A NOTE ON BOOLEAN SUBSETS OF ORTHOMODULAR POSETS[1]

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Abstract. Modelling quantum systems by orthomodular posets $\mathcal{P} = (P, \leq, ', 0, 1)$ gives rise to the question, when a finite subset A of P lies within a Boolean subalgebra of $\mathcal{P}$, in which case A is called Boolean. Boolean subsets A specify the physical subsystem represented by A to be classical. We give a characterization of a subset of P to be Boolean by only taking into account terms of elements of this subset and in such a way that an inductive algorithm can be derived.

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1. Introduction

Orthomodular lattices, and more general, orthomodular posets $\mathcal{P} = (P, \leq, ', 0, 1)$ have been intensively studied as models for quantum logics (cf., e.g., [2], [5], [6], [7] and [8]). It is well known that these algebraic structures correspond to classical mechanical systems if and only if they are Boolean algebras. However, if one deals with only a finite subset A of P the question arises whether the physical subsystem represented by A is classical, which then means that A lies within a Boolean subalgebra of $\mathcal{P}$. If this is the case, then A will be called Boolean. For this definition, cf. the pioneering paper [1], in which the question of Boolean subsets was settled for the special case of so-called algebras of numerical events. For the general case of arbitrary orthomodular posets, Boolean subsets have been

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identified by means of a generalization of the compatibility relation of two elements (cf. [3], [4] and [8]).

In this paper, we will stick to the classical commutativity relation of only two elements and will only take into account terms of elements of the given finite set of elements of P to answer the question, whether this set is Boolean. The characterization we derive will be inductive and hence will give rise to a step by step procedure.

We begin by recollecting the definition of an orthomodular poset and some of its properties.

An orthomodular poset is a ordered quintuple $\mathcal{P} = (P, \leq, ', 0, 1)$ such that $(P, \leq, 0, 1)$ is a bounded poset and $'$ is a unary operation on P satisfying the following conditions for all $x, y \in P$:

(i) $x \leq y$ implies $x' \geq y'$.

(ii) $(x')' = x$

(iii) $x \vee x' = 1$

(iv) If $x \perp y$, i. e. if $x \leq y'$, in which case x and y are said to be orthogonal (to each other), then $x \vee y$, the supremum of x and y, exists.

(v) If $x \leq y$ then the orthomodular law $y = x \vee (y \wedge x')$ holds.

$y \wedge x'$ denotes the infimum of y and x'. It is easy to see that within an orthomodular poset the existence of $x \vee y$ implies the existence of $x' \wedge y'$ and that the de Morgan laws $(x \vee y)' = x' \wedge y'$ and $(x \wedge y)' = x' \vee y'$ hold, in the sense that if one side is defined then so is the other and they are equal. Moreover, if $x \leq y$ then $x \perp y'$ and hence $x \vee y'$ exists which in turn shows the existence of $x' \wedge y = (x \vee y')'$. The existence of the term on the right-hand side of the orthomodular law is secured by the fact that $x \perp (y \wedge x')$.

Two elements a and b of an orthomodular poset $\mathcal{P} = (P, \leq, ', 0, 1)$ are said to commute (with each other) – in short, $a \mathrm{C} b$ – if there exist three mutually orthogonal elements c, d, e of P such that $a = c \vee d$ and $b = d \vee e$. It is well known (cf. e. g. [8]) that the elements c, d, e are unique if they exist, namely $c = a \wedge b'$, $d = a \wedge b$ and $e = a' \wedge b$.

There are numerous characterizations and implications concerning the property $a \mathrm{C} b$ (cf. e. g. [2] and [8]). We list the very conditions we need for our further considerations, and when this seems essential, we will refer to them.

Properties of C

For an orthomodular poset $\mathcal{P} = (P, \leq, ', 0, 1)$ the following conditions for $a, b \in P$ are equivalent:

(i) $a \mathrm{C} b$

(ii) The subset $\{a, b\}$ of P is Boolean, i. e. there exists a Boolean subalgebra of $\mathcal{P}$ containing a and b.

(iii) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ for three arbitrary but fixed elements of the four elements a, a', b, b'

(iv) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ for three arbitrary but fixed elements of the four elements a, a', b, b'

Moreover, the following holds:

(v) If a C b then x C y for all $x, y \in \{0, a, a', b, b', 1\}$.

(vi) If $a \leq b$ then a C b.

(vii) If $a \perp b$ then a C b.

(viii) If a C b then $a \vee b$ and $a \wedge b$ exist.

2. Characterizing Boolean subsets

As usual, we will write $\bigvee\limits_{i \in I} s_i$ and $\bigwedge\limits_{i \in I} s_i$ for the supremum and infimum, respectively, of the elements $s_i, i \in I$, and we will use the notation $\bigvee S$ and $\bigwedge S$ if $S = \{s_i \mid i \in I\}$. Assume that S is a finite subset of pairwise orthogonal elements of an orthomodular poset $\mathcal{P} = (P, \leq, ', 0, 1)$. Then it can immediately be seen that $\bigvee S$ exists.

The following result is well-known (cf. [3], [4] and [8]):

Lemma 2.1. *Let A be a finite subset of mutually orthogonal elements of an orthomodular poset $\mathcal{P} = (P, \leq, ', 0, 1)$. Then the subset $\{\bigvee D \mid D \subseteq A\}$ of P is Boolean.*

Now, we can prove our main theorem:

Theorem 2.2. *Let $\mathcal{P} = (P, \leq, ', 0, 1)$ be an orthomodular poset, $1 \leq k < n$ and A an n-element subset of P. If any k-element subset of A is Boolean and $(\bigwedge B)$ C $(\bigwedge D)$ for all k-element subsets B and D of A then also any $(k+1)$-element subset of A is Boolean.*

Proof. With regard to the properties of C the case $k = 1$ is obvious. Now assume $k > 1$ and let A be the set $\{a_1, ..., a_n\}$.
First, we show by induction on s that for all $s = k, k-1, ..., 1$ we have

$$a_1 \wedge ... \wedge a_s \wedge a'_{s+1} \wedge ... \wedge a'_k \text{ C } a_{k+1}.$$

Since $(a_1 \wedge ... \wedge a_k)$ C $(a_2 \wedge ... \wedge a_{k+1})$ the infimum

$$(a_1 \wedge ... \wedge a_k) \wedge (a_2 \wedge ... \wedge a_{k+1}) = a_1 \wedge ... \wedge a_{k+1}$$

exists, and due to property (iii) of C we obtain

$$\begin{aligned}
a_1\wedge\ldots\wedge a_k &= (a_1\wedge\ldots\wedge a_k)\wedge((a_2\wedge\ldots\wedge a_{k+1})\vee(a_2\wedge\ldots\wedge a_{k+1})')\\
&= ((a_1\wedge\ldots\wedge a_k)\wedge(a_2\wedge\ldots\wedge a_{k+1}))\vee((a_1\wedge\ldots\wedge a_k)\wedge(a_2'\vee\ldots\vee a_{k+1}'))\\
&= (a_1\wedge\ldots\wedge a_{k+1})\vee(a_1\wedge\ldots\wedge a_k\wedge(a_2'\vee\ldots\vee a_{k+1}'))\\
&= (a_1\wedge\ldots\wedge a_{k+1})\vee(a_1\wedge((a_2\wedge\ldots\wedge a_k)\wedge((a_2\wedge\ldots\wedge a_k)'\vee a_{k+1}')))\\
&= (a_1\wedge\ldots\wedge a_{k+1})\vee(a_1\wedge((a_2\wedge\ldots\wedge a_k)\wedge a_{k+1}'))\\
&= (a_1\wedge\ldots\wedge a_{k+1})\vee(a_1\wedge\ldots\wedge a_k\wedge a_{k+1}')\\
&= ((a_1\wedge\ldots\wedge a_k)\wedge a_{k+1})\vee((a_1\wedge\ldots\wedge a_k)\wedge a_{k+1}').
\end{aligned}$$

Hence, again by property (iii) of C, $a_1\wedge\ldots\wedge a_k \;\mathrm{C}\; a_{k+1}$, which proves our assertion for $s=k$.

Now, assume $1<s\le k$ and

$$a_1\wedge\ldots\wedge a_i\wedge a_{i+1}'\wedge\ldots\wedge a_k' \;\mathrm{C}\; a_{k+1}$$

for all $i=s,\ldots,k$. Our goal is to show that

$$a_1\wedge\ldots\wedge a_{s-1}\wedge a_s'\wedge\ldots\wedge a_k' \;\mathrm{C}\; a_{k+1}.$$

According to the properties of C,

$$a_1\wedge\ldots\wedge a_s\wedge a_{s+1}'\wedge\ldots\wedge a_k' \;\mathrm{C}\; a_{k+1}'$$

and hence

$$a_1\wedge\ldots\wedge a_s\wedge a_{s+1}'\wedge\ldots\wedge a_k'\wedge a_{k+1}' = (a_1\wedge\ldots\wedge a_s\wedge a_{s+1}'\wedge\ldots\wedge a_k')\wedge a_{k+1}'$$

exist. In the following, let us denote a by a^1 and a' by a^{-1}. For reasons of symmetry also $a_1^{i_1}\wedge\ldots\wedge a_{k+1}^{i_{k+1}}$ exists whenever

$$|\{j\in\{1,\ldots,k+1\}\mid i_j=1\}|\ge s.$$

Now, we compute by relying on the properties of C

$$\begin{aligned}
&(a_1\wedge\ldots\wedge a_{s-1}\wedge a_s'\wedge\ldots\wedge a_k')\wedge(a_{k+1}'\vee(a_1\wedge\ldots\wedge a_{s-1}\wedge a_s'\wedge\ldots\wedge a_k')')\\
&\quad= (a_1\wedge\ldots\wedge a_{s-1}\wedge a_s'\wedge\ldots\wedge a_k')\wedge(a_{k+1}'\vee a_1'\vee\ldots\vee a_{s-1}'\vee a_s\vee\ldots\vee a_k)\\
&\quad= (a_1\wedge\ldots\wedge a_{s-1}\wedge a_s'\wedge\ldots\wedge a_{k-1}')\wedge\\
&\qquad\wedge(a_k'\wedge(a_k\vee(a_1'\vee\ldots\vee a_{s-1}'\vee a_s\vee\ldots\vee a_{k-1}\vee a_{k+1}')))\\
&\quad= (a_1\wedge\ldots\wedge a_{s-1}\wedge a_s'\wedge\ldots\wedge a_{k-1}')\wedge\\
&\qquad\wedge(a_k'\wedge(a_1'\vee\ldots\vee a_{s-1}'\vee a_s\vee\ldots\vee a_{k-1}\vee a_{k+1}'))\\
&\quad= a_k'\wedge((a_1\wedge\ldots\wedge a_{s-1}\wedge a_s'\wedge\ldots\wedge a_{k-1}')\wedge\\
&\qquad\wedge((a_1\wedge\ldots\wedge a_{s-1}\wedge a_s'\wedge\ldots\wedge a_{k-1}')'\vee a_{k+1}'))\\
&\quad= a_k'\wedge((a_1\wedge\ldots\wedge a_{s-1}\wedge a_s'\wedge\ldots\wedge a_{k-1}')\wedge a_{k+1}')\\
&\quad= a_1\wedge\ldots\wedge a_{s-1}\wedge a_s'\wedge\ldots\wedge a_{k-1}'\wedge a_k'\wedge a_{k+1}'\\
&\quad= (a_1\wedge\ldots\wedge a_{s-1}\wedge a_s'\wedge\ldots\wedge a_{k-1}'\wedge a_k')\wedge a_{k+1}'
\end{aligned}$$

showing

$$a_1 \wedge \ldots \wedge a_{s-1} \wedge a'_s \wedge \ldots \wedge a'_k \text{ C } a_{k+1}$$

which completes the proof by induction.

We have shown that

$$a_1 \wedge a_2^{i_2} \wedge \ldots \wedge a_k^{i_k} \text{ C } a_{k+1}.$$

for all $i_2, \ldots, i_k \in \{-1, 1\}$ and we point out that the elements

$$a_1^{i_1} \wedge \ldots \wedge a_{k+1}^{i_{k+1}} = (a_1^{i_1} \wedge \ldots \wedge a_k^{i_k}) \wedge (a_2^{i_2} \wedge \ldots \wedge a_{k+1}^{i_{k+1}})$$

$(i_1, \ldots i_{k+1} \in \{-1, 1\})$ exist and are mutually orthogonal. Further, we obtain

$$\begin{aligned} a_1 &= \bigvee_{i_2,\ldots,i_k \in \{-1,1\}} (a_1 \wedge a_2^{i_2} \wedge \ldots \wedge a_k^{i_k}) \\ &= \bigvee_{i_2,\ldots,i_k \in \{-1,1\}} (((a_1 \wedge a_2^{i_2} \wedge \ldots \wedge a_k^{i_k}) \wedge a_{k+1}) \vee ((a_1 \wedge a_2^{i_2} \wedge \ldots \wedge a_k^{i_k}) \wedge a'_{k+1})) \\ &= \bigvee_{i_2,\ldots,i_k \in \{-1,1\}} ((a_1 \wedge a_2^{i_2} \wedge \ldots \wedge a_k^{i_k} \wedge a_{k+1}) \vee (a_1 \wedge a_2^{i_2} \wedge \ldots \wedge a_k^{i_k} \wedge a'_{k+1})) \\ &= \bigvee_{i_2,\ldots,i_{k+1} \in \{-1,1\}} (a_1 \wedge a_2^{i_2} \wedge \ldots \wedge a_{k+1}^{i_{k+1}}). \end{aligned}$$

For reasons of symmetry, we also have

$$a_j = \bigvee_{i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_{k+1} \in \{-1,1\}} (a_1^{i_1} \wedge \ldots \wedge a_{j-1}^{i_{j-1}} \wedge a_j \wedge a_{j+1}^{i_{j+1}} \wedge \ldots \wedge a_{k+1}^{i_{k+1}})$$

for all $j = 1, \ldots, k+1$ which, according to Lemma 2.1, shows that $\{a_1, \ldots, a_{k+1}\}$ is Boolean. By symmetry, it follows that every $(k+1)$-element subset of A is Boolean. ∎

Corollary 2.3. *Let $\mathcal{P} = (P, \leq, ', 0, 1)$ be an orthomodular poset, $n > 1$ and A an n-element subset of P. Then A is Boolean if and only if $(\bigwedge B) \text{ C } (\bigwedge D)$ for every $k \in \{1, \ldots, n-1\}$ and every k-element subsets B and D of A.*

Remark 2.4. According to Theorem 2.2, $\bigwedge B$ exists for all k-element subsets of A, $k \in \{1, 2, \ldots, n-1\}$.

3. Remarks about algorithmic aspects

If one wants to check whether a subset $A = \{a_1, \ldots, a_n\}$ of an orthomodular poset $\mathcal{P}$ is Boolean, one can proceed as follows:

(1) Check whether $a_i \text{ C } a_j$ for $i, j \in \{1, 2, \ldots, n\}, i \neq j$, e. g. by looking for the existence of $a_i \wedge a_j$, $a_i \wedge a'_j$ and whether $(a_i \wedge a_j) \vee (a_i \wedge a'_j) = a_i$. If these elements exist and the equation holds (which is equivalent to $a_i \text{ C } a_j$) and if $\mathcal{P}$ will be a lattice (as in the case of a Hilbert space logic) then one is done; A is Boolean. If these elements exist and the equation holds and $\mathcal{P}$ is not a lattice then

(2) for $r \in \{1, ..., \binom{n}{2}\}$, denote the elements $a_i \wedge a_j, i, j \in \{1, ..., n\}, i \neq j$, by b_r and check, whether $b_r \mathrm{C}\, b_s$ for $r, s \in \{1, ..., \binom{n}{2}\}, r \neq s$. Assuming, this is the case then

(3) replenish the elements b_r to triples $a_i \wedge a_j \wedge a_k$, $i, j, k \in \{1, ..., \binom{n}{3}\}$, $i \neq j \neq k \neq i$, and so on,

until one comes across a pair $(\bigwedge B, \bigwedge D)$ of infima of subsets B and D of A with $|B| = |D| \leq n - 1$ which do not commute, in which case A is not Boolean; otherwise, the subset A is Boolean.

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