sa \cup aS) \cap (a \cup Sa)\\
&\quad \subseteq ((a \cup a^2 \cup (aS)\,a)\,(a \cup Sa \cup aS))\,(a \cup Sa)\\
&\quad \subseteq \{S\,(a \cup Sa \cup aS)\}\,(a \cup Sa)\\
&\quad \subseteq \{Sa \cup S\,(Sa) \cup S\,(aS)\}\,(a \cup Sa)\\
&\quad = (Sa \cup aS)\,(a \cup Sa)\\
&\quad = (Sa)\,a \cup (Sa)\,(Sa) \cup (aS)\,a \cup (aS)\,(Sa)\\
&\quad = a^2S \cup a^2S \cup (aS)\,a \cup (aS)\,a\\
&\quad = a^2S \cup (aS)\,a.
\end{aligned}$$

Hence S is regular. ∎

Theorem 5 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For $(\in, \in \vee q_k)$-fuzzy bi-ideal f, $(\in, \in \vee q_k)$-fuzzy ideal g, and $(\in, \in \vee q_k)$-fuzzy left ideal h of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.*

(iii) *For $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal f, $(\in, \in \vee q_k)$-fuzzy ideal g, and $(\in, \in \vee q_k)$-fuzzy left ideal h of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.*

Proof. (i) $\Rightarrow$ (iii) Assume that f, g and h are $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal, $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy left ideal of a regular AG-groupoid S, respectively. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)\,a = (((ax)\,a)\,x)\,a = ((xa)\,(ax))\,a = (a\,((xa)\,x))\,a$. Thus,

$$\begin{aligned}
((f \circ_k g) \circ_k h)(a) &= \bigvee_{a=pq} (f \circ_k g)(p) \wedge h(q) \wedge \frac{1-k}{2}\\
&= \bigvee_{a=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge g(v) \wedge \frac{1-k}{2} \right\} \wedge h(q) \wedge \frac{1-k}{2} \right)\\
&= \bigvee_{a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{a=(a((xa)x))a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&\geq \{f(a) \wedge g((xa)x)\} \wedge h(a) \wedge \frac{1-k}{2} \\
&\geq \left\{ f(a) \wedge \left(g(a) \wedge \frac{1-k}{2} \right) \right\} \wedge h(a) \wedge \frac{1-k}{2} \\
&= \{f(a) \wedge g(a) \wedge \frac{1-k}{2}\} \wedge h(a) \wedge \frac{1-k}{2} \\
&= ((f \wedge_k g) \wedge_k h)(a).
\end{aligned}
$$

Therefore $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

(iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (i) Assume that B, I and L are bi-ideal, ideal and left ideal of S respectively. Then, by Lemma 1, $(C_B)_k$, $(C_I)_k$ and $(C_L)_k$ are $(\in, \in \vee q_k)$-fuzzy bi-ideal, $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy left ideal of S respectively. Therefore, by Lemma 2, we have, $(C_{B \cap I \cup L})_k = (C_B \wedge_k C_I) \wedge_k C_L \leq (C_B \circ_k C_I) \circ_k C_L = (C_{(BI)L})_k = (C_{(BI)L})_k$. Therefore $B \cap I \cap L \subseteq (BI)L$. Hence, by Theorem 4, S is regular. ∎

Theorem 6 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For left ideal L, ideal I and quasi-ideal Q of S, $L \cap I \cap Q \subseteq (LI)Q$.*

(iii) *$L[a] \cap I[a] \cap Q[a] \subseteq (L[a] I[a]) Q[a]$, for some a in S.*

Proof. (i) $\Rightarrow$ (ii) Assume that L, I and Q are left ideal, ideal and quasi-ideal of regular AG-groupoid S. Let $a \in L \cap I \cap Q$. This implies that $a \in L$, $a \in I$ and $a \in Q$. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)a = (((ax)a)x)a = ((xa)(ax))a = (a((xa)x))a \in (L((SI)S))Q \subseteq (LI)Q$. Thus $L \cap I \cap Q \subseteq (LI)Q$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) $L[a] = a \cup Sa$, $I[a] = a \cup Sa \cup aS$ and $Q[a] = a \cup (Sa \cap aS)$ are left ideal, ideal and quasi-ideal of S generated a respectively. Thus, by (iii), Lemma 5 and medial law we have,

$$
\begin{aligned}
(a \cup Sa) \cap (a \cup Sa \cup aS) \cap (a \cup (Sa \cap aS)) &\subseteq ((a \cup Sa)(a \cup Sa \cup aS)) \\
&\quad (a \cup (Sa \cap aS)) \\
&\subseteq \{(a \cup Sa)S\}(a \cup aS) \\
&= \{aS \cup (Sa)S\}(a \cup aS) \\
&= (aS)(a \cup aS) \\
&= (aS)a \cup (aS)(aS) \\
&= (aS)a \cup a^2 S.
\end{aligned}
$$

Hence S is regular. ∎

Theorem 7 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For $(\in,\in\vee q_k)$-fuzzy left ideal f, $(\in,\in\vee q_k)$-fuzzy ideal g, and $(\in,\in\vee q_k)$-fuzzy quasi-ideal h of S, $(f\wedge_k g)\wedge_k h\leq(f\circ_k g)\circ_k h$.*

Proof. (i) $\Rightarrow$ (ii) Assume that f, g and h are $(\in,\in\vee q_k)$-fuzzy left ideal, $(\in,\in\vee q_k)$-fuzzy ideal and $(\in,\in\vee q_k)$-fuzzy quasi-ideal of a regular AG-groupoid S, respectively. Now, since S is regular so for $a\in S$ there exist $x\in S$ such that using left invertive law and (1), we have, $a=(ax)\,a=(((ax)\,a)\,x)\,a=((xa)\,(ax))\,a=(a\,((xa)\,x))\,a$. Thus,

$$
\begin{aligned}
((f\circ_k g)\circ_k h)(a) &= \bigvee_{a=pq}(f\circ_k g)(p)\wedge h(q)\wedge\frac{1-k}{2}\\
&= \bigvee_{a=pq}\left(\left\{\bigvee_{p=uv}f(u)\wedge g(v)\wedge\frac{1-k}{2}\right\}\wedge h(q)\wedge\frac{1-k}{2}\right)\\
&= \bigvee_{a=(uv)q}\left(\{f(u)\wedge g(v)\}\wedge h(q)\wedge\frac{1-k}{2}\right)\\
&= \bigvee_{a=(a((xa)x))a=(uv)q}\left(\{f(u)\wedge g(v)\}\wedge h(q)\wedge\frac{1-k}{2}\right)\\
&\geq \{f(a)\wedge g\,((xa)\,x)\}\wedge h(a)\wedge\frac{1-k}{2}\\
&\geq \left\{f(a)\wedge\left(g(a)\wedge\frac{1-k}{2}\right)\right\}\wedge h(a)\wedge\frac{1-k}{2}\\
&= \{f\,(a)\wedge g\,(a)\wedge\frac{1-k}{2}\}\wedge h\,(a)\wedge\frac{1-k}{2}\\
&= ((f\wedge_k g)\wedge_k h)\,(a)\,.
\end{aligned}
$$

Therefore, $(f\wedge_k g)\wedge_k h\leq(f\circ_k g)\circ_k h$.

(ii) $\Longrightarrow$ (i) Assume that L, I and Q are left ideal, ideal and quasi-ideal of S respectively. Thus, by Lemma 1, $(C_L)_k$, $(C_I)_k$ and $(C_Q)_k$ are $(\in,\in\vee q_k)$-fuzzy left ideal, $(\in,\in\vee q_k)$-fuzzy ideal and $(\in,\in\vee q_k)$-fuzzy quasi-ideal of S respectively. Therefore, by Lemma 2, we have, $(C_{L\cap I\cup Q})_k=(C_L\wedge_k C_I)\wedge_k C_Q\leq(C_L\circ_k C_I)\circ_k C_Q=(C_{(LI)Q})_k=(C_{(LI)Q})_k$. Therefore $L\cap I\cap Q\subseteq(LI)\,Q$. Hence by Theorem 6, S is regular. ∎

Theorem 8 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For bi-ideal B, ideal I and quasi-ideal Q of S, $B\cap I\cap Q\subseteq(BI)\,Q$.*

(iii) $B\,[a]\cap I\,[a]\cap Q\,[a]\subseteq(B\,[a]\,I\,[a])\,Q\,[a]$, *for some a in S.*

Proof. (i) $\Rightarrow$ (ii) Assume that B, I and Q are bi-ideal, ideal and quasi-ideal of regular AG-groupoid S. Let $a \in B \cap I \cap Q$. This implies that $a \in B$, $a \in I$ and $a \in Q$. Now, since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)\,a = (((ax)\,a)\,x)\,a = ((xa)\,(ax))\,a = (a\,((xa)\,x))\,a \in (B\,((SI)\,S))\,Q \subseteq (BI)\,Q$. Thus, $B \cap I \cap Q \subseteq (BI)\,Q$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) Since $B\,[a] = a \cup a^2 \cup (aS)\,a$, $I\,[a] = a \cup Sa \cup aS$ and $Q\,[a] = a \cup (Sa \cap aS)$ are principle bi-ideal, principle ideal and principle quasi-ideal of S generated by a respectively. Thus, by (ii) and Lemma 5, (1), medial law and left invertive law, we have,

$$\begin{aligned}
&(a \cup a^2 \cup (aS)\,a) \cap (a \cup Sa \cup aS) \cap (a \cup (Sa \cap aS)) \\
&\quad \subseteq ((a \cup a^2 \cup (aS)\,a)\,(a \cup Sa \\
&\quad \cup aS))\,(a \cup (Sa \cap aS)) \\
&\quad \subseteq (S(a \cup Sa \cup aS))\,(a \cup aS) \\
&\quad = (Sa \cup S\,(Sa) \cup S\,(aS))\,(a \cup aS) \\
&\quad = (Sa \cup S\,(Sa) \cup S\,(aS))\,(a \cup aS) \\
&\quad = (aS \cup Sa)\,(a \cup aS) \\
&\quad = (aS)\,a \cup (aS)\,(aS) \cup (Sa)\,a \cup (Sa)\,(aS) \\
&\quad = (aS)\,a \cup a^2S \cup a\,(aS)\,.
\end{aligned}$$

Hence S is regular. ∎

Theorem 9 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For $(\in, \in \vee q_k)$-fuzzy bi-ideal f, $(\in, \in \vee q_k)$-fuzzy ideal g, and $(\in, \in \vee q_k)$-fuzzy quasi ideal h of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.*

(iii) *For $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal f, $(\in, \in \vee q_k)$-fuzzy ideal g, and $(\in, \in \vee q_k)$-fuzzy quasi ideal h of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.*

Proof. (i) $\Rightarrow$ (iii) Assume that f, g and h are $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal, $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy quasi ideal of a regular AG-groupoid S, respectively. Now, since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)\,a = (((ax)\,a)\,x)\,a = ((xa)\,(ax))\,a = (a\,((xa)\,x))\,a$. Thus,

$$\begin{aligned}
((f \circ_k g) \circ_k h)(a) &= \bigvee_{a=pq} (f \circ_k g)(p) \wedge h(q) \wedge \tfrac{1-k}{2} \\
&= \bigvee_{a=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge g(v) \wedge \tfrac{1-k}{2} \right\} \wedge h(q) \wedge \tfrac{1-k}{2} \right) \\
&= \bigvee_{a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \tfrac{1-k}{2} \right) \\
&= \bigvee_{a=(a((xa)x))a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \tfrac{1-k}{2} \right)
\end{aligned}$$

$$\begin{aligned}
&\geq \{f(a) \wedge g((xa)x)\} \wedge h(a) \wedge \tfrac{1-k}{2} \\
&\geq \left\{f(a) \wedge \left(g(a) \wedge \tfrac{1-k}{2}\right)\right\} \wedge h(a) \wedge \tfrac{1-k}{2} \\
&= \left\{f(a) \wedge g(a) \wedge \tfrac{1-k}{2}\right\} \wedge h(a) \wedge \tfrac{1-k}{2} \\
&= ((f \wedge_k g) \wedge_k h)(a).
\end{aligned}$$

Therefore, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

(iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (i) Assume that B, I and Q be bi-ideal, ideal and quasi-ideal of S respectively. Then, by Lemma 1, $(C_B)_k$, $(C_I)_k$ and $(C_Q)_k$ are $(\in, \in \vee q_k)$-fuzzy bi-ideal, $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S respectively. Therefore, by Lemma 2, we have, $(C_{B\cap I\cup Q})_k = (C_B \wedge_k C_I) \wedge_k C_Q \leq (C_B \circ_k C_I) \circ_k C_Q = (C_{(BI)Q})_k = (C_{(BI)Q})_k$. Therefore $B \cap I \cap Q \subseteq (BI)Q$. Hence, by Theorem 8, S is regular. ■

Theorem 10 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For an ideals I_1, I_2 and I_3 of S, $I_1 \cap I_2 \cap I_3 \subseteq (I_1 I_2) I_3$.*

(iii) *$I[a] \cap I[a] \cap I[a] \subseteq (I[a]I[a])I[a]$, for some a in S.*

Proof. (i) $\Rightarrow$ (ii) Assume that I_1, I_2, and I_2 are ideals of a regular AG-groupoid S. Let $a \in I_1 \cap I_2 \cap I_3$. This implies that $a \in I_1$, $a \in I_2$ and $a \in I_3$. Now, since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)a = (((ax)a)x)a = ((xa)(ax))a = (a((xa)x))a \in (I_1((SI_2)S))I_3 \subseteq (I_1 I_2) I_3$. Thus, $I_1 \cap I_2 \cap I_3 \subseteq (I_1 I_2) I_3$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i)

Since $I[a] = a \cup Sa \cup aS$ is a principle ideal of S generated by a. Thus, by (iii), Lemma 5, left invertive law, medial law and paramedial law we have,

$$\begin{aligned}
&(a \cup Sa \cup aS) \cap (a \cup Sa \cup aS) \cap (a \cup Sa \cup aS) \\
&\quad \subseteq ((a \cup Sa \cup aS)(a \cup Sa \cup aS)) \\
&\quad (a \cup Sa \cup aS) \\
&\quad \subseteq \{(a \cup Sa \cup aS)S\}(a \cup Sa \cup aS) \\
&\quad = \{aS \cup (Sa)S \cup (aS)S\}(a \cup Sa \cup aS) \\
&\quad = \{aS \cup Sa\}(a \cup Sa \cup aS) \\
&\quad = (aS)a \cup (aS)(Sa) \cup (aS)(aS) \cup (Sa)a \\
&\quad \cup (Sa)(Sa) \cup (Sa)(aS) \\
&\quad = (aS)a \cup a^2 S.
\end{aligned}$$

Hence S is regular. ■

Theorem 11 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For quasi-ideals Q_1, Q_2 and ideal I of S, $Q_1 \cap I \cap Q_2 \subseteq (Q_1 I)\, Q_2$.*

(iii) $Q[a] \cap I[a] \cap Q[a] \subseteq (Q[a]\, I[a])\, Q[a]$, *for some a in S.*

Proof. (i) $\Rightarrow$ (ii) Assume that Q_1 and Q are quasi-ideal and I is an ideal of a regular AG-groupoid S. Let $a \in Q_1 \cap I \cap Q_2$. This implies that $a \in Q_1$, $a \in I$ and $a \in Q_2$. Now, since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)\, a = (((ax)\, a)\, x)\, a = ((xa)\,(ax))\, a = (a\,((xa)\, x))\, a \in (Q_1\,((SI)\, S))\, Q_2 \subseteq (Q_1 I)\, Q_2$. Thus, $Q_1 \cap I \cap Q_2 \subseteq (Q_1 I)\, Q_2$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) $Q[a] = a \cup (Sa \cap aS)$ and $I[a] = a \cup Sa \cup aS$ are principle quasi-ideal and principle ideal of S generated by a respectively. Thus by (iii), left invertive law, medial law and Lemma 5, we have,

$$\begin{aligned}
&(a \cup (Sa \cap aS)) \cap (a \cup Sa \cup aS) \cap (a \cup (Sa \cap aS)) \\
&\quad \subseteq ((a \cup (Sa \cap aS))\,(a \cup Sa \cup aS)) \\
&\quad (a \cup (Sa \cap aS)) \\
&\quad \subseteq \{(a \cup aS)\, S\}\,(a \cap aS) \\
&\quad = \{aS \cup (aS)\, S\}\,(a \cap aS) \\
&\quad = (aS \cup Sa)\,(a \cap aS) \\
&\quad = \{(aS)\, a \cup (aS)\,(aS) \cup (Sa)\, a \cup (Sa)\, aSa \\
&\quad = (aS)\, a \cup a^2 S \cup a\,(aS)\,.
\end{aligned}$$

Hence, S is regular. ∎

Theorem 12 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For $(\in, \in \vee q_k)$-fuzzy quasi-ideals f, h, and $(\in, \in \vee q_k)$-fuzzy ideal g, of S, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.*

Proof. (i) $\Rightarrow$ (ii) Assume that f, h are $(\in, \in \vee q_k)$-fuzzy quasi-ideal and g is $(\in, \in \vee q_k)$-fuzzy ideal of a regular AG-groupoid S, respectively. Now since S is regular so for $a \in S$ there exist $x \in S$ such that using left invertive law and (1), we have, $a = (ax)\, a = (((ax)\, a)\, x)\, a = ((xa)\,(ax))\, a = (a\,((xa)\, x))\, a$. Thus,

$$
\begin{aligned}
((f \circ_k g) \circ_k h)(a) &= \bigvee_{a=pq} (f \circ_k g)(p) \wedge h(q) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge g(v) \wedge \frac{1-k}{2} \right\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{a=(a((xa)x))a=(uv)q} \left(\{f(u) \wedge g(v)\} \wedge h(q) \wedge \frac{1-k}{2} \right) \\
&\geq \{f(a) \wedge g((xa)x)\} \wedge h(a) \wedge \frac{1-k}{2} \\
&\geq \left\{ f(a) \wedge \left(g(a) \wedge \frac{1-k}{2} \right) \right\} \wedge h(a) \wedge \frac{1-k}{2} \\
&= \{f(a) \wedge g(a) \wedge \frac{1-k}{2}\} \wedge h(a) \wedge \frac{1-k}{2} \\
&= ((f \wedge_k g) \wedge_k h)(a).
\end{aligned}
$$

Therefore, $(f \wedge_k g) \wedge_k h \leq (f \circ_k g) \circ_k h$.

(ii) $\Longrightarrow$ (i) Assume that Q_1 and Q_2 are quasi-ideals and I is an ideal of S respectively. Thus, by Lemma 1, $(C_{Q_1})_k$, $(C_I)_k$ and $(C_{Q_2})_k$ are $(\in, \in \vee q_k)$-fuzzy quasi-ideal, $(\in, \in \vee q_k)$-fuzzy ideal and $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S respectively. Therefore, by Lemma 2, we have,

$$(C_{Q_1 \cap I \cup Q_2})_k = (C_{Q_1} \wedge_k C_I) \wedge_k C_{Q_2} \leq (C_{Q_1} \circ_k C_L) \circ_k C_{Q_2} = (C_{(Q_1 I) Q_2})_k = (C_{(Q_1 I) Q_2})_k.$$

Therefore $Q_1 \cap I \cap Q_2 \subseteq (Q_1 I) Q_2$. Hence, by Theorem 11, S is regular. ∎

Theorem 13 *For an AG-groupoid S with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For bi-ideal B, $B = (BS) B$.*

(iii) *For generalized bi-ideal B, $B = (BS) B$.*

Proof. The proof is straightforward. ∎

Theorem 14 *For an AG-groupoid S, with left identity, the following are equivalent.*

(i) *S is regular.*

(ii) *For $(\in, \in \vee q_k)$-fuzzy bi-ideal f, of S, $f_k = (f \circ_k S) \circ_k f$.*

(iii) *For $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal f, of S, $f_k = (f \circ_k S) \circ_k f$.*

Proof. (i) $\Rightarrow$ (iii) Assume that f is $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of a regular AG-groupoid S. Since S is regular so for $b \in S$ there exist $x \in S$ such that $b = (bx)\,b$. Therefore, we have,

$$\begin{aligned}
((f \circ_k S) \circ_k f)(b) &= \bigvee_{b=pq} (f \circ_k S)(p) \wedge f(q) \wedge \frac{1-k}{2} \\
&= \bigvee_{b=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge S(v) \wedge \frac{1-k}{2} \right\} \wedge f(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{b=(uv)q} \left(\{ f(u) \wedge S(v) \} \wedge f(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{b=(bx)b=(uv)q} \left(\{ f(u) \wedge S(v) \} \wedge f(q) \wedge \frac{1-k}{2} \right) \\
&\geq \{ f(b) \wedge S\,(x) \} \wedge f(b) \wedge \frac{1-k}{2} \\
&\geq f(b) \wedge 1 \wedge f(b) \wedge \frac{1-k}{2} \\
&= f\,(b) \wedge \frac{1-k}{2} = f_k\,(b)\,.
\end{aligned}$$

Thus $(f \circ_k S) \circ_k f \geq f_k$. Since f is $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of a regular AG-groupoid S. So we have,

$$\begin{aligned}
((f \circ_k S) \circ_k f)(b) &= \bigvee_{b=pq} (f \circ_k S)(p) \wedge f(q) \wedge \frac{1-k}{2} \\
&= \bigvee_{b=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge S(v) \wedge \frac{1-k}{2} \right\} \wedge f(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{b=pq} \left(\left\{ \bigvee_{p=uv} f(u) \wedge 1 \right\} \wedge f(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{b=pq} \left(\bigvee_{p=uv} f(u) \wedge f(q) \wedge \frac{1-k}{2} \right) \\
&= \bigvee_{b==pq} \left\{ \bigvee_{p=uv} \left(f(u) \wedge f(q) \wedge \frac{1-k}{2} \right) \right\} \\
&\leq \bigvee_{b==(uv)q} \left(f\,((uv)q) \wedge \frac{1-k}{2} \right) \\
&= f\,(b) \wedge \frac{1-k}{2} = f_k\,(b)\,.
\end{aligned}$$

This implies that $(f \circ_k S) \circ_k f \leq f_k$. Thus $(f \circ_k S) \circ_k f = f_k$.

(iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (i) Assume that B is a bi-ideal of S. Then, by Lemma 1, $(C_B)_k$, is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. Therefore, by *(ii)* and Lemma 2, we have, $(C_B)_k = (C_B \circ_k C_S) \circ_k C_B = (C_{(BS)B})_k$. Therefore $B = (BS)\, B$. Hence, by Theorem 13, S is regular. ∎

Acknowledgements. This research is partially supported by a grant of National Natural Science Foundation of China (61175055), Natural Science Foundation of Hubei Province (2012FFB01101) and Natural Science Foundation of Education Committee of Hubei Province (D20131903).

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Accepted: 10.12.2013