SOFT INTERSECTION h-IDEALS OF HEMIRINGS AND ITS APPLICATIONS

Xueling Ma

Jianming Zhan[1]

Department of Mathematics
Hubei University for Nationalities
Enshi, Hubei Province 445000
China

Abstract. In this paper, we introduce a new kind of soft hemirings called soft intersection hemirings and obtain some related properties. Some basic operations are also investigated. Finally, we describe some characterizations of h-hemiregular hemirings by means of SI-h-ideals.

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1. Introduction

The complexities of modeling uncertain data in economics, engineering, environmental science, sociology and many other fields can not be successfully dealt with by classical methods. In order to overcome these difficulties, Molodtsov [17] introduced the concept of soft sets as a new mathematical tool for dealing with uncertainties. Maji [15] discussed further soft set theory. Ali et al. [3] proposed some new operations on soft sets. In the same time, this theory has proven useful in many different fields such as decision making [5], [6], [7], [9], [16], data analysis, forecasting and so on. Recently, the algebraic structures of soft sets have been studied increasingly, such as soft rings [1], soft-int groups [4], soft semirings [8], soft BCK/BCI-algebras [11], soft intersection near-rings [18] and so on.

On the other hand, semirings have been found useful for dealing with problems in different areas of applied mathematics and information sciences, as the semiring structure provides an algebraic framework for modeling and investigating the key factors in these problems. We know that ideals in the semiring S do not in general coincide with the usual ring ideals if S is a ring, and so many results in ring theory have no analogues in semirings using only ideals. Consequently, some

[1]Corresponding author. Tel/Fax: 0086-718-8437732. E-mails: zjmhbmy@126.com (X. Ma); zhanjianming@hotmail.com (J. Zhan)

more restricted concepts of ideals such as k-ideals and h-ideals [2], [19], [10], [15], [16], [14], [20], [21] have been investigated. Nowadays, many researchers discussed this theory including their application. In application, hemirings are useful in automata and formal languages.

In this paper, we introduce a new kind of soft hemirings called soft intersection hemirings and obtain some related properties. Some basic operations are also investigated. Finally, we describe some characterizations of h-hemiregular hemirings by means of SI-h-ideals.

2. Preliminaries

A semiring $(S, +, \cdot)$ with zero is called a hemiring if $(S, +)$ is commutative. A subhemiring of a hemiring S is a subset A of S closed under addition and multiplication. A left (resp., right) ideal of a hemiring S is a subset A of S closed under addition such that $SA \subseteq A$ (resp., $AS \subseteq A$). A subset A is called an ideal if it is both a left ideal and a right ideal. A subhemiring (left ideal, right ideal, ideal) A of S is called an h-subhemiring (left h-ideal, right h-ideal, h-ideal) of S, respectively, if for any $x, z \in S$, and $a, b \in A$, $x+a+z = b+z$ implies $x \in A$. The h-closure $\overline{A}$ of a subset A of S is defined as $\overline{A} = \{x \in S | x+a+z = b+z \text{ for some } a, b \in A,\ z \in S\}$.

Throughout this section, S is a hemiring, U is an initial universe, E is a set of parameters, $P(U)$ is the power set of U and $A, B, C \subseteq E$.

Definition 2.1 [17] A soft set f_A of U is a set defined by $f_A : E \to P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$. Here f_A is also called an approximate function. A soft set over U can be represented by the set of ordered pairs $f_A = \{(x, f_A(x)) | x \in E, f_A(x) \in P(U)\}$. It is clear to see that a soft set is a parameterized family of subsets of the set U. Note that the set of all soft sets over U will be denoted by $S(U)$.

Definition 2.2 [6] Let $f_A, f_B \in S(U)$, then

(i) The intersection of f_A and f_B, denoted by $f_A \widetilde{\cap} f_B$, is defined as $f_A \widetilde{\cap} f_B = f_{A\widetilde{\cap}B}$, where $f_{A\widetilde{\cap}B}(x) = f_A(x) \cap f_B(x)$, for all $x \in E$;

(ii) The union of f_A and f_B, denoted by $f_A \widetilde{\cup} f_B$, is defined as $f_A \widetilde{\cup} f_B = f_{A\widetilde{\cup}B}$, where $f_{A\widetilde{\cup}B}(x) = f_A(x) \cup f_B(x)$, for all $x \in E$.

Definition 2.3 [6] Let $f_A, f_B \in S(U)$. Then $\wedge$-product and $\vee$-product of f_A and f_B, denoted by $f_A \wedge f_B$ and $f_A \vee f_B$, are defined by $f_{A\wedge B}(x, y) = f_A(x) \cap f_B(y)$, $f_{A\vee B}(x, y) = f_A(x) \cup f_B(y)$ for all $x, y \in E$, respectively.

Definition 2.4 [4] Let f_A be a soft set over U and $\alpha \subseteq U$. Then, upper α-inclusion of f_A, denoted by $U(f_A; \alpha)$, is defined as $U(f_A; \alpha) = \{x \in A | f_A(x) \supseteq \alpha\}$.

3. SI-hemirings (SI-h-ideals)

In this paper, we introduce the concepts of soft intersection hemirings (soft intersection h-ideals) and obtain some related properties.

Definition 3.1 A soft set f_S over U is called a soft intersection hemiring (briefly, SI-hemiring) of S over U if it satisfies:

(SI_1) $f_S(x+y) \supseteq f_S(x) \cap f_S(y)$,

(SI_2) $f_S(xy) \supseteq f_S(x) \cap f_S(y)$,

(SI_3) $f_S(x) \supseteq f_S(a) \cap f_S(b)$ with $x+a+z = b+z$ for all $x, a, b, z \in S$.

Example 3.2 Let $U = S = Z_6 = \{0, 1, 2, 3, 4, 5\}$ be the hemiring of non-negative integers module 6. Define a soft set f_S over U by $f_S(0) = f_S(2) = f_S(4) = \{0, 1, 2, 3, 4, 5\}$ and $f_S(1) = f_S(3) = f_S(5) = \{0, 2, 4\}$. Then one can easily check that f_S is an SI-hemiring of S over U.

From the above definition, we can obtain the following:

Proposition 3.3 *If f_S is an SI-hemiring of S over U, then $f_S(0) \supseteq f_S(x)$ for all $x \in S$.*

Definition 3.4 A soft set f_S over U is called a soft intersection left (right) h-ideal (briefly, SI-left(right) h-ideal) of S over U if it satisfies $(SI_1), (SI_3)$ and:

(SI_4) $f_S(xy) \supseteq f_S(y)$ $(f_S(xy) \supseteq f_S(x))$, for all $x, y \in S$.

A soft set over U is called a soft intersection h-ideal (briefly, SI-h-ideal) of S if it is both an SI-left h-ideal and an SI-right h-ideal of S over U.

Example 3.5 Assume that $U = Z^+$ is the universal set and $S = Z_6$ is the set of parameters. Define a soft set f_S as $f_S(0) = \{n | n \in Z^+\}$, $f_S(1) = f_S(5) = \{6n| n \in Z^+\}$, $f_S(2) = f_S(4) = \{2n|n \in Z^+\}$ and $f_S(3) = \{3n|n \in Z^+\}$. Then, one can easily check that f_S is an SI-h-ideal of S over U.

Proposition 3.6 *Let f_{S_1} and f_{S_2} be two SI-hemirings of S_1 and S_2 over U, respectively. Then $f_{S_1} \wedge f_{S_2}$ is an SI-hemiring of $S_1 \times S_2$ over U.*

Proof. Let f_{S_1} and f_{S_2} be two SI-hemirings of S_1 and S_2 over U, respectively. Then, for all $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$, we have

$$
\begin{aligned}
(i) \quad f_{S_1 \wedge S_2}((x_1, y_1) + (x_2, y_2)) &= f_{S_1 \wedge S_2}(x_1 + x_2, y_1 + y_2) \\
&= f_{S_1}(x_1 + x_2) \cap f_{S_2}(y_1 + y_2) \\
&\supseteq (f_{S_1}(x_1) \cap f_{S_1}(x_2)) \cap (f_{S_2}(y_1) \cap f_{S_2}(y_2)) \\
&= (f_{S_1}(x_1) \cap f_{S_2}(y_1)) \cap (f_{S_1}(x_2) \cap f_{S_2}(y_2)) \\
&= f_{S_1 \wedge S_2}(x_1, y_1) \cap f_{S_1 \wedge S_2}(x_2, y_2).
\end{aligned}
$$

(ii) is similar to (i).

(iii) Let $(a_1,a_2),(b_1,b_2),(x_1,x_2),(z_1,z_2) \in S_1 \times S_2$ be such that $(x_1,x_2)+(a_1,a_2)+(z_1,z_2)=(b_1,b_2)+(z_1,z_2)$, and so $x_1+a_1+z_1=b_1+z_1$ and $x_2+a_2+z_2=b_2+z_2$. Then

$$\begin{array}{rcl} f_{S_1\wedge S_2}(x_1,x_2) & = & f_{S_1}(x_1)\cap f_{S_2}(x_2) \\ & \supseteq & (f_{S_1}(a_1)\cap f_{S_1}(b_1))\cap(f_{S_2}(a_2)\cap f_{S_2}(b_2)) \\ & = & (f_{S_1}(a_1)\cap f_{S_2}(a_2))\cap(f_{S_1}(b_1)\cap f_{S_2}(b_2)) \\ & = & f_{S_1\wedge S_2}(a_1,a_2)\cap f_{S_1\wedge S_2}(b_1,b_2). \end{array}$$

Hence, $f_{S_1\wedge S_2}$ is an SI-hemiring of $S_1\times S_2$ over U. ∎

Similarly, we can obtain the following result:

Proposition 3.7 *Let f_{S_1} and f_{S_2} be two SI-h-ideals of S_1 and S_2 over U, respectively. Then $f_{S_1}\wedge f_{S_2}$ is an SI-h-ideal of $S_1\times S_2$ over U.*

Remark 3.8 Note that $f_{S_1}\vee f_{S_2}$ is not an SI-hemiring or SI-h-ideal over U.

Example 3.9 Let $U=S_3$, symmetric group, be the universal set, $S_1=Z_5=\{0,1,2,3,4\}$ and $S_2=Z_2=\{0,1\}$ be two hemirings of non-negative integers module 5 and module 2, respectively. Define two soft sets f_{S_1} and f_{S_2} over U by $f_{S_1}(0)=S_3$, $f_{S_1}(1)=f_{S_1}(4)=\{(1),(12),(132)\}$, $f_{S_1}(2)=f_{S_1}(3)=\{(12),(123),(132)\}$, $f_{S_2}(0)=S_3, f_{S_2}(1)=\{(1),(12),(132)\}$. It is clear that f_{S_1} and f_{S_2} are two SI-hemirings over U. However, we have

$$\begin{array}{rcl} f_{S_1\vee S_2}((3,1)+(1,0)) & = & f_{S_1\vee S_2}(4,1) \\ & = & f_{S_1}(4)\cup f_{S_2}(1) \\ & = & \{(1),(12),(132)\}, \end{array}$$

but

$$\begin{array}{rcl} f_{S_1\vee S_2}(3,1)\cap f_{S_1\vee S_2}(1,0) & = & (f_{S_1}(3)\cup f_{S_2}(1))\cap(f_{S_1}(1)\cup f_{S_2}(0)) \\ & = & \{(1),(12),(123),(132)\}\cap S_3 \\ & = & \{(1),(12),(123),(132)\}. \end{array}$$

This implies that $f_{S_1\vee S_2}((3,1)+(1,0)) \not\supseteq f_{S_1\vee S_2}(3,1)\cap f_{S_1\vee S_2}(1,0)$. Hence, $f_{S_1\vee S_2}$ is not an SI-hemiring over U.

Theorem 3.10 *Let f_S and g_S be two SI-hemirings of S over U. Then $f_S\widetilde{\cap}g_S$ is also an SI-hemiring of S over U.*

Proof. Let f_S and g_S be two SI-hemirings of S over U. Then for all $x,y\in S$, we have

$$\begin{array}{rrcl} (i) & (f_S\widetilde{\cap}g_S)(x+y) & = & f_S(x+y)\cap g_S(x+y) \\ & & \supseteq & (f_S(x)\cap f_S(y))\cap(g_S(x)\cap g_S(y)) \\ & & = & (f_S(x)\cap g_S(x))\cap(f_S(y)\cap g_S(y)) \\ & & = & (f_S\widetilde{\cap}g_S)(x)\cap(f_S\widetilde{\cap}g_S)(y). \end{array}$$

(ii) is similar to (i).

(iii) Let $a, b, x, z \in S$ be such that $x + a + z = b + z$. Then

$$\begin{aligned}(f_S \widetilde{\cap} g_S)(x) &= f_S(x) \cap g_S(x) \\ &\supseteq (f_S(a) \cap f_S(b)) \cap (g_S(a) \cap g_S(b)) \\ &= (f_S(a) \cap g_S(a)) \cap (f_S(b) \cap g_S(b)) \\ &= (f_S \widetilde{\cap} g_S)(a) \cap (f_S \widetilde{\cap} g_S)(b).\end{aligned}$$

Hence $f_S \widetilde{\cap} g_S$ is an SI-hemiring of S over U. ■

Similarly, we can obtain the following theorem:

Theorem 3.11 *Let f_S and g_S be two SI-h-ideals of S over U, then $f_S \widetilde{\cap} g_S$ is also an SI-h-ideal of S over U.*

4. h-hemiregular hemirings

In this section, we describe the characterizations of h-hemiregular hemirings by means of SI-h-ideals.

Definition 4.1 [22] A hemiring S is called h-hemiregular if for each $x \in S$, these exist $a_1, a_2, z \in S$ such that $x + xa_1x + z = xa_2x + z$.

Lemma 4.2 [22] If A and B, are, respectively, a right and a left h-ideal of S, then $\overline{AB} \subseteq A \cap B$.

Lemma 4.3 [22] A hemiring S is h-hemiregular if and only if for any right h-ideal A and any left h-ideal B, we have $\overline{AB} = A \cap B$.

Definition 4.4 Let $f_S, g_S \in S(U)$. Define soft h-sum and soft h-product of f_S and g_S as follows:

(1) $(f_S +_h g_S)(x) = \bigcup\limits_{x+a_1+b_1+z=a_2+b_2+z} (f_S(a_1) \cap f_S(a_2) \cap g_S(b_1) \cap g_S(b_2))$ and $(f_S +_h g_S)(x) = \emptyset$ if x cannot be expressed as $x + a_1 + b_1 + z = a_2 + b_2 + z$.

(2) $(f_S \circ_h g_S)(x) = \bigcup\limits_{x+a_1b_1+z=a_2b_2+z} (f_S(a_1) \cap f_S(a_2) \cap g_S(b_1) \cap g_S(b_2))$ and $(f_S \circ_h g_S)(x) = \emptyset$ if x cannot be expressed as $x + a_1b_1 + z = a_2b_2 + z$.

Lemma 4.5 *Let f_S and g_S be an SI-right h-ideal and an SI-left h-ideal of S over U, respectively, then $f_S \circ_h g_S \widetilde{\subseteq} f_S \widetilde{\cap} g_S$.*

Proof. If $(f_S \circ_h g_S)(x) = \emptyset$, then it is clear that $f_S \circ_h g_S \widetilde{\subseteq} f_S \widetilde{\cap} g_S$. Otherwise, we have

$$\begin{aligned}
(f_S \circ_h g_S)(x) &= \bigcup_{x+a_1b_1+z=a_2b_2+z} (f_S(a_1) \cap f_S(a_2) \cap g_S(b_1) \cap g_S(b_2)) \\
&\subseteq \bigcup_{x+a_1b_1+z=a_2b_2+z} (f_S(a_1b_1) \cap f_S(a_2b_2) \cap g_S(a_1b_1) \cap g_S(a_2b_2)) \\
&\subseteq \bigcup_{x+a_1b_1+z=a_2b_2+z} (f_S(x) \cap g_S(x)) \\
&= f_S(x) \cap g_S(x) \\
&= (f_S \cap g_S)(x),
\end{aligned}$$

which implies, $f_S \circ_h g_S \widetilde{\subseteq} f_S \widetilde{\cap} g_S$. ∎

Definition 4.6 Let $A \subseteq S$. We denote $\mathcal{S}_A$ the soft characteristic function of A and define as

$$\mathcal{S}_A(x) = \begin{cases} U & \text{if } x \in A, \\ \emptyset & \text{if } x \notin A. \end{cases}$$

The following proposition is obvious and we omit the details.

Proposition 4.7 *Let $A, B \subseteq S$. Then the following hold:*

(1) $A \subseteq B \Rightarrow \mathcal{S}_A \widetilde{\subseteq} \mathcal{S}_B$.

(2) $\mathcal{S}_A \widetilde{\cap} \mathcal{S}_B = \mathcal{S}_{A \cap B}$.

(3) $\mathcal{S}_A \circ_h \mathcal{S}_B = \mathcal{S}_{\overline{AB}}$.

Theorem 4.8 *For any hemiring S, then the following are equivalent:*

(1) *S is h-hemiregular;*

(2) *$f_S \circ_h g_S = f_S \widetilde{\cap} g_S$ for any SI-right h-ideal f_S and SI-left h-ideal g_S of S over U.*

Proof. (1) $\Longrightarrow$ (2): Let S be an h-hemiregular hemiring, f_S and g_S an SI-right h-ideal and an SI-left h-ideal of S over U, respectively. By Lemma 4.5, we have $f_S \circ_h g_S \widetilde{\subseteq} f_S \widetilde{\cap} g_S$. Let $x \in S$, then there exist $a, a', z \in S$ such that $x + xax + z = xa'x + z$ since S is h-hemiregular. Thus, we have

$$\begin{aligned}
(f_S \circ_h g_S)(x) &= \bigcup_{x+a_1b_1+z=a_2b_2+z} (f_S(a_1) \cap f_S(a_2) \cap g_S(b_1) \cap g_S(b_2)) \\
&\supseteq f_S(xa) \cap f_S(xa') \cap g_S(x) \\
&\supseteq f_S(x) \cap g_S(x) \\
&= (f_S \cap g_S)(x),
\end{aligned}$$

which implies $f_S \circ_h g_S \widetilde{\supseteq} f_S \widetilde{\cap} g_S$. Thus, $f_S \circ_h g_S = f_S \widetilde{\cap} g_S$.

(2) $\Longrightarrow$ (1): Let R and L be any right h-ideal and left h-ideal of S, respectively. Then, by Lemma 4.2, we have $\overline{RL} \subseteq R \cap L$. Moreover, it is easy to check that $\mathcal{S}_R$ and $\mathcal{S}_L$ are an SI-right h-ideal and an SI-left h-ideal of S over U, respectively. Let $x \in R \cap L$, then, by Proposition 4.8, we have

$$\mathcal{S}_{\overline{RL}}(x) = (\mathcal{S}_R \circ_h \mathcal{S}_L)(x) = (\mathcal{S}_R \widetilde{\cap} \mathcal{S}_L)(x) = \mathcal{S}_{R \cap L}(x) = U,$$

and so $x \in \overline{RL}$. Then, $R \cap L \subseteq \overline{RL}$. Thus, $R \cap L = \overline{RL}$. It follows from Lemma 4.3 that S is h-hemiregular. ∎

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