SOFT INTERSECTION \(h\)-IDEALS OF HEMIRINGS 
AND ITS APPLICATIONS

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Abstract. In this paper, we introduce a new kind of soft hemirings called soft intersection hemirings and obtain some related properties. Some basic operations are also investigated. Finally, we describe some characterizations of \(h\)-hemiregular hemirings by means of \(SI-h\)-ideals.

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1. Introduction

The complexities of modeling uncertain data in economics, engineering, environmental science, sociology and many other fields can not be successfully dealt with by classical methods. In order to overcome these difficulties, Molodtsov [17] introduced the concept of soft sets as a new mathematical tool for dealing with uncertainties. Maji [15] discussed further soft set theory. Ali et al. [3] proposed some new operations on soft sets. In the same time, this theory has proven useful in many different fields such as decision making [5], [6], [7], [9], [16], data analysis, forecasting and so on. Recently, the algebraic structures of soft sets have been studied increasingly, such as soft rings [1], soft-int groups [4], soft semirings [8], soft BCK/BCI-algebras [11], soft intersection near-rings [18] and so on.

On the other hand, semirings have been found useful for dealing with problems in different areas of applied mathematics and information sciences, as the semiring structure provides an algebraic framework for modeling and investigating the key factors in these problems. We know that ideals in the semiring \(S\) do not in general coincide with the usual ring ideals if \(S\) is a ring, and so many results in ring theory have no analogues in semirings using only ideals. Consequently, some

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more restricted concepts of ideals such as \( k \)-ideals and \( h \)-ideals \[2\], \[19\], \[10\], \[15\], \[16\], \[14\], \[20\], \[21\] have been investigated. Nowadays, many researchers discussed this theory including their application. In application, hemirings are useful in automata and formal languages.

In this paper, we introduce a new kind of soft hemirings called soft intersection hemirings and obtain some related properties. Some basic operations are also investigated. Finally, we describe some characterizations of \( h \)-hemiregular hemirings by means of SI-\( h \)-ideals.

2. Preliminaries

A semiring \((S, +, \cdot)\) with zero is called a hemiring if \((S, +)\) is commutative. A subhemiring of a hemiring \(S\) is a subset \(A\) of \(S\) closed under addition and multiplication. A left (resp., right) ideal of a hemiring \(S\) is a subset \(A\) of \(S\) closed under addition such that \(SA \subseteq A\) (resp., \(AS \subseteq A\)). A subset \(A\) is called an ideal if it is both a left ideal and a right ideal. A subhemiring (left ideal, right ideal, ideal) \(A\) of \(S\) is called an \( h \)-subhemiring (left \( h \)-ideal, right \( h \)-ideal, \( h \)-ideal) of \(S\), respectively, if for any \(x, z \in S\), and \(a, b \in A\), \(x + a + z = b + z\) implies \(x \in A\). The \( h \)-closure \( \overline{A} \) of a subset \(A\) of \(S\) is defined as \( \overline{A} = \{x \in S|x + a + z = b + z\} \) for some \(a, b \in A\), \(z \in S\).

Throughout this section, \(S\) is a hemiring, \(U\) is an initial universe, \(E\) is a set of parameters, \(P(U)\) is the power set of \(U\) and \(A, B, C \subseteq E\).

**Definition 2.1** \[17\] A soft set \(f_A\) of \(U\) is a set defined by \(f_A : E \rightarrow P(U)\) such that \(f_A(x) = \emptyset\) if \(x \notin A\). Here \(f_A\) is also called an approximate function. A soft set over \(U\) can be represented by the set of ordered pairs \(f_A = \{(x, f_A(x))|x \in E, f_A(x) \in P(U)\}\). It is clear to see that a soft set is a parameterized family of subsets of the set \(U\). Note that the set of all soft sets over \(U\) will be denoted by \(S(U)\).

**Definition 2.2** \[6\] Let \(f_A, f_B \in S(U)\), then

(i) The intersection of \(f_A\) and \(f_B\), denoted by \(f_A \cap f_B\), is defined as \(f_A \cap f_B = f_{A \cap B}\), where \(f_{A \cap B}(x) = f_A(x) \cap f_B(x)\), for all \(x \in E\);

(ii) The union of \(f_A\) and \(f_B\), denoted by \(f_A \cup f_B\), is defined as \(f_A \cup f_B = f_{A \cup B}\), where \(f_{A \cup B}(x) = f_A(x) \cup f_B(x)\), for all \(x \in E\).

**Definition 2.3** \[6\] Let \(f_A, f_B \in S(U)\). Then \(\land\)-product and \(\lor\)-product of \(f_A\) and \(f_B\), denoted by \(f_A \land f_B\) and \(f_A \lor f_B\), are defined by \(f_{A \land B}(x, y) = f_A(x) \land f_B(y), f_{A \lor B}(x, y) = f_A(x) \lor f_B(y)\) for all \(x, y \in E\), respectively.

**Definition 2.4** \[4\] Let \(f_A\) be a soft set over \(U\) and \(\alpha \subseteq U\). Then, upper \(\alpha\)-inclusion of \(f_A\), denoted by \(U(f_A; \alpha)\), is defined as \(U(f_A; \alpha) = \{x \in A|f_A(x) \supseteq \alpha\}\).

3. SI-hemirings (SI-\( h \)-ideals)

In this paper, we introduce the concepts of soft intersection hemirings (soft intersection \( h \)-ideals) and obtain some related properties.
**Definition 3.1** A soft set $f_S$ over $U$ is called a soft intersection hemiring (briefly, SI-hemiring) of $S$ over $U$ if it satisfies:

1. $(SI_1)$ $f_S(x + y) \supseteq f_S(x) \cap f_S(y),$
2. $(SI_2)$ $f_S(xy) \supseteq f_S(x) \cap f_S(y),$
3. $(SI_3)$ $f_S(x) \supseteq f_S(a) \cap f_S(b)$ with $x + a + z = b + z$ for all $x, a, b, z \in S.$

**Example 3.2** Let $U = S = Z_6 = \{0, 1, 2, 3, 4, 5\}$ be the hemiring of non-negative integers module 6. Define a soft set $f_S$ over $U$ by $f_S(0) = f_S(2) = f_S(4) = \{0, 1, 2, 3, 4, 5\}$ and $f_S(1) = f_S(3) = f_S(5) = \{0, 2, 4\}$. Then one can easily check that $f_S$ is an SI-hemiring of $S$ over $U$.

From the above definition, we can obtain the following:

**Proposition 3.3** If $f_S$ is an SI-hemiring of $S$ over $U$, then $f_S(0) \supseteq f_S(x)$ for all $x \in S$.

**Definition 3.4** A soft set $f_S$ over $U$ is called a soft intersection left (right) $h$-ideal (briefly, SI-left(right) $h$-ideal) of $S$ over $U$ if it satisfies $(SI_1), (SI_3)$ and:

1. $(SI_4)$ $f_S(xy) \supseteq f_S(y) (f_S(xy) \supseteq f_S(x))$, for all $x, y \in S.$

A soft set over $U$ is called a soft intersection $h$-ideal (briefly, SI-$h$-ideal) of $S$ if it is both an SI-left $h$-ideal and an SI-right $h$-ideal of $S$ over $U$.

**Example 3.5** Assume that $U = Z^+$ is the universal set and $S = Z_6$ is the set of parameters. Define a soft set $f_S$ as $f_S(0) = \{n|n \in Z^+\}$, $f_S(1) = f_S(5) = \{6n|n \in Z^+\}$, $f_S(2) = f_S(4) = \{2n|n \in Z^+\}$ and $f_S(3) = \{3n|n \in Z^+\}$. Then, one can easily check that $f_S$ is an SI-$h$-ideal of $S$ over $U$.

**Proposition 3.6** Let $f_{S_1}$ and $f_{S_2}$ be two SI-hemirings of $S_1$ and $S_2$ over $U$, respectively. Then $f_{S_1} \wedge f_{S_2}$ is an SI-hemiring of $S_1 \times S_2$ over $U$.

**Proof.** Let $f_{S_1}$ and $f_{S_2}$ be two SI-hemirings of $S_1$ and $S_2$ over $U$, respectively. Then, for all $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$, we have

(i) $f_{S_1 \wedge S_2}(x_1, y_1) + (x_2, y_2)) = f_{S_1 \wedge S_2}(x_1 + x_2, y_1 + y_2)$

$= f_{S_1}(x_1 + x_2) \cap f_{S_2}(y_1 + y_2)$

$\supseteq (f_{S_1}(x_1) \cap f_{S_1}(x_2)) \cap (f_{S_2}(y_1) \cap f_{S_2}(y_2))$

$= (f_{S_1}(x_1) \cap f_{S_2}(y_1)) \cap (f_{S_1}(x_2) \cap f_{S_2}(y_2))$

$= f_{S_1 \wedge S_2}(x_1, y_1) \cap f_{S_1 \wedge S_2}(x_2, y_2).$

(ii) is similar to (i).
(iii) Let \((a_1, a_2), (b_1, b_2), (x_1, x_2), (z_1, z_2) \in S_1 \times S_2\) be such that \((x_1, x_2) + (a_1, a_2) + (z_1, z_2) = (b_1, b_2) + (z_1, z_2)\), and so \(x_1 + a_1 + z_1 = b_1 + z_1\) and \(x_2 + a_2 + z_2 = b_2 + z_2\). Then

\[
\begin{align*}
    f_{S_1 \land S_2}(x_1, x_2) &= f_{S_1}(x_1) \cap f_{S_2}(x_2) \\
                              &\supseteq (f_{S_1}(a_1) \cap f_{S_1}(b_1)) \cap (f_{S_2}(a_2) \cap f_{S_2}(b_2)) \\
                              &= (f_{S_1}(a_1) \cap f_{S_2}(a_2)) \cap (f_{S_1}(b_1) \cap f_{S_2}(b_2)) \\
                              &= f_{S_1 \land S_2}(a_1, a_2) \cap f_{S_1 \land S_2}(b_1, b_2).
\end{align*}
\]

Hence, \(f_{S_1 \land S_2}\) is an SI-hemiring of \(S_1 \times S_2\) over \(U\).

Similarly, we can obtain the following result:

**Proposition 3.7** Let \(f_{S_1}\) and \(f_{S_2}\) be two SI-h-ideals of \(S_1\) and \(S_2\) over \(U\), respectively. Then \(f_{S_1} \land f_{S_2}\) is an SI-h-ideal of \(S_1 \times S_2\) over \(U\).

**Remark 3.8** Note that \(f_{S_1} \lor f_{S_2}\) is not an SI-hemiring or SI-h-ideal over \(U\).

**Example 3.9** Let \(U = S_3\), symmetric group, be the universal set, \(S_1 = Z_5 = \{0, 1, 2, 3, 4\}\) and \(S_2 = Z_2 = \{0, 1\}\) be two hemirings of non-negative integers module 5 and module 2, respectively. Define two soft sets \(f_{S_1}\) and \(f_{S_2}\) over \(U\) by \(f_{S_1}(0) = S_3\), \(f_{S_1}(1) = f_{S_1}(4) = \{(1), (12), (132)\}\), \(f_{S_1}(2) = f_{S_1}(3) = \{(12), (123), (132)\}\), \(f_{S_2}(0) = S_3\), \(f_{S_2}(1) = \{(1), (12), (132)\}\). It is clear that \(f_{S_1}\) and \(f_{S_2}\) are two SI-hemirings over \(U\). However, we have

\[
\begin{align*}
    f_{S_1 \lor S_2}((3, 1) + (1, 0)) &= f_{S_1}(4, 1) \\
                                   &= f_{S_1}(4) \lor f_{S_2}(1) \\
                                   &= \{(1), (12), (132)\},
\end{align*}
\]

but

\[
\begin{align*}
    f_{S_1 \lor S_2}((3, 1) \cap f_{S_1 \lor S_2}(1, 0) &= (f_{S_1}(3) \lor f_{S_1}(1)) \cap (f_{S_1}(1) \lor f_{S_2}(0)) \\
                                                   &= \{(1), (12), (123), (132)\} \cap S_3 \\
                                                   &= \{(1), (12), (123), (132)\}.
\end{align*}
\]

This implies that \(f_{S_1 \lor S_2}((3, 1) + (1, 0)) \not\supseteq f_{S_1 \lor S_2}((3, 1) \cap f_{S_1 \lor S_2}(1, 0)\). Hence, \(f_{S_1 \lor S_2}\) is not an SI-hemiring over \(U\).

**Theorem 3.10** Let \(f_S\) and \(g_S\) be two SI-hemirings of \(S\) over \(U\). Then \(f_S \complement g_S\) is also an SI-hemiring of \(S\) over \(U\).

**Proof.** Let \(f_S\) and \(g_S\) be two SI-hemirings of \(S\) over \(U\). Then for all \(x, y \in S\), we have

\[
\begin{align*}
    (i) \quad (f_S \complement g_S)(x + y) &= f_S(x + y) \cap g_S(x + y) \\
                                       &\supseteq (f_S(x) \cap f_S(y)) \cap (g_S(x) \cap g_S(y)) \\
                                       &= (f_S(x) \cap g_S(x)) \cap (f_S(y) \cap g_S(y)) \\
                                       &= (f_S \complement g_S)(x) \cap (f_S \complement g_S)(y).
\end{align*}
\]
(ii) is similar to (i).
(iii) Let $a, b, x, z \in S$ be such that $x + a + z = b + z$. Then

$$(f_S\cap g_S)(x) = f_S(x) \cap g_S(x) \supseteq (f_S(a) \cap f_S(b)) \cap (g_S(a) \cap g_S(b)) = (f_S(a) \cap g_S(a)) \cap (f_S(b) \cap g_S(b)) = (f_S\cap g_S)(a) \cap (f_S\cap g_S)(b).$$

Hence $f_S\cap g_S$ is an $SI$-hemiring of $S$ over $U$.

Similarly, we can obtain the following theorem:

**Theorem 3.11** Let $f_S$ and $g_S$ be two $SI$-$h$-ideals of $S$ over $U$, then $f_S\cap g_S$ is also an $SI$-$h$-ideal of $S$ over $U$.

4. $h$-hemiregular hemirings

In this section, we describe the characterizations of $h$-hemiregular hemirings by means of $SI$-$h$-ideals.

**Definition 4.1** [22] A hemiring $S$ is called $h$-hemiregular if for each $x \in S$, these exist $a_1, a_2, z \in S$ such that $x + xa_1x + z = xa_2x + z$.

**Lemma 4.2** [22] If $A$ and $B$, are, respectively, a right and a left $h$-ideal of $S$, then $AB \subseteq A \cap B$.

**Lemma 4.3** [22] A hemiring $S$ is $h$-hemiregular if and only if for any right $h$-ideal $A$ and any left $h$-ideal $B$, we have $AB = A \cap B$.

**Definition 4.4** Let $f_S, g_S \in S(U)$. Define soft $h$-sum and soft $h$-product of $f_S$ and $g_S$ as follows:

1. $$(f_S +_h g_S)(x) = \bigcup_{x + a_1 + b_1 + z = a_2 + b_2 + z} (f_S(a_1) \cap f_S(a_2) \cap g_S(b_1) \cap g_S(b_2))$$ and $$(f_S +_h g_S)(x) = \emptyset$$ if $x$ cannot be expressed as $x + a_1 + b_1 + z = a_2 + b_2 + z$.

2. $$(f_S \circ_h g_S)(x) = \bigcup_{x + a_1 b_1 + z = a_2 b_2 + z} (f_S(a_1) \cap f_S(a_2) \cap g_S(b_1) \cap g_S(b_2))$$ and $$(f_S \circ_h g_S)(x) = \emptyset$$ if $x$ cannot be expressed as $x + a_1 b_1 + z = a_2 b_2 + z$.

**Lemma 4.5** Let $f_S$ and $g_S$ be an $SI$-right $h$-ideal and an $SI$-left $h$-ideal of $S$ over $U$, respectively, then $f_S \circ_h g_S \subseteq f_S \cap g_S$. 
Proof. If \((f_S \circ_h g_S)(x) = \emptyset\), then it is clear that \(f_S \circ_h g_S \subseteq f_S \tilde{\cap} g_S\). Otherwise, we have
\[
(f_S \circ_h g_S)(x) = \bigcup_{x + a_1 b_1 + z = a_2 b_2 + z} (f_S(a_1) \cap f_S(a_2) \cap g_S(b_1) \cap g_S(b_2)) \\
\subseteq \bigcup_{x + a_1 b_1 + z = a_2 b_2 + z} (f_S(a_1 b_1) \cap f_S(a_2 b_2) \cap g_S(a_1 b_1) \cap g_S(a_2 b_2)) \\
\subseteq \bigcup_{x + a_1 b_1 + z = a_2 b_2 + z} (f_S(x) \cap g_S(x)) \\
= f_S(x) \cap g_S(x) \\
= (f_S \cap g_S)(x),
\]
which implies, \(f_S \circ_h g_S \subseteq f_S \tilde{\cap} g_S\). \(\blacksquare\)

Definition 4.6 Let \(A \subseteq S\). We denote \(S_A\) the soft characteristic function of \(A\) and define as
\[
S_A(x) = \begin{cases} 
U & \text{if } x \in A, \\
\emptyset & \text{if } x \notin A.
\end{cases}
\]

The following proposition is obvious and we omit the details.

Proposition 4.7 Let \(A, B \subseteq S\). Then the following hold:

1. \(A \subseteq B \Rightarrow S_A \subseteq S_B\).
2. \(S_A \tilde{\cap} S_B = S_{A \cap B}\).
3. \(S_A \circ_h S_B = S_{T \cap U}\).

Theorem 4.8 For any hemiring \(S\), then the following are equivalent:

1. \(S\) is \(h\)-hemiregular;
2. \(f_S \circ_h g_S = f_S \tilde{\cap} g_S\) for any \(SI\)-right \(h\)-ideal \(f_S\) and \(SI\)-left \(h\)-ideal \(g_S\) of \(S\) over \(U\).

Proof. (1) \(\Rightarrow\) (2): Let \(S\) be an \(h\)-hemiregular hemiring, \(f_S\) and \(g_S\) an \(SI\)-right \(h\)-ideal and an \(SI\)-left \(h\)-ideal of \(S\) over \(U\), respectively. By Lemma 4.5, we have \(f_S \circ_h g_S \subseteq f_S \tilde{\cap} g_S\). Let \(x \in S\), then there exist \(a, a', z \in S\) such that \(x + xax + z = xa'x + z\) since \(S\) is \(h\)-hemiregular. Thus, we have
\[
(f_S \circ_h g_S)(x) = \bigcup_{x + a_1 b_1 + z = a_2 b_2 + z} (f_S(a_1) \cap f_S(a_2) \cap g_S(b_1) \cap g_S(b_2)) \\
\subseteq f_S(xa) \cap f_S(xa') \cap g_S(x) \\
\subseteq f_S(x) \cap g_S(x) \\
= (f_S \cap g_S)(x),
\]
which implies \(f_S \circ_h g_S \subseteq f_S \tilde{\cap} g_S\). Thus, \(f_S \circ_h g_S = f_S \tilde{\cap} g_S\).
(2) $\implies$ (1): Let $R$ and $L$ be any right $h$-ideal and left $h$-ideal of $S$, respectively. Then, by Lemma 4.2, we have $RL \subseteq R \cap L$. Moreover, it is easy to check that $S_R$ and $S_L$ are an $SI$-right $h$-ideal and an $SI$-left $h$-ideal of $S$ over $U$, respectively. Let $x \in R \cap L$, then, by Proposition 4.8, we have

$$S_{RL}(x) = (S_R \circ_h S_L)(x) = (S_R \cap S_L)(x) = S_{RL}(x) = U,$$

and so $x \in RL$. Then, $R \cap L \subseteq RL$. Thus, $R \cap L = RL$. It follows from Lemma 4.3 that $S$ is $h$-hemiregular.

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### References


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