SOME DERIVATIONS ON THE BOUNDS FOR THE ZEROS OF ENTIRE FUNCTIONS BASED ON SLOWLY CHANGING FUNCTIONS

Sanjib Kumar Datta

Department of Mathematics
University of Kalyani
Kalyani, Dist-Nadia, PIN–741235
West Bengal
India
email: sanjib_kr_datta@yahoo.co.in

Dilip Chandra Pramanik

Department of Mathematics
University of North Bengal
Raja Rammohunpur, Dist-Darjeeling, PIN–734013
West Bengal
India
email: dcpramanik@gmail.com

Abstract. A single valued function of one complex variable which is analytic in the finite complex plane is called an entire function. The purpose of this paper is to establish the bounds for the moduli of zeros of entire functions in the light of slowly changing functions.

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1. Introduction, definitions and notations

Let

$$P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots\dots + a_{n-1} z^{n-1} + a_n z^n; |a_n| \neq 0$$

be a polynomial of degree n. Datt and Govil [2], Govil and Rahaman [4], Marden [8], Mohammad [9], Chattopadhyay, Das, Jain and Konwer [1], Joyal, Labelle and Rahaman [5], Jain [6],[7], Sun and Hsieh [12], Zilovic, Roytman, Combettes and Swamy [14], Das and Datta [3] etc. worked in the theory of the distribution of the zeros of polynomials and obtained some newly developed results.

In this paper we intend to establish some of sharper results concerning the theory of distribution of zeros of entire functions in the light of slowly changing functions.

The following definitions are well known :

Definition 1. The order ρ and lower order λ of an entire function f are defined as

$$\rho = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad \text{and} \quad \lambda = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r},$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

Let $L \equiv L(r)$ be a positive continuous function increasing slowly, i.e., $L(ar) \sim L(r)$ as $r \longrightarrow \infty$ for every positive constant a. Singh and Barker[10] defined it in the following way:

Definition 2. [10] A positive continuous function $L(r)$ is called a *slowly changing function* if, for ε (> 0),

$$\frac{1}{k^{\varepsilon}} \leq \frac{L(kr)}{L(r)} \leq k^{\varepsilon} \quad \text{for } r > r(\varepsilon)$$

and uniformly for $k (\geq 1)$.

If, further, $L(r)$ is differentiable, the above condition is equivalent to

$$\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0.$$

Somasundaram and Thamizharasi [11] introduced the notions of L-order and L-lower order for entire functions defined in the open complex plane $\mathbb{C}$ as follows:

Definition 3. [11] The L-order ρ^L and the L-lower order λ^L of an entire function f are defined as

$$\rho^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]} \quad \text{and} \quad \lambda^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]}.$$

The more generalized concept for L-order and L-lower order are L^*-order and L^*-lower order respectively. Their definitions are as follows:

Definition 4. The L^*-order ρ^{L^*} and the L^*-lower order λ^{L^*} of an entire function f are defined as

$$\rho^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]} \quad \text{and} \quad \lambda^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]}.$$

2. Lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 1. *If $f(z)$ is an entire function of L-order ρ^L, then for every $\varepsilon > 0$ the inequality*

$$N(r) \leq [rL(r)]^{\rho^L+\varepsilon}$$

holds for all sufficiently large r where $N(r)$ is the number of zeros of $f(z)$ in $|z| \leq [rL(r)]$.

Proof. Let us suppose that $f(0) = 1$. This supposition can be made without loss of generality because if $f(z)$ has a zero of order $'m'$ at the origin then we may consider $g(z) = c \cdot \frac{f(z)}{z^m}$ where c is so chosen that $g(0) = 1$. Since the function $g(z)$ and $f(z)$ have the same order therefore it will be unimportant for our investigations that the number of zeros of $g(z)$ and $f(z)$ differ by m.

We further assume that $f(z)$ has no zeros on $|z| = 2[rL(r)]$ and the zeros z_i's of $f(z)$ in $|z| < [rL(r)]$ are in non decreasing order of their moduli so that $|z_i| \leq |z_{i+1}|$. Also let ρ^L suppose to be finite.

Now, we shall make use of Jenson's formula as state below

$$\log|f(0)| = -\sum_{i=1}^{n} \log\frac{R}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(R\, e^{i\phi}\right)\right| d\phi. \tag{1}$$

Let us replace R by $2r$ and n by $N(2r)$ in (1)

$$\log|f(0)| = -\sum_{i=1}^{N(2r)} \log\frac{2r}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(2r\, e^{i\phi}\right)\right| d\phi.$$

Since $f(0) = 1, \log|f(0)| = \log 1 = 0$.

$$\therefore \sum_{i=1}^{N(2r)} \log\frac{2r}{|z_i|} = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(2r\, e^{i\phi}\right)\right| d\phi. \tag{2}$$

$$\text{L.H.S.} = \sum_{i=1}^{N(2r)} \log\frac{2r}{|z_i|} \geq \sum_{i=1}^{N(r)} \log\frac{2r}{|z_i|} \geq N(r)\log 2 \tag{3}$$

because for large values of r,

$$\log\frac{2r}{|z_i|} \geq \log 2.$$

$$\begin{aligned} \text{R.H.S} &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(2r\, e^{i\phi}\right)\right| d\phi \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \log M(2r)\, d\phi = \log M(2r). \end{aligned} \tag{4}$$

Again, by definition of order ρ^L of $f(z)$, we have for every $\varepsilon > 0$, and as $L(2r) \sim L(r)$,

$$\begin{aligned} \log M(2r) &\leq [2rL(2r)]^{\rho^L+\varepsilon/2} \\ \log M(2r) &\leq [2rL(r)]^{\rho^L+\varepsilon/2}. \end{aligned} \tag{5}$$

Hence, from (2) by the help of $(3), (4)$ and (5), we have

$$\begin{aligned} N(r)\log 2 &\leq [2rL(r)]^{\rho^L+\varepsilon/2} \\ \text{i.e.,} \qquad N(r) &\leq \frac{2^{\rho^L+\varepsilon/2}}{\log 2}\cdot\frac{(rL(r))^{\rho^L+\varepsilon}}{(rL(r))^{\varepsilon/2}} \leq [rL(r)]^{\rho^L+\varepsilon}. \end{aligned}$$

This proves the lemma. ∎

In the line of Lemma 1, we may state the following lemma:

Lemma 2. *If $f(z)$ is an entire function of L^*-order ρ^{L^*}, then for every $\varepsilon > 0$ the inequality*

$$N(r) \leq [re^{L(r)}]^{\rho^{L^*}+\varepsilon}$$

holds for all sufficiently large r where $N(r)$ is the number of zeros of $f(z)$ in $|z| \leq [re^{L(r)}]$.

Proof. With the initial assumptions as laid down in Lemma 1, let us suppose that $f(z)$ has no zeros on $|z| = 2[re^{L(r)}]$ and the zeros z_i's of $f(z)$ in $|z| < [re^{L(r)}]$ are in non decreasing order of their moduli so that $|z_i| \leq |z_{i+1}|$. Also, let ρ^{L^*} supposed to be finite.

In view of (1), (2), (3) and (4), by definition of ρ^{L^*} and as $L(2r) \sim L(r)$, we get for every $\varepsilon > 0$ that

$$\begin{aligned} \log M(2r) &\leq [2re^{L(2r)}]^{\rho^{L^*}+\varepsilon/2} \\ \text{i.e., } \log M(2r) &\leq [2re^{L(r)}]^{\rho^{L^*+\varepsilon/2}}. \end{aligned} \tag{6}$$

Hence, by the help of $(3), (4)$ and (6), we obtain from (2) that

$$\begin{aligned} N(r)\log 2 &\leq [2re^{L(r)}]^{\rho^{L^*}+\varepsilon/2} \\ N(r) &\leq \frac{2^{\rho^{L^*}+\varepsilon/2}}{\log 2}\cdot\frac{[re^{L(r)}]^{\rho^{L^*}+\varepsilon}}{[rL(r)]^{\varepsilon/2}} \leq [re^{L(r)}]^{\rho^{L^*}+\varepsilon}. \end{aligned}$$

Thus, the lemma is established. ∎

3. Theorems

In this section, we present the main results of the paper.

Theorem 1. *Let $P(z)$ be an entire function having L-order ρ^L in the disc $|z| \leq [rL(r)]$ for sufficiently large r. Also, let the Taylor's series expansion of $P(z)$ be given by*

$$P(z) = a_0 + a_{p_1}z^{p_1} + \cdots + a_{p_m}z^{p_m} + a_{N(r)}z^{N(r)},\ a_0 \neq 0,\ a_{N(r)} \neq 0$$

with $1 \leq p_1 < p_2 < \cdots < p_m \leq N(r) - 1$, p_i's are integers such that for $\rho^L > 0$,

$$|a_0|(\rho^L)^{N(r)} \geq |a_{p_1}|(\rho^L)^{N(r)-p_1} \geq \cdots \geq |a_{p_m}|(\rho^L)^{N(r)-p_m} \geq \left|a_{N(r)}\right|.$$

Then, all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{\rho^L\left(1+\frac{|a_{p_1}|}{|a_0|(\rho^L)^{p_1}}\right)}<|z|<\frac{1}{\rho^L}\left(1+\frac{|a_0|}{\left|a_{N(r)}\right|}(\rho^L)^{N(r)}\right).$$

Proof. Given that

$$P(z)=a_0+a_{p_1}z^{p_1}+\cdots+a_{p_m}z^{p_m}+a_{N(r)}z^{N(r)}$$

where p_i's are integers and $1\leq p_1<p_2<\cdots<p_m\leq N(r)-1$. Then for $\rho^L>0$,

$$|a_0|\,(\rho^L)^{N(r)}\geq|a_{p_1}|\,(\rho^L)^{N(r)-p_1}\geq\cdots\geq|a_{p_m}|\,(\rho^L)^{N(r)-p_m}\geq\left|a_{N(r)}\right|.$$

Let us consider

$$\begin{aligned}Q(z)&=(\rho^L)^{N(r)}P\left(\frac{z}{\rho^L}\right)\\&=(\rho^L)^{N(r)}\left\{a_0+a_{p_1}\tfrac{z^{p_1}}{(\rho^L)^{p_1}}+\cdots+a_{p_m}\frac{z^{p_m}}{(\rho^L)^{p_m}}+a_{N(r)}\frac{z^{N(r)}}{(\rho^L)^{N(r)}}\right\}\\&=a_0(\rho^L)^{N(r)}+a_{p_1}(\rho^L)^{N(r)-p_1}z^{p_1}+\cdots+a_{p_m}(\rho^L)^{N(r)-p_m}z^{p_m}+a_{N(r)}z^{N(r)}.\end{aligned}$$

Therefore,

$$\begin{aligned}|Q(z)|\ \geq\ &\left|a_{N(r)}z^{N(r)}\right|-\\&\left|a_0(\rho^L)^{N(r)}+a_{p_1}(\rho^L)^{N(r)-p_1}z^{p_1}+\cdots+a_{p_m}(\rho^L)^{N(r)-p_m}z^{p_m}\right|.\end{aligned}\tag{7}$$

Now, using the given condition of Theorem 1 we obtain that

$$\begin{aligned}&\left|a_0(\rho^L)^{N(r)}+a_{p_1}(\rho^L)^{N(r)-p_1}z^{p_1}+\cdots+a_{p_m}(\rho^L)^{N(r)-p_m}z^{p_m}\right|\\&\leq|a_0|\,(\rho^L)^{N(r)}+|a_{p_1}|\,(\rho^L)^{N(r)-p_1}\,|z|^{p_1}+\cdots+|a_{p_m}|\,(\rho^L)^{N(r)-p_m}\,|z|^{p_m}\\&\leq|a_0|\,(\rho^L)^{N(r)}\,|z|^{N(r)}\left(\frac{1}{|z|^{N(r)-p_m}}+\cdots+\frac{1}{|z|^{N(r)}}\right)\ \text{ for }\ |z|\neq0.\end{aligned}$$

Using (7), we get for $|z|\neq0$ that

$$|Q(z)|\geq\left|a_{N(r)}\right|\,|z|^{N(r)}-|a_0|\,(\rho^L)^{N(r)}\,|z|^{N(r)}\left(\frac{1}{|z|^{N(r)-p_m}}+\cdots+\frac{1}{|z|^{N(r)}}\right)$$

$$\begin{aligned}&>\left|a_{N(r)}\right|\,|z|^{N(r)}-|a_0|\,(\rho^L)^{N(r)}\,|z|^{N(r)}\left(\frac{1}{|z|^{N(r)-p_m}}+\cdots+\frac{1}{|z|^{N(r)}}+\cdots\right)\\&=\left|a_{N(r)}\right|\,|z|^{N(r)}-|a_0|\,(\rho^L)^{N(r)}\,|z|^{N(r)}\left(\sum_{k=1}^{\infty}\frac{1}{|z|^k}\right).\end{aligned}\tag{8}$$

The geometric series $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is convergent for

$$\frac{1}{|z|} < 1$$

$$\text{i.e., for } |z| > 1$$

and converges to

$$\frac{1}{|z|} \frac{1}{1 - \frac{1}{|z|}} = \frac{1}{|z| - 1}.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \quad \text{for } |z| > 1.$$

Using (8), we get from above that for $|z| > 1$

$$\begin{aligned} |Q(z)| &> \left|a_{N(r)}\right| |z|^{N(r)} - |a_0| \left(\rho^L\right)^{N(r)} |z|^{N(r)} \left(\frac{1}{|z| - 1}\right) \\ &= |z|^{N(r)} \left(\left|a_{N(r)}\right| - \frac{|a_0| \left(\rho^L\right)^{N(r)}}{|z| - 1}\right). \end{aligned}$$

Now, for $|z| > 1$,

$$\begin{aligned} |Q(z)| > 0 \text{ if } \quad & \left|a_{N(r)}\right| - \frac{|a_0| \left(\rho^L\right)^{N(r)}}{|z| - 1} \geq 0 \\ \text{i.e., if } \quad & \left|a_{N(r)}\right| \geq \frac{|a_0| \left(\rho^L\right)^{N(r)}}{|z| - 1} \\ \text{i.e., if } \quad & |z| - 1 \geq \frac{|a_0| \left(\rho^L\right)^{N(r)}}{\left|a_{N(r)}\right|} \\ \text{i.e., if } \quad & |z| \geq 1 + \frac{|a_0| \left(\rho^L\right)^{N(r)}}{\left|a_{N(r)}\right|} > 1. \end{aligned}$$

Therefore, $|Q(z)| > 0$ if

$$|z| \geq 1 + \frac{|a_0| \left(\rho^L\right)^{N(r)}}{\left|a_{N(r)}\right|}.$$

Therefore, $Q(z)$ does not vanish for

$$|z| \geq 1 + \frac{|a_0| \left(\rho^L\right)^{N(r)}}{\left|a_{N(r)}\right|}.$$

So, all the zeros of $Q(z)$ lie in

$$|z| < 1 + \frac{|a_0| \left(\rho^L\right)^{N(r)}}{\left|a_{N(r)}\right|}.$$

Let $z = z_0$ be any zero of $P(z)$. Therefore, $P(z_0) = 0$. Clearly, $z_0 \neq 0$ as $a_0 \neq 0$. Putting $z = \rho^L z_0$ in $Q(z)$, we get that

$$Q(\rho^L z_0) = (\rho^L)^{N(r)}.P(z_0) = (\rho^L)^{N(r)}.0 = 0.$$

So $z = \rho^L z_0$ is a zero of $Q(z)$. Hence,

$$\begin{aligned} \left|\rho^L z_0\right| &< 1 + \frac{|a_0| (\rho^L)^{N(r)}}{\left|a_{N(r)}\right|} \\ \text{i.e.,} \qquad |z_0| &< \frac{1}{\rho^L}\left(1 + \frac{|a_0| (\rho^L)^{N(r)}}{\left|a_{N(r)}\right|}\right). \end{aligned}$$

Since z_0 is an arbitrary zero of $P(z)$, therefore, all the zeros of $Q(z)$ lie in

$$|z| < \frac{1}{\rho^L}\left(1 + \frac{|a_0| (\rho^L)^{N(r)}}{\left|a_{N(r)}\right|}\right). \tag{9}$$

Again, let us consider

$$R(z) = (\rho^L)^{N(r)} z^{N(r)} P\left(\frac{1}{\rho^L z}\right).$$

Therefore,

$$\begin{aligned} R(z) &= (\rho^L)^{N(r)} z^{N(r)} \left\{ a_0 + a_{p_1}\frac{1}{(\rho^L)^{p_1} z^{p_1}} + \cdots + a_{p_m}\frac{1}{(\rho^L)^{p_m} z^{p_m}} + a_{N(r)}\frac{1}{(\rho^L)^{N(r)} z^{N(r)}} \right\} \\ &= a_0 (\rho^L)^{N(r)} z^{N(r)} + a_{p_1}(\rho^L)^{N(r)-p_1} z^{N(r)-p_1} + \cdots + a_{p_m}(\rho^L)^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}. \end{aligned}$$

Now,

$$\begin{aligned} |R(z)| \geq{}& \left|a_0 (\rho^L)^{N(r)} z^{N(r)}\right| \\ & - \left|a_{p_1}(\rho^L)^{N(r)-p_1} z^{N(r)-p_1} + \cdots + a_{p_m}(\rho^L)^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}\right|. \end{aligned} \tag{10}$$

Also

$$\begin{aligned} &\left|a_{p_1}(\rho^L)^{N(r)-p_1} z^{N(r)-p_1} + \cdots + a_{p_m}(\rho^L)^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}\right| \\ &\leq \left|a_{p_1}(\rho^L)^{N(r)-p_1} z^{N(r)-p_1}\right| + \cdots + \left|a_{p_m}(\rho^L)^{N(r)-p_m} z^{N(r)-p_m}\right| + \left|a_{N(r)}\right| \\ &= \left|a_{p_1}\right| (\rho^L)^{N(r)-p_1} |z|^{N(r)-p_1} + \cdots + \left|a_{p_m}\right| (\rho^L)^{N(r)-p_m} |z|^{N(r)-p_m} + \left|a_{N(r)}\right| \\ &\leq \left|a_{p_1}\right| (\rho^L)^{N(r)-p_1} \left(|z|^{N(r)-p_1} + \cdots + |z|^{N(r)-p_m} + 1\right). \end{aligned} \tag{11}$$

Using (11), we get from (10) that for $|z| \neq 0$

$$\begin{aligned} |R(z)| &\geq |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - \left|a_{p_1}\right| (\rho^L)^{N(r)-p_1} \left(|z|^{N(r)-p_1} + \cdots + |z|^{N(r)-p_m} + 1\right) \\ &= |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - \left|a_{p_1}\right| (\rho^L)^{N(r)-p_1} |z|^{N(r)} \left(\frac{1}{|z|^{p_1}} + \cdots + \frac{1}{|z|^{p_m}} + \frac{1}{|z|^{N(r)}}\right) \\ &> |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \\ &\quad - \left|a_{p_1}\right| (\rho^L)^{N(r)-p_1} |z|^{N(r)} \left(\tfrac{1}{|z|^{p_1}} + \cdots + \tfrac{1}{|z|^{p_m}} + \tfrac{1}{|z|^{N(r)}} + \cdots\right). \end{aligned}$$

Therefore, for $|z| \neq 0$,

$$|R(z)| > |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - |a_{p_1}| (\rho^L)^{N(r)-p_1} |z|^{N(r)} \left(\sum_{k=1}^{\infty} \frac{1}{|z|^k} \right). \tag{12}$$

Now, the geometric series $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is convergent for

$$\frac{1}{|z|} < 1$$
$$\text{i.e., for} \quad |z| > 1$$

and converges to

$$\frac{1}{|z|} \frac{1}{1 - \frac{1}{|z|}} = \frac{1}{|z| - 1}.$$

So

$$\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z| - 1} \text{ for } |z| > 1.$$

Therefore, for $|z| > 1$,

$$\begin{aligned} |R(z)| &> |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - |a_{p_1}| (\rho^L)^{N(r)-p_1} |z|^{N(r)} \left(\tfrac{1}{|z|-1} \right) \\ &= |z|^{N(r)} (\rho^L)^{N(r)-p_1} \left(|a_0| (\rho^L)^{p_1} - \tfrac{|a_{p_1}|}{|z|-1} \right). \end{aligned}$$

i.e., for $|z| > 1$

$$|R(z)| > |z|^{N(r)} (\rho^L)^{N(r)-p_1} \left(|a_0| (\rho^L)^{p_1} - \frac{|a_{p_1}|}{|z| - 1} \right).$$

Now,

$$\begin{aligned} |R(z)| > 0 \text{ if} \quad & |a_0| (\rho^L)^{p_1} - \frac{|a_{p_1}|}{|z| - 1} \geq 0 \\ \text{i.e., if} \quad & |a_0| (\rho^L)^{p_1} \geq \frac{|a_{p_1}|}{|z| - 1} \\ \text{i.e., if} \quad & |z| - 1 \geq \frac{|a_{p_1}|}{|a_0| (\rho^L)^{p_1}} \\ \text{i.e., if} \quad & |z| \geq 1 + \frac{|a_{p_1}|}{|a_0| (\rho^L)^{p_1}} > 1. \end{aligned}$$

Therefore,

$$|R(z)| > 0 \text{ if } |z| \geq 1 + \frac{|a_{p_1}|}{|a_0| (\rho^L)^{p_1}}.$$

Since $R(z)$ does not vanish in

$$|z| \geq 1 + \frac{|a_{p_1}|}{|a_0| (\rho^L)^{p_1}},$$

all the zeros of $R(z)$ lie in

$$|z| < 1 + \frac{|a_{p_1}|}{|a_0|\,(\rho^L)^{p_1}}.$$

Let $z = z_0$ be any zero of $P(z)$. Therefore, $P(z_0) = 0$. Clearly, $z_0 \neq 0$ as $a_0 \neq 0$. Putting $z = 1/\rho^L z_0$ in $R(z)$, we obtain that

$$\begin{aligned} R\left(\frac{1}{\rho^L z_0}\right) &= (\rho^L)^{N(r)}\left(\frac{1}{\rho^L z_0}\right)^{N(r)} \cdot P(z_0) \\ &= \left(\frac{1}{z_0}\right)^{N(r)} \cdot 0 = 0. \end{aligned}$$

So

$$\begin{aligned} \left|\frac{1}{\rho^L z_0}\right| &< 1 + \frac{|a_{p_1}|}{|a_0|\,(\rho^L)^{p_1}} \\ \text{i.e.,} \qquad \left|\frac{1}{z_0}\right| &< \rho^L\left(1 + \frac{|a_{p_1}|}{|a_0|\,(\rho^L)^{p_1}}\right) \\ \text{i.e.,} \qquad |z_0| &> \frac{1}{\rho^L\left(1 + \frac{|a_{p_1}|}{|a_0|(\rho^L)^{p_1}}\right)}. \end{aligned}$$

As z_0 is an arbitrary zero of $P(z)$, all the zeros of $P(z)$ lie in

$$|z| > \frac{1}{\rho^L\left(1 + \frac{|a_{p_1}|}{|a_0|(\rho^L)^{p_1}}\right)}. \tag{13}$$

So, from (9) and (13), we may conclude that all the zeros of $P(z)$ lie in the proper ring shaped region

$$\frac{1}{\rho^L\left(1 + \frac{|a_{p_1}|}{|a_0|(\rho^L)^{p_1}}\right)} < |z| < \frac{1}{\rho^L}\left(1 + \frac{|a_0|}{|a_{N(r)}|}(\rho^L)^{N(r)}\right).$$

This proves the theorem. ∎

In the line of Theorem 1, we may state the following theorem in view of Lemma 2:

Theorem 2. *Let $P(z)$ be an entire function having L^*-order ρ^{L^*} in the disc $|z| \leq [re^{L(r)}]$ for sufficiently large r. Also, let the Taylor's series expansion of $P(z)$ be given by*

$$P(z) = a_0 + a_{p_1}z^{p_1} + \cdots + a_{p_m}z^{p_m} + a_{N(r)}z^{N(r)},\ a_0 \neq 0,\ a_{N(r)} \neq 0$$

with $1 \le p_1 < p_2 < \cdots < p_m \le N(r) - 1$, p_i*'s are integers such that for* $\rho^{L^*} > 0$,

$$|a_0| \, (\rho^{L^*})^{N(r)} \ge |a_{p_1}| \, (\rho^{L^*})^{N(r)-p_1} \ge \cdots \ge |a_{p_m}| \, (\rho^{L^*})^{N(r)-p_m} \ge \left|a_{N(r)}\right|.$$

Then, all the zeros of $P(z)$ *lie in the ring shaped region*

$$\frac{1}{\rho^{L^*}\left(1 + \frac{|a_{p_1}|}{|a_0|(\rho^{L^*})^{p_1}}\right)} < |z| < \frac{1}{\rho^{L^*}}\left(1 + \frac{|a_0|}{\left|a_{N(r)}\right|}(\rho^{L^*})^{N(r)}\right).$$

The proof is omitted.

Corollary 1. *In view of Theorem* 1*, we may conclude that all the zeros of*

$$P(z) = a_0 + a_{p_1} z^{p_1} + \cdots + a_{p_m} z^{p_m} + a_n z^n$$

of degree n *with* $1 \le p_1 < p_2 < \cdots < p_m \le n - 1$, p_i*'s are integers such that for* $\rho^L > 0$,

$$|a_0| \ge |a_{p_1}| \ge \cdots \ge |a_n|$$

lie in the ring shaped region

$$\frac{1}{\left(1 + \frac{|a_{p_1}|}{|a_0|}\right)} < |z| < \left(1 + \frac{|a_0|}{|a_n|}\right)$$

on putting $\rho^L = 1$ *in Theorem* 1.

Corollary 2. *In view of Theorem* 2*, we may conclude that all the zeros of*

$$P(z) = a_0 + a_{p_1} z^{p_1} + \cdots + a_{p_m} z^{p_m} + a_n z^n$$

of degree n *with* $1 \le p_1 < p_2 < \cdots < p_m \le n - 1$, p_i*'s are integers such that for* $\rho^{L^*} > 0$,

$$|a_0| \ge |a_{p_1}| \ge ...... \ge |a_n|$$

lie in the ring shaped region

$$\frac{1}{\left(1 + \frac{|a_{p_1}|}{|a_0|}\right)} < |z| < \left(1 + \frac{|a_0|}{|a_n|}\right)$$

on putting $\rho^{L^*} = 1$ *in Theorem* 2.

Theorem 3. *Let* $P(z)$ *be an entire function having L-order* ρ^L*. For sufficiently large* r *in the disc* $|z| \le [rL(r)]$*, the Taylor's series expansion of* $P(z)$ *be given by* $P(z) = a_0 + a_1 z + \cdots + a_{N(r)} z^{N(r)}$, $a_0 \ne 0$*. Further, for* $\rho^L > 0$,

$$|a_0| \, (\rho^L)^{N(r)} \ge |a_1| \, (\rho^L)^{N(r)-1} \ge \cdots \ge \left|a_{N(r)}\right|.$$

Then, all the zeros of $P(z)$ lie in the ring shaped region.

$$\frac{1}{\rho^L t_0'} < |z| < \frac{1}{\rho^L} t_0,$$

where t_0 and t_0' are the greatest roots of

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)+1} - \left(\left|a_{N(r)}\right| + (\rho^L)^{N(r)} |a_0|\right) t^{N(r)} + (\rho^L)^{N(r)} |a_0| = 0$$

and

$$f(t) \equiv |a_0| \rho^L t^{N(r)+1} - \left(|a_0| \rho^L + |a_1|\right) t^{N(r)} + |a_1| = 0.$$

Proof. Let

$$P(z) = a_0 + a_1 z + \cdots + a_{N(r)} z^{N(r)}$$

by applying Lemma 1 and in view of Taylor's series expansion of $P(z)$. Also

$$|a_0| (\rho^L)^{N(r)} \geq |a_1| (\rho^L)^{N(r)-1} \geq \cdots \geq \left|a_{N(r)}\right|.$$

Let us consider

$$\begin{aligned} Q(z) &= (\rho^L)^{N(r)} P\left(\frac{z}{\rho^L}\right) \\ &= \left(\rho^L\right)^{N(r)} \left\{a_0 + a_1 \frac{z}{\rho^L} + a_2 \frac{z^2}{(\rho^L)^2} + \cdots + a_{N(r)} \frac{z^{N(r)}}{(\rho^L)^{N(r)}}\right\} \\ &= a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + \cdots + a_{N(r)} z^{N(r)}. \end{aligned}$$

Now

$$|Q(z)| \geq \left|a_{N(r)}\right| |z|^{N(r)} - \left|a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + \cdots + a_{N(r)-1} z^{N(r)-1}\right|.$$

Also, applying the condition $|a_0| (\rho^L)^{N(r)} \geq |a_1| (\rho^L)^{N(r)-1} \geq \cdots \geq \left|a_{N(r)}\right|$, we get from above that

$$\begin{aligned} &\left|a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + \cdots + a_{N(r)-1} z^{N(r)-1}\right| \\ &\quad \leq |a_0| (\rho^L)^{N(r)} + |a_1| (\rho^L)^{N(r)-1} |z| + \cdots + \left|a_{N(r)-1}\right| |z|^{N(r)-1} \\ &\quad \leq |a_0| (\rho^L)^{N(r)} \left(1 + |z| + \cdots + |z|^{N(r)-1}\right) \\ &\quad = |a_0| (\rho^L)^{N(r)} \frac{|z|^{N(r)} - 1}{|z| - 1} \text{ for } |z| \neq 1. \end{aligned}$$

Therefore, it follows from above that

$$|Q(z)| \geq \left|a_{N(r)}\right| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} \cdot \frac{|z|^{N(r)} - 1}{|z| - 1}.$$

Now

$$|Q(z)| > 0 \text{ if } \left|a_{N(r)}\right| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} \cdot \frac{|z|^{N(r)} - 1}{|z| - 1} > 0$$

$$\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)} > |a_0| (\rho^L)^{N(r)} . \frac{|z|^{N(r)} - 1}{|z| - 1}$$

$$\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)} (|z| - 1) > |a_0| (\rho^L)^{N(r)} \left(|z|^{N(r)} - 1\right)$$

$$\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)+1} - \left(\left|a_{N(r)}\right| + |a_0| (\rho^L)^{N(r)}\right) |z|^{N(r)} + |a_0| (\rho^L)^{N(r)} > 0.$$

Let us consider

$$(14)\quad g(t) \equiv \left|a_{N(r)}\right| t^{N(r)+1} - \left(\left|a_{N(r)}\right| + |a_0| (\rho^L)^{N(r)}\right) t^{N(r)} + |a_0| (\rho^L)^{N(r)} = 0.$$

The maximum number of positive roots of (14) is two because maximum number of changes of sign in $g(t) = 0$ is two and if it is less, less by two. Clearly, $t = 1$ is a positive root of $g(t) = 0$. Therefore, $g(t) = 0$ must have exactly one positive root other than 1. Let the positive root of $g(t)$ be t_1. Let us take $t_0 = \max\{1, t_1\}$. Clearly, for $t > t_0, g(t) > 0$. If not for some $t_2 > t_0$, $g(t_2) < 0$. Also $g(\infty) > 0$. Therefore $g(t) = 0$ has another positive root in (t_2, ∞) which gives a contradiction. So, for $t > t_0, g(t) > 0$. Also $t_0 \geq 1$. Therefore, $|Q(z)| > 0$ if $|z| > t_0$. So, $Q(z)$ does not vanish in $|z| > t_0$. Hence, all the zeros of $Q(z)$ lie in $|z| \leq t_0$.

Let $z = z_0$ be a zero of $P(z)$. So, $P(z_0) = 0$. Clearly, $z_0 \neq 0$ as $a_0 \neq 0$. Putting $z = \rho^L z_0$ in $Q(z)$, we get that

$$Q(\rho^L z_0) = \left(\rho^L\right)^{N(r)} P(z_0) = \left(\rho^L\right)^{N(r)} .0 = 0.$$

Therefore, $z = \rho^L z_0$ is a zero of $Q(z)$. So, $\left|\rho^L z_0\right| \leq t_0$ or $|z_0| \leq \frac{1}{\rho^L} t_0$. As z_0 is an arbitrary zero of $P(z)$,

$$(15)\qquad \text{all the zeros of } P(z) \text{ lie in the region } |z| \leq \frac{1}{\rho^L} t_0.$$

In order to prove the lower bound of Theorem 3, let us consider

$$R(z) = (\rho^L)^{N(r)} z^{N(r)} P\left(\frac{1}{\rho^L z}\right).$$

Then

$$\begin{aligned} R(z) &= (\rho^L)^{N(r)} z^{N(r)} \left(a_0 + \frac{a_1}{\rho^L z} + \cdots + a_{N(r)} \frac{1}{(\rho^L)^{N(r)} z^{N(r)}}\right) \\ &= a_0 (\rho^L)^{N(r)} z^{N(r)} + a_1 (\rho^L)^{N(r)-1} z^{N(r)-1} + \cdots + a_{N(r)}. \end{aligned}$$

Now

$$|R(z)| \geq |a_0| (\rho^L)^{N(r)} |z|^{N(r)} - \left|a_1 (\rho^L)^{N(r)-1} z^{N(r)-1} + \cdots + a_{N(r)}\right|.$$

Also

$$\left|a_1(\rho^L)^{N(r)-1}z^{N(r)-1}+\cdots+a_{N(r)}\right| \le |a_1|\,(\rho^L)^{N(r)-1}\,|z|^{N(r)-1}+\cdots+\left|a_{N(r)}\right|.$$

So, applying the condition $|a_0|\,(\rho^L)^{N(r)} \ge |a_1|\,(\rho^L)^{N(r)-1} \ge \cdots \ge \left|a_{N(r)}\right|$, we get from above that

$$\begin{aligned}
&-\left|a_1(\rho^L)^{N(r)-1}z^{N(r)-1}+\cdots+a_{N(r)}\right| \\
&\ge -|a_1|\,(\rho^L)^{N(r)-1}\,|z|^{N(r)-1}-\cdots-\left|a_{N(r)}\right| \\
&\ge -|a_1|\,(\rho^L)^{N(r)-1}\left(|z|^{N(r)-1}+\cdots+1\right) \\
&= -|a_1|\,(\rho^L)^{N(r)-1}\frac{|z|^{N(r)-1}-1}{|z|-1} \text{ for } |z|\neq 1.
\end{aligned}$$

Using (16), we get for $|z|\neq 1$ that

$$|R(z)| \ge (\rho^L)^{N(r)-1}\left(|a_0|\,\rho^L\,|z|^{N(r)}-|a_1|\,\frac{|z|^{N(r)-1}-1}{|z|-1}\right). \tag{16}$$

Now

$$\begin{aligned}
|R(z)|>0 \text{ if } \quad &(\rho^L)^{N(r)-1}\left(|a_0|\,\rho^L\,|z|^{N(r)}-|a_1|\,\frac{|z|^{N(r)-1}-1}{|z|-1}\right)>0 \\
\text{i.e., if } \quad &|a_0|\,\rho^L\,|z|^{N(r)}-|a_1|\,\frac{|z|^{N(r)-1}-1}{|z|-1}>0 \\
\text{i.e., if } \quad &|a_0|\,\rho^L\,|z|^{N(r)}>|a_1|\,\frac{|z|^{N(r)-1}-1}{|z|-1} \\
\text{i.e., if } \quad &|a_0|\,\rho^L\,|z|^{N(r)}(|z|-1)>|a_1|\left(|z|^{N(r)-1}-1\right) \\
\text{i.e., if } \quad &|a_0|\,\rho^L\,|z|^{N(r)+1}-\left(|a_0|\,\rho^L+|a_1|\right)|z|^{N(r)}+|a_1|>0.
\end{aligned}$$

Let us consider

$$f(t)\equiv |a_0|\,\rho^L t^{N(r)+1}-\left(|a_0|\,\rho^L+|a_1|\right)t^{N(r)}+|a_1|=0.$$

Clearly, $f(t)=0$ has two positive roots, because the number of changes of sign of $f(t)$ is two. If it is less, less by two. Also, $t=1$ is the one of the positive roots of $f(t)=0$. Let us suppose that $t=t_2$ be the other positive root. Also, let $t_0'=\max\{1,t_2\}$ and so $t_0'\ge 1$. Now $t>t_0'$ implies $f(t)>0$. If not, then there exists some $t_3>t_0'$ such that $f(t_3)<0$. Also, $f(\infty)>0$. Therefore, there exists another positive root in (t_3,∞) which is a contradiction. So, $|R(z)|>0$ if $|z|>t_0'$. Thus $R(z)$ does not vanish in $|z|>t_0'$. In other words, all the zeros of $R(z)$ lie in $|z|\le t_0'$.

Let $z = z_0$ be any zero of $P(z)$. So, $P(z_0) = 0$. Clearly, $z_0 \neq 0$ as $a_0 \neq 0$. Putting $z = \dfrac{1}{\rho^L z_0}$ in $R(z)$, we get that

$$R\left(\frac{1}{\rho^L z_0}\right) = (\rho^L)^{N(r)} \left(\frac{1}{\rho^L z_0}\right)^{N(r)} P(z_0) = \left(\frac{1}{z_0}\right)^{N(r)} \cdot 0 = 0$$

Therefore, $\dfrac{1}{\rho^L z_0}$ is a root of $R(z)$. So, $\left|\dfrac{1}{\rho^L z_0}\right| \leq t_0'$ implies $|z_0| \geq \dfrac{1}{\rho^L t_0'}$. As z_0 is an arbitrary zero of $P(z) = 0$,

$$\text{all the zeros of } P(z) \text{ lie in } |z| \geq \frac{1}{\rho^L t_0'}. \tag{17}$$

From (15) and (18) we have all the zeros of $P(z)$ lie in the ring shaped region given by

$$\frac{1}{\rho^L t_0'} \leq |z| \leq \frac{1}{\rho^L} t_0$$

where t_0 and t_0' are the greatest positive roots of $g(t) = 0$ and $f(t) = 0$ respectively.

This proves the theorem. ∎

In the line of Theorem 3, we may state the following theorem in view of Lemma 2:

Theorem 4. *Let $P(z)$ be an entire function having L^*-order ρ^{L^*}. For sufficiently large r in the disc $|z| \leq [re^{L(r)}]$, the Taylor's series expansion of $P(z)$ be given by*

$$P(z) = a_0 + a_1 z + \cdots + a_{N(r)} z^{N(r)}, \ a_0 \neq 0.$$

Further, for $\rho^{L^*} > 0$,

$$|a_0| (\rho^{L^*})^{N(r)} \geq |a_1| (\rho^{L^*})^{N(r)-1} \geq \cdots \geq \left|a_{N(r)}\right|.$$

Then all the zeros of $P(z)$ lie in the ring shaped region.

$$\frac{1}{\rho^{L^*} t_0'} < |z| < \frac{1}{\rho^{L^*}} t_0$$

where t_0 and t_0' are the greatest roots of

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)+1} - \left(\left|a_{N(r)}\right| + (\rho^{L^*})^{N(r)} |a_0|\right) t^{N(r)} + (\rho^{L^*})^{N(r)} |a_0| = 0$$

and

$$f(t) \equiv |a_0| \rho^{L^*} t^{N(r)+1} - \left(|a_0| \rho^{L^*} + |a_1|\right) t^{N(r)} + |a_1| = 0.$$

The proof is omitted.

Corollary 3. *From Theorem 3, we can easily conclude that all the zeros of*

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$

of degree n with property $|a_0| \geq |a_1| \geq \cdots \geq |a_n|$ lie in the ring shaped region

$$\frac{1}{t_0'} \leq |z| \leq t_0$$

where t_0 and t_0' are the greatest positive roots of

$$g(t) \equiv |a_n|\, t^{n+1} - (|a_n| + |a_0|)t^n + |a_0| = 0$$

and

$$f(t) \equiv |a_0|\, t^{n+1} - (|a_0| + |a_1|)t^n + |a_1| = 0$$

respectively by putting $\rho^L = 1$.

Corollary 4. *From Theorem 4, we can easily conclude that all the zeros of*

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$

of degree n with property $|a_0| \geq |a_1| \geq \cdots \geq |a_n|$ lie in the ring shaped region

$$\frac{1}{t_0'} \leq |z| \leq t_0$$

where t_0 and t_0' are the greatest positive roots of

$$g(t) \equiv |a_n|\, t^{n+1} - (|a_n| + |a_0|)t^n + |a_0| = 0$$

and

$$f(t) \equiv |a_0|\, t^{n+1} - (|a_0| + |a_1|)t^n + |a_1| = 0$$

respectively by putting $\rho^{L^*} = 1$.

Corollary 5. *Under the conditions of Theorem 3 and*

$$P(z) = a_0 + a_{p_1} z^{p_1} + \cdots + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}$$

with

$$1 \leq p_1 \leq p_2 \leq \cdots \leq p_m \leq N(r) - 1,$$

where p_i's are integers and $a_0, a_{p_1}, ..., a_{N(r)}$ are non vanishing coefficients with

$$|a_0|\, (\rho^L)^{N(r)} \geq |a_{p_1}|\, (\rho^L)^{N(r)-p_1} \geq \cdots \geq |a_{p_m}|\, (\rho^L)^{N(r)-p_m} \geq \left|a_{N(r)}\right|$$

then we can show that all the zeros of $P(z)$ lie in

$$\frac{1}{\rho^L t_0'} \leq |z| \leq \frac{1}{\rho^L} t_0$$

where t_0 and t_0' are the greatest positive roots of

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)+1} - \left(\left|a_{N(r)}\right| + |a_0|\, (\rho^L)^{N(r)}\right) t^{N(r)} + |a_0|\, (\rho^L)^{N(r)} = 0$$

and

$$f(t) \equiv |a_0|\, (\rho^L)^{p_1} t^{N(r)+1} - \left(|a_0|\, (\rho^L)^{p_1} + |a_{p_1}|\right) t^{N(r)} - |a_{p_1}| = 0 \text{ respectively.}$$

Corollary 6. *Under the conditions of Theorem* 4 *and*

$$P(z) = a_0 + a_{p_1} z^{p_1} + ... + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}$$

with

$$1 \leq p_1 \leq p_2 \leq \cdots \leq p_m \leq N(r) - 1,$$

where p_i*'s are integers and* $a_0, a_{p_1}, ..., a_{N(r)}$ *are non vanishing coefficients with*

$$|a_0| (\rho^{L^*})^{N(r)} \geq |a_{p_1}| (\rho^{L^*})^{N(r)-p_1} \geq \cdots \geq |a_{p_m}| (\rho^{L^*})^{N(r)-p_m} \geq \left| a_{N(r)} \right|$$

then we can show that all the zeros of $P(z)$ *lie in*

$$\frac{1}{\rho^{L^*} t_0'} \leq |z| \leq \frac{1}{\rho^{L^*}} t_0$$

where t_0 *and* t_0' *are the greatest positive roots of*

$$g(t) \equiv \left| a_{N(r)} \right| t^{N(r)+1} - \left(\left| a_{N(r)} \right| + |a_0| (\rho^{L^*})^{N(r)} \right) t^{N(r)} + |a_0| (\rho^{L^*})^{N(r)} = 0$$

and

$$f(t) \equiv |a_0| (\rho^{L^*})^{p_1} t^{N(r)+1} - \left(|a_0| (\rho^{L^*})^{p_1} + |a_{p_1}| \right) t^{N(r)} - |a_{p_1}| = 0 \text{ respectively.}$$

Corollary 7. *If we put* $\rho^L = 1$ *in Corollary* 5*, then all the zeros of*

$$P(z) = a_0 + a_{p_1} z^{p_1} + \cdots + a_{p_m} z^{p_m} + a_n z^n$$

lie in the ring shaped region

$$\frac{1}{t_0'} \leq |z| \leq t_0$$

where t_0 *and* t_0' *are the greatest positive roots of*

$$g(t) \equiv |a_n| t^{n+1} - (|a_n| + |a_0|) t^n + |a_0| = 0$$

and

$$f(t) \equiv |a_0| t^{n+1} - (|a_0| + |a_{p_1}|) t^n - |a_{p_1}| = 0 \quad \text{respectively}$$

provided

$$|a_0| \geq |a_{p_1}| \geq ..... \geq |a_{p_m}| \geq |a_n| .$$

Corollary 8. *If we put* $\rho^{L^*} = 1$ *in Corollary* 6*, then all the zeros of*

$$P(z) = a_0 + a_{p_1} z^{p_1} + \cdots + a_{p_m} z^{p_m} + a_n z^n$$

lie in the ring shaped region

$$\frac{1}{t_0'} \leq |z| \leq t_0$$

where t_0 *and* t_0' *are the greatest positive roots of*

$$g(t) \equiv |a_n|\, t^{n+1} - (|a_n| + |a_0|)\, t^n + |a_0| = 0$$

and

$$f(t) \equiv |a_0|\, t^{n+1} - (|a_0| + |a_{p_1}|)\, t^n - |a_{p_1}| = 0 \quad \textit{respectively}$$

provided

$$|a_0| \geq |a_{p_1}| \geq ..... \geq |a_{p_m}| \geq |a_n|\,.$$

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